Algebraic Geometry: Filling in the Gaps

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Contents

Introduction

Spec **and the Structure Sheaf**

1.1 Spec **of a Ring**

Throughout this expository paper, we take our rings to be commutative with identity. We begin with the definition of Spec:

Definition 1.1.1. If R is a commutative ring, then $\operatorname{Spec} R$ is a priori a set defined by:

$$\operatorname{Spec} R = \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal of } R\}$$

Example 1.1.1. Let $R = \mathbb{Z}$, then we have that $\operatorname{Spec} \mathbb{Z}$ can be identified with the set of all prime numbers. Moreover, if R is a field, then $\operatorname{Spec} R$ is the singleton set consisting only of the zero ideal $\langle 0 \rangle$. If $R = \mathbb{R}[x]$ is the polynomial ring with real coefficients, then:

$$\operatorname{Spec} R = \{\langle 0 \rangle, \langle x - r \rangle, \langle x^2 + bx + c \rangle : r, b, c \in \mathbb{R}, b^2 - 4c < 0\}$$

Note that if $\phi : A \to B$ is a ring homomorphism between commutative rings A and B, we have that there is induced set map:

$$\begin{aligned} \psi : \operatorname{Spec} B &\longrightarrow \operatorname{Spec} A \\ \mathfrak{q} &\longmapsto \phi^{-1}(q) \end{aligned}$$

turning Spec into a contravariant functor from the category of commutative rings to the category of sets. We will shortly put a topology on $\operatorname{Spec} R$ so that the induced set map is actually a continuous map between topological spaces.

Definition 1.1.2. Let I be an ideal of a commutative ring R, then we define the set $\mathbb{V}(I)$ to be:

$$\mathbb{V}(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subset \mathfrak{p}\}$$

If $f \in A$, then we take $\mathbb{V}(f) := \mathbb{V}(\langle f \rangle)$, and clearly we have that:

$$\mathbb{V}(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \in \mathfrak{p}\}$$

Similarly for any set S, we define $\mathbb{V}(S) := \mathbb{V}(\langle S \rangle)$.

We now have the following:

Proposition 1.1.1. *Taking the closed sets of* $\operatorname{Spec} R$ *to be* $\mathbb{V}(I)$ *defines a topology on* $\operatorname{Spec} R$ *such that the induced map* $\psi : \operatorname{Spec} B \to \operatorname{Spec} A$ *from* $\phi : A \to B$ *is continuous.*

Proof. We need to check that the finite unions of closed sets are closed, that infinite intersections of closed sets are closed, and that $\emptyset$ and $\operatorname{Spec} R$ are closed. We begin with the latter, note that:

$$\mathbb{V}(\operatorname{Spec} R) = \{\mathfrak{p} \in \operatorname{Spec} R : \operatorname{Spec} R \subset \mathfrak{p}\} = \emptyset$$

and that:

$$\mathbb{V}(\langle 0 \rangle) = \{\mathfrak{p} \in \operatorname{Spec} R : 0 \in \mathfrak{p}\} = \operatorname{Spec} R$$

so the emptyset and $\operatorname{Spec} R$ are closed. Now suppose that I and J are two ideals, then:

$$\mathbb{V}(I) \cup \mathbb{V}(J) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \in \mathbb{V}(I) \text{ or } \mathfrak{p} \in \mathbb{V}(J)\}$$

We claim that this equal to $\mathbb{V}(I \cap J)$. Suppose that $\mathfrak{p} \in \mathbb{V}(I) \cup (J)$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$, if $I \subset \mathfrak{p}$, then we have that $I \cap J \subset I \subset \mathfrak{p}$, and similarly for J, hence $\mathfrak{p} \in \mathbb{V}(I \cap J)$. If $\mathfrak{p} \in \mathbb{V}(I \cap J)$, then $I \cap J \subset \mathfrak{p}$, so let $r \in I \cdot J$, then $r = i \cdot j$ for some $i \in I$ and $j \in J$. It follows that $r \in I \cap J$, so $I \cdot J \subset \mathfrak{p}$. Now suppose that $I \not\subset \mathfrak{p}$, then there exists at least one $i \notin \mathfrak{p}$. It follows that for all $j \in J$, that $i \cdot j \in \mathfrak{p}$, hence $J \subset \mathfrak{p}$. The same argument for J then implies that if $J \not\subset \mathfrak{p}$, then $I \subset \mathfrak{p}$. Note that if neither $I \subset \mathfrak{p}$, nor $J \subset \mathfrak{p}$, we have that there exists an $i \in I$, and a $j \in J$, such that $i, j \notin \mathfrak{p}$, but $i \cdot j\mathfrak{p}$ contradicting the fact that $\mathfrak{p}$ is prime. It follows that if $I \cap J \subset \mathfrak{p}$, then $I \cdot J \subset \mathfrak{p}$, and thus either $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$, implying that $\mathfrak{p} \in \mathbb{V}(I) \cup \mathbb{V}(J)$, hence $\mathbb{V}(I \cap J) = \mathbb{V}(I) \cup \mathbb{V}(J)$, as desired.

Now let I_α be an infinite family of ideals, we claim that:

$$\bigcap_\alpha \mathbb{V}(I_\alpha) = \mathbb{V}\left(\sum_\alpha I_\alpha\right)$$

where $\sum_\alpha I_\alpha$ is the smallest ideal containing I_α. In other words, it is the ideal generated by $\cup_\alpha I_\alpha$. Suppose that $\mathfrak{p} \in \cap_\alpha \mathbb{V}(I_\alpha)$, then we have that $I_\alpha \subset \mathfrak{p}$ for all α. Now since an $i \in \sum_\alpha I_\alpha$, can be written as the a finite sum $\sum_{j=1}^n r_j$, where each $r_j \in I_\alpha \subset \mathfrak{p}$, we have that $i \in \mathfrak{p}$, so $\bigcap_\alpha \mathbb{V}(I_\alpha) \subset \mathbb{V}\left(\sum_\alpha I_\alpha\right)$. Now suppose that $\mathfrak{p} \in \mathbb{V}\left(\sum_\alpha I_\alpha\right)$, then $\sum_\alpha I_\alpha \subset \mathfrak{p}$. It follows that for all α, $I_\alpha \subset \sum_\alpha I_\alpha$, so $I_\alpha \subset \mathfrak{p}$, hence $\mathfrak{p} \in \mathbb{V}(I_\alpha)$ for all α. It follows that $\mathfrak{p} \in \cap_\alpha \mathbb{V}(I_\alpha)$ implying the claim.

Let $\phi : A \to B$ be a ring homomorphism, and $\psi : \operatorname{Spec} B \to \operatorname{Spec} A$ be the corresponding set map. We need only show that for each $I \subset A$, that $\psi^{-1}(\mathbb{V}(I))$ is a closed set in $\operatorname{Spec} B$. We have that:

$$\begin{aligned}
\psi^{-1}(\mathbb{V}(I)) =& \{\mathfrak{q} \in \operatorname{Spec} B : \psi(\mathfrak{q}) \in \mathbb{V}(I)\} \\
=& \{\mathfrak{q} \in \operatorname{Spec} B : \phi^{-1}(\mathfrak{q}) \in \mathbb{V}(I)\} \\
=& \{\mathfrak{q} \in \operatorname{Spec} B : I \subset \phi^{-1}(\mathfrak{q})\} \\
=& \{\mathfrak{q} \in \operatorname{Spec} B : \phi(I) \subset \mathfrak{q}\} \\
=& \{\mathfrak{q} \in \operatorname{Spec} B : \langle \phi(I) \rangle \subset \mathfrak{q}\} \\
=& \mathbb{V}(\langle \phi(I) \rangle) \\
=& \mathbb{V}(\phi(I))
\end{aligned}$$

so ψ is a continuous map. □

The above topology is called the **Zariski topology** on $\operatorname{Spec} R$. We also have the following helpful lemma:

Lemma 1.1.1. *Let R be a commutative ring, then the following relations hold:*

a) $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$

b) $J \subset I \Longrightarrow \mathbb{V}(J) \supset \mathbb{V}(I)$

c) $\mathbb{V}(I) \subset \mathbb{V}(J) \Longleftrightarrow \sqrt{I} \supset \sqrt{J}$

Proof. First note that the radical of I is defined by:

$$\sqrt{I} = \{r \in R : \exists n \in \mathbb{Z}^+, r^n \in I\} = \bigcap_{\mathfrak{p} \in \mathbb{V}(I)} \mathfrak{p}$$

If $\mathfrak{p} \in \mathbb{V}(I)$, then we have that $I \subset \mathfrak{p}$. Suppose that $r \in \sqrt{I}$, then we have that $r^n \in I$, so $r^n \in \mathfrak{p}$. We can write $r^n = r^{n-1} \cdot r$, so either $r^{n-1} \in \mathfrak{p}$ or $r \in \mathfrak{p}$. If $r \in \mathfrak{p}$, then we are done. If $r^{n-1} \in \mathfrak{p}$, then we repeat the process until we come to conclusion that $r^2 \in \mathfrak{p}$, implying $r \in \mathfrak{p}$, so $\sqrt{I} \subset \mathfrak{p}$, hence $\mathbb{V}(I) \subset \mathbb{V}(\sqrt{I})$. If $\mathfrak{p} \in \mathbb{V}(\sqrt{I})$, then we have that $\sqrt{I} \subset \mathfrak{p}$, however clearly $I \subset \sqrt{I}$, so $I \subset \mathfrak{p}$, hence $\mathbb{V}(\sqrt{I}) \subset \mathbb{V}(I)$, implying *a*).

Now suppose that $J \subset I$, and let $\mathfrak{p} \in \mathbb{V}(I)$. It follows that $I \subset \mathfrak{p}$, so $J \subset I \subset \mathfrak{p}$, implies that $\mathfrak{p} \in \mathbb{V}(J)$, so $\mathbb{V}(J) \supset \mathbb{V}(I)$, hence we have *b*).

Finally suppose $\mathbb{V}(I) \subset \mathbb{V}(J)$. By definition:

$$\sqrt{J} = \bigcap_{\mathfrak{p} \in \mathbb{V}(J)} \mathfrak{p} \qquad \text{and} \qquad \sqrt{I} = \bigcap_{\mathfrak{p} \in \mathbb{V}(I)} \mathfrak{p}$$

Suppose that $r \in \sqrt{J}$, then $r \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathbb{V}(J)$. Since all $\mathfrak{p} \in \mathbb{V}(I)$ lie in $\mathbb{V}(J)$ as well, it follows that $r \in \sqrt{I}$ hence $\sqrt{J} \subset \sqrt{I}$. If $\sqrt{J} \subset \sqrt{I}$, then by *b*) and *a*) we have that $\mathbb{V}(I) \subset \mathbb{V}(J)$ implying *c*).

□

We want to develop a basis for the Zariski topology on $\operatorname{Spec} R$.

Definition 1.1.3. For each $r \in R$, define the **distinguished open** to be:

$$U_f = \mathbb{V}(f)^c = \operatorname{Spec} R \setminus \mathbb{V}(f)$$

Lemma 1.1.2. *The set of distinguished opens form a basis for the Zariski topology on* $\operatorname{Spec} R$*.*

Proof. Suppose that $U \subset \operatorname{Spec} R$ is an open subset, then for some I we have that:

$$\begin{aligned} U =& \mathbb{V}(I)^c \\ =& \mathbb{V}\left(\sum_{i \in I} \langle i \rangle\right)^c \\ =& \left(\bigcap_{i \in I} \mathbb{V}(i)\right)^c \\ =& \bigcup_{i \in I} U_i \end{aligned}$$

so any open set is the arbitrary union of distinguished opens, hence the distinguished opens generate the Zariski topology on $\operatorname{Spec} R$. □

Note that if $\mathfrak{q} \in U_f \cap U_g$, then $f \notin \mathfrak{q}$ and $g \notin \mathfrak{q}$, so $fg \notin \mathfrak{q}$, hence $\mathfrak{q} \in U_{fg}$. We thus have that the intersection of two distinguished opens is again a distinguished open. We also have the following lemma, akin to Lemma 1.1.1:

Lemma 1.1.3. *For all $f, g \in R$, the distinguished opens satisfy:*

a) $U_{f^n} = U_f$

b) $U_f \subset U_g \iff \sqrt{\langle f \rangle} \subset \sqrt{\langle g \rangle}$

c) $U_f \subset U_g \iff \exists m \in \mathbb{Z}^+, r \in R, f^m = r \cdot g$

Proof. Suppose that $\mathfrak{q} \in U_{f^n}$, then $f^n \notin \mathfrak{q}$, however this implies that both f^{n-1} and f are not in $\mathfrak{q}$, so $\mathfrak{q} \in U_f$. Now suppose that $\mathfrak{q} \in U_f$, then $f \notin \mathfrak{q}$, so $f^2 \notin \mathfrak{q}$. Assume that $f^n \notin \mathfrak{q}$, then $f^{n+1} = f^n \cdot f \notin \mathfrak{q}$, so $f^n \notin \mathfrak{q}$ by induction. This then implies *a*).

Suppose that $U_f \subset U_g$, then we have that:

$$\mathbb{V}(f)^c \subset \mathbb{V}(g)^c \implies \mathbb{V}(f) \supset \mathbb{V}(g)$$

It follows from Lemma 1.1.1 that $\sqrt{\langle f \rangle} \subset \sqrt{\langle g \rangle}$. Suppose that $\sqrt{\langle f \rangle} \subset \sqrt{\langle g \rangle}$, then again from Lemma 1.1.1, we have that $\mathbb{V}(g) \subset \mathbb{V}(f)$, taking compliments we thus have shown *b*).

For *c*), we see that if $U_f \subset U_g$, then $\sqrt{\langle f \rangle} \subset \sqrt{\langle g \rangle}$, implying that $f \in \sqrt{\langle g \rangle}$, so there exists some $m \in \mathbb{Z}^+$, and some $r \in R$ such that $f^m = r \cdot g$. Conversely, if we have that $f^m = r \cdot g$, then we have that $f \in \sqrt{\langle () \, g \rangle}$. So suppose that $a \in \sqrt{\langle f \rangle}$, then $a^k = p \cdot f$ for some $k \in \mathbb{Z}^+$, and some $p \in R$. It follows that:

$$(a^k)^m = p^m \cdot f^m = (p^m \cdot r) \cdot g \in \langle g \rangle$$

so $\sqrt{\langle f \rangle} \subset \sqrt{\langle g \rangle}$, and by *b*) we have that $U_f \subset U_g$, implying *c*). □

We now want to show that each U_f is actually homeomorphic to Spec of some ring. We begin with the following definition:

Definition 1.1.4. Let A be a commutative ring, S be a multiplicatively closed set, then the **localization** of A by S, denoted $S^{-1}A$, is a ring equipped with a morphism $\pi : A \to S^{-1}A$, such that for all $s \in S$ $\pi(s)$ is invertible in $S^{-1}A$, and for any homomorphism $\phi : A \to B$ where $\phi(s)$ is a unit for all $s \in S$, there exists a unique homomorphism $\theta : S^{-1}A \to B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \\ S^{-1}A \arrow[ur, "\theta"'] & \end{tikzcd}$$

Our first goal is to show that such a ring exists.

Proposition 1.1.2. *Let A be a ring, and S be a multiplicatively closed set. Then $S^{-1}A$ exists, and is unique up to unique isomorphism.*

Proof. We define an equivalence relation on the set $A \times S$ by:

$$(a, s) \sim (b, t) \iff \exists u \in S, u(at - sb) = 0$$

It is clear that this relation is symmetric, reflexive, and with some work transitive, hence it it indeed defines an equivalence relation. We claim that $A \times S/\sim$ has the structure of a ring. We define addition by:

$$[a, s] + [b, t] = [at + bs, ts]$$

We check that this well defined. Suppose that $[f, v] = [a, s]$, then we need to show that:

$$[ft + bv, tv] = [at + bs, ts]$$

so we need to find a u such that:

$$u(ft^2s + bvts - at^2v + btsv) = u(ft^2s - at^2v) = 0$$

Note that there exists a w such that $w(fs - av) = 0$, hence with $u = w$ we have that:

$$w(ft^2s - at^2v) = t^2(w[fs - av]) = t^2 \cdot 0 = 0$$

so addition is well defined. It is then clear that for any s, the zero element is given by $[0, s]$, and that any $[a, s]$, has inverse given by $[-a, s]$, so $A \times S/\sim$ is an abelian group. We define a ring structure on $A \times S/\sim$ by:

$$[a, s] \cdot [b, t] = [ab, st]$$

We again wish to check that this well defined, so let $[f, v] = [a, s]$, then:

$$[f, v] \cdot [b, t] = [fb, vt]$$

We again want to find a u such that:

$$u \cdot (fbst - abvt) = 0$$

Let $u = w$, then we have that:

$$wfbst - wabvt = bt(wfs - wav) = 0$$

so multiplication is well defined. It is then clear that the multiplicative identity is given by $[1, 1]$ which is then clearly equivalent to $[s, s]$ for any $s \in S$.

Let the map $\pi : A \to A \times S/\sim$ be given by:

$$a \longmapsto [a, 1]$$

This is then clearly a ring homomorphism, and we see that for:

$$\theta \circ \pi = \phi$$

we must have that:

$$\theta([a,1]) = \phi(a)$$

for all $a \in A$. We thus define θ by:

$$\theta([a,s]) = \phi(a) \cdot \phi(s)^{-1}$$

where $\phi(s)^{-1}$ exists, as $\phi(S)$ is a set of units in B. This uniquely determines θ, so long as it is well defined. We check that this well defined, let $[a,s] = [f,v]$, then $wav = wfs$, so we have that:

$$\phi(w) \cdot \phi(a) \cdot \phi(v) = \phi(w) \cdot \phi(f) \cdot \phi(s)$$

Since $\phi(w), \phi(s), \phi(v)$ are all units, we then have that by multiplying both sides by $\phi(w)^{-1}$, $\phi(s)^{-1}$, and $\phi(v^{-1})$:

$$\theta([a,s])\phi(a) \cdot \phi(s)^{-1} = \phi(f) \cdot \phi(v)^{-1} = \theta([f,v])$$

To see this is a ring homomorphism, we note that:

$$\theta([a,s]) + \theta([b,t]) = \phi(a)\phi(s)^{-1} + \phi(b) \cdot \phi(t)^{-1}$$

while:

$$\theta([at+bs,st]) = \phi(at+bs) \cdot \phi(st)^{-1} = \phi(a) \cdot \phi(s)^{-1} + \phi(b) \cdot \phi(t)^{-1}$$

so θ respects addition. Moreover,

$$\theta([a,s]) \cdot \theta([b,t]) = \phi(a) \cdot \phi(b) \cdot \phi(s)^{-1} \cdot \phi(t)^{-1}$$

while:

$$\theta([ab,st]) = \phi(ab)\phi(st)^{-1} = \phi(a) \cdot \phi(b) \cdot \phi(s)^{-1} \cdot \phi(t)^{-1}$$

so θ respects multiplication as well, and is thus a ring homomorphism such that $\theta \circ \pi = \phi$. It follows that $A \times S/\sim$ satisfies Definition 1.1.4, and so $S^{-1}A$ exists, and is unique up to unique isomorphism as it is defined by a universal property[1]. $\square$

Note that the localization of A by S is easily seen to mimic multiplication and addition of fractions, it is for the purpose that going forward we denote the equivalence classes $[a,s]$ by:

$$\frac{a}{s}$$

Moreover, if $f \in A$, we denote by A_f the localization of A by the multiplicatively closed subset $\{1, f, f^2, \dots\}$, and if $\mathfrak{p}$ is a prime ideal of A, we denote by $A_{\mathfrak{p}}$ the localization of A by the multiplicatively closed subset $(A - \mathfrak{p})$. Moreover, A_f can be thought of as the polynomial ring:

$$A_f = A[1/f]$$

We have the following lemma:

Lemma 1.1.4. *Let A be a commutative ring, and $f, g \in A$, then there exist unique isomorphisms:*

$$(A_f)_g \cong A_{fg} \cong (A_g)_f$$

where in the first and third terms g and f are really the equivalence classes $g/1$ and $f/1$. Moreover, if $\sqrt{\langle g \rangle} = \sqrt{\langle f \rangle}$, then there exists a unique isomorphism:

$$A_f \cong A_g$$

[1] Note that $\pi(S)$ is a set of units in $S^{-1}A$, so one can apply the universal property to any other object satisfying said property and get a unique isomorphism between the two.

Proof. Clearly we need only prove that $(A_f)_g \cong A_{fg}$, as the proof of the other isomorphism will be identical. We first note that the map the natural map $\pi_{fg} : A \to A_{fg}$ maps:

$$f \longmapsto \frac{f}{1}$$

This is clearly a unit in A_{fg} where $(f/1)^{-1}$ is given by g/fg. It follows that there exists a unique map $\omega : A_f \to A_{fg}$ given by:

$$\frac{a}{f^k} \longmapsto \frac{a}{1} \cdot \frac{g^k}{f^k g^k} = \frac{ag^k}{f^k g^k}$$

Now suppose that $\phi : A_f \to B$ is any map such that $g/1$ is a unit in B, we want to show that there exists a unique map $\theta : A_{fg} \to B$ such that:

$$\theta \circ \omega = \phi$$

However, note that:

$$\omega \circ \pi_f(a) = \frac{a}{1}$$

so:

$$\omega \circ \pi_f = \pi_{fg}$$

Moreover, we obtain a unique map $\psi : A \to B$ such that both f and g are units in B, defined by:

$$\psi = \phi \circ \pi_f$$

We thus have the following diagram:

$$\begin{tikzcd}
A \arrow[d, "\pi_f"] \arrow[drr, "\psi"] \arrow[dd, bend right, "\pi_{fg}"'] & & \\
A_f \arrow[d, "\omega"] \arrow[rr, "\phi"] & & B \\
A_{fg} \arrow[urr, dotted, "\exists!\theta"'] & &
\end{tikzcd}$$

where θ is the unique homomorphism such that $\theta \circ \pi_{fg} = \psi$. We need to check that θ satisfies:

$$\theta \circ \omega = \phi$$

Let $\frac{a}{f^k} \in A_f$, then by definition we have that:

$$\phi(a/f^k) = \psi(a) \cdot \psi(f^k)^{-1}$$

Meanwhile:

$$\theta \circ \omega(a/f^k) = \theta(ag^k/(g^k f^k)) = \psi(ag^k) \cdot \psi(g^k f^k)^{-1} = \psi(a) \cdot \psi(f^k)^{-1}$$

so θ is the unique map which satisfies $\theta \circ \omega = \phi$. It follows that A_{fg} satisfies the universal property of the localization of A_f by $g/1$, then $(A_f)_g$ is uniquely isomorphic to A_{fg}.

Note that if $\sqrt{\langle f \rangle} = \sqrt{\langle g \rangle}$, then there exists elements $u, v \in A$ and $m, n > 0$ such that:

$$f^m = ug \qquad \text{and} \qquad g^n = vf$$

It follows that $\pi_g(f)$ is a unit in A_g and that $\pi_f(g)$ is a unit in A_f with inverses given by v/g^n and u/f^m respectively. We thus have the following commutative diagram:

$$\begin{tikzcd} A \arrow[rr, "\pi_g"'] \arrow[dd, "\pi_f"'] & & A_g \arrow[ddll, shift left, "\theta_g"] \\ & & \\ A_f \arrow[uurr, shift left, "\theta_f"] & & \end{tikzcd}$$

Let $a/f^k \in A_f$, then:

$$\theta_g(a/f^k) = \frac{av^k}{g^{nk}}$$

and then:

$$\theta_f \circ \theta_g(a/f^k) = \frac{av^k u^{nk}}{f^{nkm}}$$

so we need to find a K such that:

$$f^K(av^k u^{nk} f^k - f^{nkm} a) = 0$$

However we see that:

$$av^k u^{nk} f^k = ag^{nk} u^{nk} = af^{nkm}$$

so $K = 0$ will do, and we see that $\theta_f \circ \theta_g = \mathrm{Id}$. The same argument shows that $\theta_g \circ \theta_f = \mathrm{Id}$, so $A_f \cong A_g$ are isomorphic as desired. $\square$

Proposition 1.1.3. *Let A be a commutative ring, and $f \in A$, then the distinguished open set U_f is homeomorphic to* $\operatorname{Spec} A_f$.

Proof. We have a ring homomorphism $\pi : A \to A_f$, which induces a continuous map $\psi : \operatorname{Spec} A_f \to \operatorname{Spec} A$. We first want to show that $\operatorname{im} \psi = U_f$. Suppose that $\mathfrak{p} \in \operatorname{im} \psi$, then $\mathfrak{p}$ is of the form $\pi^{-1}(\mathfrak{q})$ for some $\mathfrak{q} \in \operatorname{Spec} A_f$. Note that:

$$\pi^{-1}(\mathfrak{q}) = \{a \in A : \pi(a) \in \mathfrak{q}\}$$

If $f \in \pi^{-1}(q)$, then we have that $\pi(f) = f/1 \in \mathfrak{q}$, implying that $1 \in \mathfrak{q}$ so it follows that that $f \notin \pi^{-1}(q)$, hence $\mathfrak{p} \in U_f$. Now suppose that $\mathfrak{p} \in U_f$, we want to show that there exists a prime ideal $\mathfrak{q} \in A_f$ such that $\pi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Define $\mathfrak{q}$ by:

$$\mathfrak{q} = \left\{ \frac{p}{f^k} \in A_f : p \in \mathfrak{p}, k \geq 0 \right\}$$

We see that this is an ideal, $-p \in \mathfrak{p}$, $0 \in p$, and for any $b/f^m \in A_f$, we have that $f^{m+k} \in S$, and $pb \in \mathfrak{p}$, hence $bp/(f^{k+m}) \in \mathfrak{q}$. It is prime as if:

$$\frac{a}{f^k} \cdot \frac{b}{f^m} \in \mathfrak{q}$$

then we have that:

$$\frac{ab}{f^{k+m}} = \frac{p}{f^n}$$

for some $p \in \mathfrak{p}$ an $n > 0$. This implies that there exists a $j \geq 0$ such that:

$$f^j(abf^n - pf^{k+m}) = 0$$

We then see that:

$$abf^{j+n} = pf^{k+m_j}$$

implying that $abf^{j+n} \in \mathfrak{p}$. We have that $f^{j+n} \notin \mathfrak{p}$, so $ab \in \mathfrak{p}$, implying either a or b is in $\mathfrak{p}$, so $\mathfrak{q}$ is a prime ideal. It is then clear that:

$$\pi^{-1}(\mathfrak{q}) = \mathfrak{p}$$

as if $a \in \pi^{-1}(\mathfrak{q})$, then we have that $a/1 \in \mathfrak{q}$, so:

$$\frac{a}{1} = \frac{p}{f^k} \Longrightarrow af^k = p$$

so $a \in \mathfrak{p}$. If $a \in \mathfrak{p}$, then we have $a/1 \in \mathfrak{q}$, and $\pi(a) = a/1$, so $a \in \pi^{-1}(\mathfrak{q})$.

The map ψ is then a continuos surjection onto it's image by definition, so we define an inverse map $\eta : U_f \to \operatorname{Spec} A_f$, by:

$$\eta(\mathfrak{p}) = \left\{ \frac{p}{f^k} \in A_f : p \in \mathfrak{p}, k \geq 0 \right\}$$

which as we have just shown is a prime ideal in A_f. We check that these are inverses, let $\mathfrak{p} \in U_f$, then our argument showing that $\operatorname{im} \psi = U_f$ demonstrates:

$$\psi \circ \eta(\mathfrak{p}) = \mathfrak{p}$$

Now suppose that $\mathfrak{q} \in \operatorname{Spec} A_f$, we have that:

$$\eta \circ \psi(\mathfrak{q}) = \eta(\pi^{-1}(\mathfrak{q})) = \left\{ \frac{p}{f^k} \in A_f : p \in \pi^{-1}(q), k \geq 0 \right\} := I$$

Let $p/f^k \in \mathfrak{q}$, then $p/1 \in \mathfrak{q}$, so $p \in \pi^{-1}(\mathfrak{q})$. It follows that $p/f^k \in I$. Now suppose that $p/f^k \in I$, then $p \in \pi^{-1}(q)$, so $p/1 \in \mathfrak{q}$, hence $p/f^k \in \mathfrak{q}$. It follows that $I = \mathfrak{p}$, the two are inverses of one another.

We need to show that η is continuous, it suffices to check that on basis open sets. First note that $U_{a/f^k} = U_{a/1} \subset \operatorname{Spec} A_f$, as if $\mathfrak{q} \in U_{a/f^k}$, then we have that $a/f^k \notin \mathfrak{q}$. Since $f/1 \notin \mathfrak{q}$, we have that $a/f^k \cdot f^k = a/1 \notin \mathfrak{q}$, hence $\mathfrak{q} \in U_{a/1}$. Moreover, if $\mathfrak{q} \in U_{a/1}$, then $a/1 \notin \mathfrak{q}$, and since $f \notin \mathfrak{q}$, we have that $a \cdot 1/f^k \notin \mathfrak{q}$, so $\mathfrak{q} \in U_{a/f^k}$. It thus suffices to check this on distinguished opens of the form U_g for some $g/1 \in A_f$. We see that:

$$\eta^{-1}(U_g) = \{\mathfrak{p} \in U_f : \eta(\mathfrak{p}) \in U_g\}$$

We claim that:

$$\eta^{-1}(U_g) = U_{fg} = U_f \cap U_g \subset \operatorname{Spec} A$$

and would thus be open in the subspace topology on U_f. Let $\mathfrak{p} \in U_{fg}$, then $\mathfrak{p} \in U_f \cap U_g$, so neither g nor f lie in $\mathfrak{p}$. Now, we see that:

$$\eta(\mathfrak{p}) = \left\{ \frac{p}{f^k} : p \in \mathfrak{p}, k \geq 0 \right\}$$

Since $g \notin \mathfrak{p}$, it follows that $g/1 \notin \eta(\mathfrak{p})$, hence $\eta(\mathfrak{p}) \in U_g \subset A_f$. If $\mathfrak{p} \in \eta^{-1}(U_g)$, then we have that $g/1 \notin \eta(\mathfrak{p})$, implying that $g \notin \mathfrak{p}$, so $\mathfrak{p} \in U_f \cap U_g = U_{fg}$. It follows that η is a continuous map, and in particular, the inverse ψ, hence U_f is homeomorphic to $\operatorname{Spec} A_f$, as desired. □

1.2 Some Category Theory: Sheaves, Stalks, Germs, and all that

In this section we go over the basics of sheaf theory, and attempt to take a categorical approach wherever possible. We begin by fixing a topological space X, and a category denoted $\mathscr{C}_X$, who's objects are open sets of X, and morphisms are inclusion maps $\iota_U^V : U \to V$, whenever $U \subset V$. Note that this puts a partial order on $\mathscr{C}_X$, where $U < V \Leftrightarrow U \supset V$[2].

[2]The reason for the reverse inclusion is to due the contravariant nature of a presheaf/sheaf.

Definition 1.2.1. A **pre sheaf** is a contravariant functor $\mathscr{F} : \mathscr{C}_X \to D$ where D is generally one of the following categories: Set, Ab, or Ring[3]. We call the object $\mathscr{F}(U)$ **sections** over U, and the induced maps $\theta^V_U : \mathscr{F}(V) \to \mathscr{F}(U)$, the **restriction maps**. A **sheaf**, is then presheaf such that:

i) Let U_i be an open cover for U, then if $s, t \in \mathscr{F}(U)$ such that $\theta^U_{U_i}(s) = \theta^U_{U_i}(t)$ for all i, then $s = t$.

ii) If U_i is an open cover of U, and there exists $s_i \in \mathscr{F}(U_i)$ such that:

$$\theta^{U_i}_{U_i \cap U_j}(s_i) = \theta^{U_j}_{U_i \cap U_j}(s_j)$$

for all i and j, then there exists a section $s \in \mathscr{F}(U)$ such that $\theta^U_{U_i}(s) = s_i$.

We have the following example:

Example 1.2.1. If X is a topological space, then let $\mathscr{F} = C^0$ assign to each open set of X the ring of continuous real valued functions. This obviously defines a presheaf on X, where the restriction maps are given by $f \in C^0(V) \mapsto f \circ \iota^V_U \in C^0(U)$. Now suppose that U_i is an open cover of U, and $f \circ \iota^U_{U_i} = 0$ for all U_i. Well this implies that $f(p) = 0$ for all $p \in U$, as all $p \in U$ lie in U_i for some i, and $f \circ \iota^U_{U_i}(p) = 0 = f(p)$ by definition, so sheaf axiom one is satisfied[4]. Now suppose that U_i covers U, and $f_i \in C^0(U_i)$ satisfy $f \circ \iota^{U_i}_{U_i \cap U_j} = f_j \circ \iota^{U_i}_{U_i \cap U_j}$. We define a map f by:

$$f(p) = f_i(p)$$

when $p \in U_i$. If $p \in U_i \cap U_j$, then since f_i and f_j agree on the overlap we have that $f_i = f_j$. We show that this is continuous, Let $W \subset \mathbb{R}$ be open, then

$$\begin{aligned} f^{-1}(W) =& \{p \in U : f(p) \in W\} \\ =& \bigcup_i \{p \in U_i : f_i(p) \in W\} \\ =& \bigcup_i f_i^{-1}(W) \end{aligned}$$

however, each f_i is continuos, hence $f \in C^0(U)$, and satisfies $\theta^U_{U_i}(f) = f_i$.

Example 1.2.2. Let X be a smooth manifold. A similar argument shows that the contravariant functor $\mathscr{F} = C^\infty_X$, which assigns to each open set of X the ring of smooth functions $C^\infty(U)$, is a sheaf. Moreover though, if $E \to X$ is a smooth vector bundle over X, then the $\mathscr{F} = \Gamma$, which is the functor that assigns to each open set of X the ring of smooth local sections of E is also a sheaf. Indeed, the restriction maps are just composition of with the inclusions, and sheaf axiom one is satisfied in the same as in Example 1.2.1. Now suppose that U_i is an open cover of U, and $\phi_i \in \Gamma(U_i)$ are smooth sections such that $\theta^{U_i} U_i \cap U_j(\phi_i) = \theta^{U_i} U_i \cap U_j(\phi_i)$. We let ψ_i be a partition of unity subordinate to the open cover U_i, then we see that:

$$\xi_i = \begin{cases} \psi_i \phi_i & \forall x \in U_i \\ 0 & \forall x \notin U_i \end{cases}$$

defines an element $\xi_i \in \Gamma(U)$. We define $\phi \in \Gamma(U)$ by:

$$\phi = \sum_i \xi_i$$

Then this satisfies $\theta^U_{U_k}(\phi) = \phi_k$, as for all $p \in U_k$, we have that for some n:

$$\phi(p) = \sum_{i : U_i \cap U_k \neq \emptyset} \psi_i(p)\phi_i(p) = \sum_{j=1}^n \psi_j(p)\phi_j(p)$$

since all ϕ_j agree with ϕ_k on $U_j \cap U_k$, and $p \in U_k$, this becomes:

$$\phi(p) = \phi_k(p) \cdot \sum_j \psi_j(p) = \phi_k(p)$$

as a partition of unity always sums to one. It follows that $\phi \circ \iota^U_{U_k} = \phi_k$ as desired, so Γ is a sheaf.

[3]By Ring we always mean commutative rings.

[4]If addition is well defined, and a group operation in the objects of your target category, sheaf axiom one is equivalent to the case where s restricted U_i is zero for all i implies that s is zero.

In the case where are sheafs are literally rings/groups of maps to another topological or smooth space, the sheaf axioms encode a sort of locality condition that mimics continuity, and smoothness. When we turn to studying schemes, and locally ringed spaces in generality, it will be good to keep this picture in mind. At times we write $s|_U$, for $\theta_U^V(s)$ when it is understood that $s \in \mathscr{F}(V)$.

Definition 1.2.2. Let X be a topological space, and $\mathscr{F} : \mathscr{C}_X \to D$ a pre sheaf, the **stalk** of $\mathscr{F}$ at $x \in X$, denoted $\mathscr{F}_x$ is an object in D satisfying the following conditions:

a) For all $U \subset V$ where $x \in U$ and V, there exist morphisms $\psi_U : \mathscr{F}(U) \to \mathscr{F}_x$, $\psi_V : \mathscr{F}(V) \to \mathscr{F}_x$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[rr, "\theta_U^V"] \arrow[dr, "\psi_V"'] & & \mathscr{F}(U) \arrow[dl, "\psi_U"] \\
& \mathscr{F}_x &
\end{tikzcd}$$

b) If G is another object in D, equipped with morphisms $\phi_U : \mathscr{F}(U) \to G$, $\phi_V : \mathscr{F}(V) \to G$ for U, V where $x \in U, V$, such that a similar diagram commutes, then there exists a unique map $\phi_x : \mathscr{F}_x \to G$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[rr, "\theta_U^V"] \arrow[dr, "\psi_V"] \arrow[ddr, bend right, "\phi_V"'] & & \mathscr{F}(U) \arrow[dl, "\psi_U"'] \arrow[ddl, bend left, "\phi_U"] \\
& \mathscr{F}_x \arrow[d, "\exists!\phi_x"] & \\
& G &
\end{tikzcd}$$

Elements of $\mathscr{F}_x$ are called **germs**

The astute will notice that this definition is equivalent to the definition of the colimit, or direct limit. In other words, we have that:

$$\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F}(U)$$

As always, when defining something by a universal property, it is important to check that such an object exists.

Proposition 1.2.1. *Let X be a topological space, and $\mathscr{F}$ a presheaf, then for all $x \in X$ the stalk $\mathscr{F}_x$ exists.*

Proof. We work in the category $D = \text{Ring}$, as the proof in this case will imply the others. Define $\mathscr{F}_x$ as the following set:

$$F = \{(U, s) : x \in U, s \in \mathscr{F}(U)\}$$

modulo the equivalence relation

$$(U, s) \sim (V, t)$$

if and only there exists a $W \in U \cap V$ such that $x \in W$ and:

$$s|_W = t|_W$$

One easily checks that this is indeed an equivalence relation on F, thus we set:

$$\mathscr{F}_x = F/\sim$$

We first check that $\mathscr{F}_x$ is indeed a ring. We define addition on $\mathscr{F}_x$ by:

$$[U, s] + [V, t] = [U \cap V, s|_{U \cap V} + t|_{U \cap V}]$$

We need to check that this well defined. Suppose that $[Z, f] = [U, s]$, then we need to show that:

$$[Z \cap V, s|_{Z \cap V} + t|_{Z \cap V}] = [U \cap V, s|_{U \cap V} + t|_{U \cap V}]$$

Well, consider $W = U \cap Z \cap V$, and note that by the functorial properties of the restriction maps we have that:

$$(s|_{Z \cap V} + t|_{Z \cap V})|_W = s|_W + t|_W = (s|_{U \cap V} + t|_{U \cap V})|_W$$

so addition is well defined. Moreover the zero element is given by $[U, 0]$ for any open set U which contains x. Indeed, we have clearly have that:

$$[U, 0] + [V, s] = [U \cap V, s|_{U \cap V}] = [V, s]$$

The inverse of any element $[U, s]$ is then easily seen to be $[U, -s]$, so $\mathscr{F}_x$ is indeed an abelian group. We define a ring structure in the same:

$$[U, s] \cdot [V, t] = [U \cap V, s|_{U \cap V} \cdot t|_{U \cap V}]$$

and the same argument demonstrates that this well defined, and that $[U, 1]$ is the multiplicative identity of $\mathscr{F}_x$, so $\mathscr{F}_x$ is a ring.

For all open sets U, we define a map $\psi_U : \mathscr{F}(U) \to \mathscr{F}_x$ by:

$$s \longmapsto [U, s]$$

Let $V \cap U$, and $\theta_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$ be the restriction map, then we have that:

$$\psi_V \circ \theta_V^U(s) = [V, s|_V]$$

However, we see that $U \cap V = V$, so tautologically we have that:

$$[U, s] = [V, s|_V]$$

it follows that property a) of Definition 1.2.2 is satisfied. Now suppose that for all open U we have ring homomorphisms $\phi_U : \mathscr{F}(U) \to G$, such that $\phi_U = \phi_V \circ \theta_V^U$, then we see that if $\phi_x : \mathscr{F}_x \to G$ exists it must satisfy:

$$\phi_x \circ \psi_U = \phi_U$$

so we define ϕ_x by:

$$\phi_x([U, s]) = \phi_U(s) \tag{1.2.1}$$

We need to check that this well defined; let $[U, s] = [V, t]$, then there exists a $W \subset U \cap V$ such that $s|_W = t|_W$. It follows that:

$$\phi_W(s|_W) = \phi_W(t|_W)$$

however:

$$\phi_W(s|_W) = \phi_U(s)$$

and:

$$\phi_W(t|_W) = \phi_V(t)$$

so $\phi_U(s) = \phi_V(t)$ hence $\phi_x([U, s]) = \phi_x([V, t])$. We check that ϕ_x is a ring homomorphism, let $[U, s]$ and $[V, t] \in \mathscr{F}_x$, then:

$$\phi_x([U, s] + [V, t]) = \phi_{U \cap V}(s|_{U \cap V} + t|_{U \cap V})$$

while:

$$\phi_x([U,s]) + \phi_x([V,t]) = \phi_U(s) + \phi_V(t) = \phi_{U\cap V}(s|_{U\cap V}) + \phi_{U\cap V}(t|_{U\cap V})$$

so by the fact $\phi_{U\cap V}$ is a ring homomorphism, we have that ϕ_x respects addition. The same argument shows that ϕ_x respects multiplication, and sends 0 and 1 to 0 and 1 respectively so ϕ_x is a ring homomorphism. It is unique, as any other ring homomorphism that makes the diagram in *b*) commute must satisfy (1.1). It follows that $F/\sim$ satisfies the properties of Definition 1.2.2, so $\mathscr{F}_x$ exists and is unique up to unique isomorphism. ☐

Definition 1.2.3. Let X be a topological space, and $\mathscr{F}$ and $\mathscr{G}$ be sheaves (pre sheaves) on X. A **morphism of (pre) sheaves** is a natural transformation $F : \mathscr{F} \to \mathscr{G}$. In particular, the a morphism of (pre) sheaves is a family of morphisms $F_U : \mathscr{F}(U) \to \mathscr{G}(U)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "F_U"] \arrow[d, "\theta_V^U"'] & \mathscr{G}(U) \arrow[d, "\theta_V^U"] \\
\mathscr{F}(V) \arrow[r, "F_V"'] & \mathscr{G}(V)
\end{tikzcd}$$

A **isomorphism of sheaves** (**presheaves**) is a natural transformation in which every morphism F_U is an isomorphism. We denote the category of presheaves on X by $\mathrm{PSh}(X)$.

Lemma 1.2.1. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves or sheaves, then there exists a unique map on stalks $F_x : \mathscr{F}_x \to \mathscr{G}_x$.*

Proof. Clearly, we need only define maps $\phi_U : \mathscr{F}(U) \to \mathscr{G}_x$, satisfying $\phi_U = \phi_V \circ \theta_V^U$, then by the universal property of the colimit, we will have a unique map F_x. We define ϕ_U by:

$$\phi_U(s) = [U, F_U(s)]$$

We see that this is clearly a ring homomorphism by our previous work, and that:

$$\phi_V(s|_V) = [V, F_V(s|_v)] = [V, F_U(s)|_V] = [U, F_(s)]$$

implying the claim. ☐

Note that we have that:

$$F_x([U,s]) = [U, F_U(s)]$$

If $s \in \mathscr{F}(U)$, we often denotes its image in $\mathscr{F}_x$ as s_x. Moreover, if it is not understood which stalk $[U,s]$ belongs to, we write $[U,s]_x$. Importantly this lemma implies the following:

Corollary 1.2.1. *Let $F : \mathscr{F} \to \mathscr{G}$ and $G : \mathscr{G} \to \mathscr{H}$ be morphisms of (pre) sheaves, then for all $x \in X$:*

$$(G \circ F)_x = G_x \circ F_x$$

Proof. We have that $G \circ F$ is a morphism $\mathscr{F} \to \mathscr{H}$, so there exists a unique map $(G \circ F)_x : \mathscr{F}_x \to \mathscr{H}_x$ such that:

$$(G\circ F)_x([U,s]) = [U, (G\circ F)_U(s)] = [U, G_U \circ F_U(s)] = G_x([U, F_U(s))) = G_x \circ F_x([U,s])$$

implying the claim. ☐

Lemma 1.2.2. *If $\mathscr{F}$ is a sheaf, then the natural homomorphism:*

$$\mathscr{F}(U) \longrightarrow \prod_{x\in U} \mathscr{F}_x$$
$$s \longmapsto (s_x)$$

is injective.

Proof. Suppose that $s, t \in \mathscr{F}(U)$ and $(s_x) = (t_x)$. Then for each x we have that:

$$[U, s]_x = [U, t]_x$$

implying that there exists $W_x \subset U$ such that:

$$s|_{W_x} = t|_{W_x}$$

We then obtain an open cover $\{W_x\}$ of U such that $s|_{W_x} = t|_{W_x}$, so sheaf axiom one implies the claim. □

Proposition 1.2.2. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X , then F is an isomorphism if and only if $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism for all $x \in X$.*

Proof. If $F : \mathscr{F} \to \mathscr{G}$ is an isomorphism, the there exists an inverse natural transformation given F^{-1} : $\mathscr{F} \to \mathscr{G}$. Let $[U, s] \in \mathscr{F}_x$, then:

$$F_x^{-1} \circ F_x([U, s]) = F_x^{-1}([U, F_U(s)]) = [U, F_U^{-1} \circ F_U(s)] = [U, s]$$

Similarly if $[U, s] \in \mathscr{G}_x$, then we have that:

$$F_x \circ F_x^{-1}([U, s]) = [U, s]$$

so we have that F_x is an isomorphism.

For the converse, note that since the target category of our functors $\mathscr{F}$ and $\mathscr{G}$ is either Set, Ab, or Ring, it suffices to check that F_U is injective and surjective for all U. Note that we get an induced isomorphism:

$$\begin{aligned} \prod_{x \in U} \mathscr{F}_x &\longrightarrow \prod_{x \in U} \mathscr{G}_x \\ (s_x) &\longmapsto (F_x(s_x)) \end{aligned} \tag{1.2.2}$$

as F_x is an isomorphism for all x. Suppose that $F_U(s) = F_U(t)$, then we have that by definition of the stalk map $(F_U(s))_x = F_x(s_x) = F_x(t_x) = (F_U(t))_x$ for all $x \in U$. Since $F_x(s_x) = F_x(t_x)$ for all U, we have that $(s_x) = (t_x)$ so by Lemma 1.2.2 F_U is injective.

Now let $g \in \mathscr{G}(U)$, then by the isomorphism (1.2), we have that there exists a unique sequence $(s_x) \in \prod_{x \in U} \mathscr{F}_x$ such that $(F_x(s_x)) = (g_x)$. Write $[V_x, f^x]_x$ for each s_x in the sequence, and without loss of generality let $V_x \subset U$[5]. Then note that:

$$F_x([V_x, f^x]_x) = [V_x, F_{V_x}(f^x)]_x = [U, g]_x$$

so there exists a $W_x \subset V_x$ such that $F_{V_x}(f^x)|_{W_x} = g|_{W_x}$. Cover U by $\{W_x\}$, then we have sections $f^x|_{W_x} \in \mathscr{F}(W_x)$. We see that:

$$F_{W_x \cap W_y}(f^x|_{W_x \cap W_y}) = g|_{W_x \cap W_y} = F_{W_x \cap W_y}(f^y|_{W_x \cap W_y})$$

and since F is injective, it follows that:

$$f^x|_{W_x \cap W_y} = f^x|_{W_x \cap W_y}$$

so we have a global section $f \in \mathscr{F}(U)$ by sheaf axiom two. We see that:

$$F_U(f)|_{W_x} = F_{W_x}(f|_{W_x}) = F_{W_x}(f^x|_{W_x}) = g|_{W_x}$$

so by sheaf axiom one $F_U(f) = g$, implying that F_U is surjective for all U, and thus F is a natural isomorphism as desired. □

One can easily check that presheaves with values in Ab form an abelian category, as one easily define the kernel and cokernel of a presheaf morphism to be the functor on X that takes U to $\ker F_U$ and $\operatorname{coker} F_U$. This does not work with sheaves, however there is a workaround.

[5] We can always further restrict to $U \cap V_x$ to make this true.

Definition 1.2.4. Let $\mathscr{F}$ be a presheaf on X, then sheafification of $\mathscr{F}$, denoted $\mathscr{F}^\sharp$, is the a sheaf equipped with a morphism $\mathrm{sh} : \mathscr{F} \to \mathscr{F}^\sharp$, such that for all morphisms $\phi : \mathscr{F} \to \mathscr{G}$, where $\mathscr{G}$ is a sheaf, then there exists a unique morphism $\phi^\sharp : \mathscr{F}^\sharp \to \mathscr{G}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\phi"] \arrow[d, "\mathrm{sh}"'] & \mathscr{G} \\
\mathscr{F}^\sharp \arrow[ur, "\phi^\sharp"'] &
\end{tikzcd}$$

As always, we check that such a construction exists, and is thus unique up to unique isomorphism.

Proposition 1.2.3. *Let $\mathscr{F}$ be a presheaf on X, then the sheafification of $\mathscr{F}^\sharp$ exists.*

Proof. We define $\mathscr{F}^\sharp$ on open sets by:

$$\mathscr{F}^\sharp(U) = \left\{ (s_x) \in \prod_{x \in U} \mathscr{F}_x : \forall p \in U, \exists V_p \subset U, \text{ and a } f \in \mathscr{F}(V_p), \text{such that } f_q = s_q, \forall q \in V_x \right\}$$

All this is saying, is that for each $p \in U$, we can find an open neighborhood of p, and a section on that open neighborhood such that the germ of that section at every point agrees with s_q. With that in mind, we quickly check that this is a subring of $\prod_{x \in U} \mathscr{F}_x$. Clearly, $\mathscr{F}^\sharp(U)$ contains the zero section and the multiplicative identity. Moreover, if $(s_x) \in \mathscr{F}^\sharp(U)$, then it's inverse $(-s_x) \in \mathscr{F}^\sharp(U)$, as we just take $-f$ to cover $(-s_x)$ for each $V_p \subset U$. It is closed under addition, and multiplication as if $(s_x), (t_x) \in \mathscr{F}^\sharp(U)$, then we have that:

$$(s_x) + (t_x) = (s_x + t_x) \qquad \text{and} \qquad (s_x) \cdot (t_x) = (s_x \cdot t_x)$$

Let $p \in U$, then there exists W_p, Z_p, $s^p \in \mathscr{F}(W_p)$ and $t^p \in \mathscr{F}(Z_p)$ such that $s^p_q = s_q$ and $t^p_q = t_q$ for all $q \in W_p$ and Z_p respectively. We then see that:

$$(s^p|_{W_p \cap Z_p} + t^p|_{W_p \cap Z_p})_q = (s_q) + (t_q) = (s_q + t_q)$$

and similarly for multiplication. It follows that for all p there exists sections on small enough neighborhoods that agree with addition or multiplication of two elements in $\mathscr{F}^\sharp(U)$, so $\mathscr{F}^\sharp(U)$ is a subring of $\prod_{x \in U} \mathscr{F}_x$, and thus a ring.

We check that $U \mapsto \mathscr{F}^\sharp(U)$ is a contravariant functor. Define restriction maps θ^U_V in the obvious way:

$$\begin{aligned} \theta^U_V : \prod_{x \in U} \mathscr{F}_x &\longrightarrow \prod_{x \in V} \mathscr{F}_x \\ (s_x) &\longmapsto (s_x) \end{aligned}$$

which is clearly a ring homomorphism, as we essentially just toss out the elements in the (s_x) where $x \in\notin U$. Restricting the restriction maps to $\mathscr{F}^\sharp(U)$, it is clear that θ^U_V has image in $\mathscr{F}^\sharp(V)$ as restricting sections commutes with the map from sections to stalks. It is then clear that:

$$\theta^U_U = \mathrm{Id} \text{ and } \theta^V_W \circ \theta^U_V = \theta^U_W$$

so $\mathscr{F}^\sharp$ is a presheaf.

To see that $\mathscr{F}^\sharp$ is a sheaf, let $\{U_i\}$ be an open cover of U, and $(s_x) \in \mathscr{F}^\sharp(U)$ such that $(s_x)|_{U_i} = 0$. Then clearly by the definition of the restriction map, $s_x = 0$ for all $x \in U$, so $(s_x) = 0$ and sheaf axiom one is satisfied. Now suppose that we have sections $(s^i_x) \in \mathscr{F}^\sharp(U_i)$ such that $(s^i_x)|_{U_i \cap U_j} = (s^j_x)|_{U_i \cap U_j}$, then we define a section $(s_x) \in \mathscr{F}^\sharp(U)$ by:

$$(s_x) = (s^i_x)$$

whenever $x \in U_i$. If $x \in U_i \cap U_j$, then since $(s^i_x)|_{U_i \cap U_j} = (s^j_x)|_{U_i \cap U_j}$ implies that for all $p \in U_i \cap U_j$ we have $s^i_p = s^j_p$, it is clear that this assignment is well defined. Moreover, (s_x) lies in $\mathscr{F}^\sharp(U)$, as for all $p \in U$, there exists a U_i such that $p \in U_i$, and $(s^i_x) \in \mathscr{F}^\sharp(U_i)$ with $s^i_x = s_x$ for all $x \in U_i$, so there must exist a section f on each open neighborhood of $x \in U_i$ such that $f_q = s^i_q = s_q$, hence $(s_x) \in \mathscr{F}^\sharp(U)$. Moreover, we have that by construction $(s_x)|_{U_i} = (s^i_x)$. It follows that $\mathscr{F}^\sharp$ is a sheaf.

We define the natural transformation $\mathrm{sh} : \mathscr{F} \to \mathscr{F}^\sharp$ by:

$$\begin{aligned} \mathrm{sh}_U : \mathscr{F}(U) &\longrightarrow \mathscr{F}^\sharp(U) \\ s &\longmapsto (s_x) \end{aligned}$$

which has image in $\mathscr{F}^\sharp$ essentially by construction, i.e. take $V_p = U$ for all $p \in U$, then $s \in \mathscr{F}(U)$ satisfies $s_q = s_q$ tautologically. Moreover, this clearly commutes with restriction maps, and is thus a natural transformation.

We construct the natural transformation $\phi^\sharp$ for all U as follows; let $(s_x) \in \mathscr{F}^\sharp(U)$ then for all $p \in U$ there exists V_p and $f^p \in \mathscr{F}(V_p)$ such that $[V_p, f^p]_q = s_q$ for all $q \in V_p$. We thus obtain an open cover U by $\{V_p\}$ and section $\phi_{V_p}(f^p) \in \mathscr{G}(V_p)$. Now consider overlaps $W = V_x \cap V_y$, then:

$$\phi_{V_x}(f^x)|_W = \phi_W(f^x|_W)$$

By the universal property of the colimit, we have a unique map $\phi_q : \mathscr{F}_q \to \mathscr{G}^q$ for all q such that:

$$\phi_q(f^x_q) = [V_x, \phi_{V_x}(f^x)]_q = [W, \phi_W(f^x|_W)]_q = (\phi_W(f^x|_W))_q$$

However, for all $q \in W$, we have that $f^x_q = f^y_q$, hence:

$$(\phi_W(f^x|_W))_q = \phi_q(f^x_q) = \phi_q(f^y_q) = (\phi_W(f^y|_W))$$

implying that:

$$(\phi_{V_x}(f^x)|_W)_q = (\phi_{V_y}(f^y)|_W)_q$$

for all $q \in W$. However, $\mathscr{G}$ is a sheaf, so by Lemma 1.2.2, we have that $\phi_{V_x}(f^x)|_W = \phi_{V_y}(f^y)|_W$. So by sheaf axiom two, the $\phi_{V_x}(f^x)$ glue together to form a unique global section $g \in \mathscr{G}(U)$. We thus define $\phi^\sharp_U$ by:

$$\phi^\sharp_U((s_x)) = g$$

This is well defined, since if we had some other set of functions on e^p on some other open cover Z_p, repeating the same process yields a section $h \in \mathscr{G}(U)$. For all $q \in U$, we then have that:

$$h_q = \phi_q(s_q) = g_q$$

so by Lemma 1.2.2, it follows that $g = h$. This is clearly a ring homomorphism as if $(s_x), (t_x) \in \mathscr{F}^\sharp(U)$, then we have that:

$$\phi^\sharp_U(s_x) + \phi^\sharp_U(t_x) = g + h$$

where $g = \phi^\sharp_U(s_x)$ and $h = \phi^\sharp_U(t_x)$. Now suppose that:

$$\phi^\sharp_U(s_x + t_x) = f$$

for some $f \in \mathscr{G}(U)$. Then we have that:

$$f_q = \phi_q(s_q + t_q) = \phi_q(s_q) + \phi_q(t_q) = g_q + h_q = (g + h)_q$$

Since this holds for all q, we have again by Lemma 1.2.2 that:

$$\phi^\sharp_U(s_x + t_x) = \phi^\sharp_U(s_x) + \phi^\sharp_U(t_x)$$

The same argument shows that $\phi^\sharp_U$ respects multiplication.

Finally, we check that that $\phi^\sharp \circ \mathrm{sh} = \phi$. Let $s \in \mathscr{F}(U)$, then $\mathrm{sh}_U(s) = (s_x)$. Since $\phi^\sharp$ is independent of the choice of cover we use to obtain a section, chose the trivial cover U with $s \in \mathscr{F}(U)$, then we have clearly have that:

$$\phi^\sharp_U \circ \mathrm{sh}_U(s) = \phi_U(s)$$

so $\mathscr{F}^\sharp$ satisfies the universal property, implying the claim. □

Importantly if $\mathscr{F}$ is already a sheaf, we have that $\mathscr{F}^\sharp$ is uniquely isomorphic to $\mathscr{F}$.

Lemma 1.2.3. *Suppose that $\mathscr{F}$ is a sheaf, then $\mathscr{F} \cong \mathscr{F}^\sharp$*

Proof. We simply check that $\mathscr{F}, \mathrm{Id}$ satisfies the universal property of sheafification. Let ϕ by a morphism $\mathscr{F} \to \mathscr{G}$, then we have the following diagram:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\phi"'] \arrow[d, "\mathrm{Id}"'] & \mathscr{G} \\ \mathscr{F} \arrow[ur, "\exists!\phi^\sharp?"'] & \end{tikzcd}$$

Clearly, for this diagram to commute we must have that $\phi^\sharp = \phi$, but that morphism exists, and is unique so $\mathscr{F}, \mathrm{Id}$ satisfies the universal property of sheafification and is thus uniquely isomorphic to $\mathscr{F}^\sharp$. □

Lemma 1.2.4. *Suppose that $\mathscr{F}$ and $\mathscr{G}$ are presheaves, and $\phi : \mathscr{F} \to \mathscr{G}$ a morphism between them. Then there exists unique isomorphisms $\mathrm{sh}_q : \mathscr{F}_q \to \mathscr{F}^\sharp_q$, $\mathrm{sh}_q : \mathscr{G}_q \to \mathscr{G}^\sharp_q$[6], such that $\phi^\sharp_q \circ \mathrm{sh}_q = \mathrm{sh}_q \circ \phi_q$.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\mathrm{sh}"'] \arrow[d, "\phi"'] \arrow[dr, dashed, "\mathrm{sh} \circ \phi" description] & \mathscr{F}^\sharp \\ \mathscr{G} \arrow[r, "\mathrm{sh}"'] & \mathscr{G}^\sharp \end{tikzcd}$$

So there exists a unique morphism $(\phi)^\sharp$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\mathrm{sh}"'] \arrow[d, "\phi"'] \arrow[dr, dashed, "\mathrm{sh} \circ \phi" description] & \mathscr{F}^\sharp \arrow[d, "\phi^\sharp"] \\ \mathscr{G} \arrow[r, "\mathrm{sh}"'] & \mathscr{G}^\sharp \end{tikzcd}$$

It follows that $\mathrm{sh} \circ \phi = \phi^\sharp \circ \mathrm{sh}$, so we need only show that the unique map $\mathrm{sh}_q : \mathscr{F}_q \to \mathscr{F}^\sharp_q$ is an isomorphism. We have that:

$$\mathrm{sh}_q([U, s]) = [U, \mathrm{sh}_U(s)] = [U, (s_x)]$$

Suppose that $\mathrm{sh}_q([U, s]) = \mathrm{sh}_q([V, t])$, then we have that there exists a $W \ni q \subset U \cap V$, such that:

$$(s_x)|_W = (t_x)|_W$$

implying that for all $x \in W$ $s_x = t_x$. Since $q \in W$, it follows that $s_q = t_q$ so $[U, s] = [U, t]$. Now let $[U, (s_x)] \in \mathscr{F}^\sharp_q$, and take $(s_x) \in \mathscr{F}^\sharp(U)$. It follows that there exists an open neighborhood V_q of q, and a section $f \in \mathscr{F}(U)$ such that $f_x = s_x$ for $x \in V_q$. We see that:

$$\mathrm{sh}_q([V_x, f]) = [V_x, (f_x)] = [V_x, (s_x)|_{V_x}] = [U, (s_x)]$$

so sh_q is surjective. It follows that sh_q is an isomorphism, as the $\mathscr{F}_q$ and $\mathscr{F}^\sharp_q$ are either sets, groups, or rings. □

[6]Abuse of notation alert! We are using the same notation to refer to two different sheafification map.

Example 1.2.3. Let X be a topological space, and denote by $\mathbb{Z}$ the constant presheaf which assigns to each open non empty set the abelian group $\mathbb{Z}$, and to $\emptyset$ the trivial group. The restriction maps are either the identity, or the trivial morphism. This is not necessarily a sheaf, as if U open is the disjoint union of two open sets U_1, U_2, then we have that $s \in \mathbb{Z}(U_1)$, and $t \in \mathbb{Z}(U_2)$, such that $s \neq t$, it follows that $s|_{U_1 \cap U_2} = t|_{U_1 \cap U_2}$, but clearly s and t can't glue together to form a section of U restricting to s and t.

We want to find the sheafification of $\mathbb{Z}$. Define $\mathbb{Z}^\sharp$ by:

$$\mathbb{Z}^\sharp(U) = \{\text{locally constant functions } s : U \to \mathbb{Z}\}$$

i.e. if U is connected then $s : U \to \mathbb{Z}$ is a constant function, and if U is disconnected then s is constant on each connected component. The restriction maps are just the restriction of the function s to a smaller domain. This is then clearly a sheaf, as if U_i is a cover for U, and each $s|_{U_i} = 0$, then at each point in U $s(p) = 0$ so $s = 0$. Moreover, if we have $s_i \in U_i$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then the same construction in Example 1.2.1 gives a section on U that restricts to s_i.

We need only show that $\mathbb{Z}^\sharp$ satisfies the universal property of sheafification. Define sh on each open set by:

$$\mathrm{sh}_U(a) = s_a$$

where $s_a : U \to \mathbb{Z}$ is the constant function $s_a(p) = a$. This clearly commutes with restriction maps, hence defines a natural transformation $\mathbb{Z} \to \mathbb{Z}^\sharp$. Let $\phi : \mathbb{Z} \to \mathscr{G}$ be any morphism, where $\mathscr{G}$ is a sheaf. We see that $\phi^\sharp$ must satisfy:

$$\phi^\sharp \circ \mathrm{sh} = \phi$$

Let $s \in \mathbb{Z}^\sharp(U)$, then note that $s^{-1}(a)$ is open in U as s is locally constant. Indeed, if s is locally constant, then for each $x \in U$ there exists an open neighborhood of x such that s is constant. The preimage $s^{-1}(a)$ is then the union of all such open neighborhoods which is certainly open. Moreover, we see that $s^{-1}(a) \cap s^{-1}(b) = \emptyset$ for all $a \neq b$, and that $\{s^{-1}(a)\}_{a \in \mathbb{Z}}$ forms an open cover of U. For each $a \in \mathbb{Z}$, we choose the section $a \in \mathbb{Z}(s^{-1}(a))$, and then see that:

$$\phi_{s^{-1}(a)}(a)|_{s^{-1}(a) \cap s^{-1}(b)} = \phi_{s^{-1}(b)}(b)|_{s^{-1}(a) \cap s^{-1}(b)}$$

as the restrictions map to the empty set. It follows that since $\phi_{s^{-1}(a)}(a)$ glue together to give a global section $g \in \mathscr{G}(U)$. We thus define $\phi^\sharp$ by:

$$\phi^\sharp(s) = g$$

We thus see that if $a \in \mathbb{Z}(U)$, then $\mathrm{sh}(a) = s_a$:

$$\phi^\sharp(s_a) = \phi(a)$$

as $s_a^{-1}(a) = U$. It follows that $\phi^\sharp$ is unique, and well defined by the same argument in Proposition 1.2.3, so $\mathbb{Z}^\sharp$ is the sheafification of $\mathbb{Z}$. Going forward, we call $\mathbb{Z}^\sharp$ the **constant sheaf with values in** $\mathbb{Z}$[7], and denote by $\mathbb{Z}$.

We now go out of our way to explicitly explain the kernel sheaf, cokernel, sheaf, and the image sheaf. We work entirely with sheafs of abelian groups, though similar objects can be defined in the category of rings, the resulting sheafs just don't necessarily stay in the category of rings. The zero sheaf, will be denoted by 0, and is the sheaf that sends every open set to the trivial group, and the trivial transformation will be denote 0.

Definition 1.2.5. Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheafs, then the **kernel sheaf**, denoted $(\ker F, \iota)$ is a sheaf equipped with a natural transformation $\iota : \ker F \to \mathscr{F}$ such that $F \circ \iota = 0$, and for all $\psi : \mathscr{H} \to \mathscr{F}$ such that $F \circ \psi = 0$, there exists a unique $\theta : \mathscr{H} \to \ker \mathscr{F}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{H} \arrow[rr, "\psi"] \arrow[rrrr, bend left, "0"] \arrow[dr, dashed, "\exists!\theta"'] & & \mathscr{F} \arrow[rr, "F"] & & \mathscr{G} \\
& \ker F \arrow[ur, "\iota"'] & & &
\end{tikzcd}$$

[7]We can also use the same construction to obtain the constant sheaf with values in any set, abelian group, or ring

Proposition 1.2.4. *Let $F : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism, then* $\ker F$ *exists and is unique up to unique isomorphism.*

Proof. We define $\ker F$ by:

$$(\ker F)(U) = \ker F_U$$

This is easily seen to be a presheaf. We check that it is a sheaf. Let U_i cover U, and $s \in \ker F_U$ such that $s|_{U_i} = 0$. However, each $s|_{U_i} \in \ker F_{U_i} \subset \mathscr{F}(U_i)$, and $s \in \ker F_U \subset \mathscr{F}(U)$ hence $s = 0$. Now suppose we have $s_i \in \ker F_{U_i}$ such that:

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

These glue together to form an $s \in \mathscr{F}(U)$, however we need to check that $s \in \ker F_U$. Note that:

$$F_U(s)|_{U_i} = F_{U_i}(s|_{U_i}) = 0$$

and since $\mathscr{G}$ is a sheaf we have that $F(s) = 0$, so $s \in \ker F_U$. It follows that $\ker F$ is a sheaf.

Define $\iota : \ker F \to \mathscr{F}$ by $\iota_U(s) = s$, i.e. ι_U is just the natural inclusion of abelian groups. It is clear that $\phi \circ \iota = 0$. Let $\psi : \mathscr{H} \to \mathscr{F}$ be a morphism such that $F \circ \psi = 0$, then we need a morphism θ such that for all U :

$$\iota_U \circ \theta_U = \psi_U$$

We note that since $\phi_U \circ \psi_U = 0$, so ψ_U has in $\ker F_U$. We thus define:

$$\theta_U(s) = \psi_U(s)$$

with restricted target. This is readily seen to be a natural transformation, and is unique and well defined, hence $\ker F$ satisfies the universal property of a sheaf kernel. It follows that $\ker F$ is unique up to unique isomorphism, implying the claim. □

We have a similar definition for the cokernel, but with arrows reversed:

Definition 1.2.6. Let $F : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism, then the **sheaf cokernel**, denoted $(\operatorname{coker} F, \pi)$ is a sheaf equipped with a morphism $\pi : \mathscr{G} \to \operatorname{coker} F$, such that $\pi \circ F = 0$, and for all morphisms $\psi : \mathscr{G} \to \mathscr{H}$ such that $\psi \circ F = 0$, there exists a unique morphism $\theta : \operatorname{coker} F \to \mathscr{H}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "F"] \arrow[rrr, bend left, "0"] & \mathscr{G} \arrow[rr, "\psi"] \arrow[dr, "\pi"] & & \mathscr{H} \\
& & \operatorname{coker} F \arrow[ur, dashed, "\exists!\theta"] &
\end{tikzcd}$$

Proposition 1.2.5. *Let $F : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism, then the sheaf cokernel* $(\operatorname{coker} F, \pi)$ *exists and is unique up to unique isomorphism.*

Proof. Note that in the category of abelian groups, the cokernel of F_U is given by:

$$\mathscr{G}(U)/\operatorname{im} F_U$$

Using this assignment as the cokernel is problematic, as $\operatorname{coker} F$ will then fail be to a sheaf. In particular, the gluing property does not always hold. We thus define the presheaf:

$$\operatorname{coker}^p F : U \longmapsto \operatorname{coker} F_U = \mathscr{G}(U)/\operatorname{im} F_U$$

with π^p to be the natural transformation defined as the projection map $\mathscr{G}(U) \to \mathscr{G}(U)/\operatorname{im} F_U$ for all U, and define the cokernel sheaf to be:

$$\operatorname{coker} \mathscr{F} = (\operatorname{coker}^p F)^\sharp$$

with $\pi = \operatorname{sh} \circ \pi^p$. Suppose that $\operatorname{coker}^p F$ satisfies the universal property of the cokernel in the category of presheaves, then for all morphisms $\psi : \mathscr{G} \to \mathscr{H}$, we obtain the following commutative diagram by the universal property of sheafification:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "F"] \arrow[rrr, bend left, "0"] & \mathscr{G} \arrow[rr, "\psi"] \arrow[dr, "\pi^p"] & & \mathscr{H} \\
& & \operatorname{coker}^p F \arrow[ur, "\theta"] \arrow[r, "\operatorname{sh}"] & \operatorname{coker} F \arrow[u, "\theta^\sharp"]
\end{tikzcd}$$

It would then follow that $(\operatorname{coker} F, \pi)$ satisfies the universal property of cokernels in the category of sheaves. We now show that coker^p is a presheaf, and π^p is a natural transformation. First note that for all $U \subset V$ we have the following diagram:

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[r, "\theta^U_V"] \arrow[d, "\pi^p_U"] & \mathscr{G}(V) \arrow[d, "\pi^p_V"] \\
\mathscr{G}(U)/\operatorname{im} F_U & \mathscr{G}(V)/\operatorname{im} F_V
\end{tikzcd}$$

Note that $\pi^p_V \circ \theta^U_V$ is a morphism $\mathscr{G}(U) \to \mathscr{G}(V)/\operatorname{im} F_V$. Suppose that $g \in \operatorname{im} F_U$, then $g = F_U(s)$ for some $s \in \mathscr{F}(U)$, and we see that

$$\pi^p_V \circ \theta^U_V(F_U(s)) = \pi^p_V(F_V(s|_V)) = 0$$

so $\operatorname{im} F_V \subset \ker \theta^U_V \circ \pi^p_V$. It follows that there exists a unique map which we also denote by $\theta^U_V : \mathscr{G}(U)/\operatorname{im} F_U \to \mathscr{G}(V)/\operatorname{im} F_V$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[r, "\theta^U_V"] \arrow[d, "\pi^p_U"] & \mathscr{G}(V) \arrow[d, "\pi^p_V"] \\
\mathscr{G}(U)/\operatorname{im} F_U \arrow[r, "\theta^U_V"] & \mathscr{G}(V)/\operatorname{im} F_V
\end{tikzcd}$$

This implies that if coker^p is a presheaf, then π^p_U is a natural transformation. We need to check that $\theta^U_U = \operatorname{Id}$. Examine the diagram:

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[r, "\theta^U_U"] \arrow[d, "\pi^p_U"] & \mathscr{G}(U) \arrow[d, "\pi^p_U"] \\
\mathscr{G}(U)/\operatorname{im} F_U \arrow[r, "\theta^U_U"] & \mathscr{G}(U)/\operatorname{im} F_U
\end{tikzcd}$$

The top θ^U_U is the identity, so the only way for the bottom θ^U_U to make the diagram commute is for $\theta^U_U = \operatorname{Id}$ as well. We need to show that $\theta^V_W \circ \theta^U_V = \theta^U_W$; examine the diagram:

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[r, "\theta^U_V"] \arrow[d, "\pi^p_U"] & \mathscr{G}(V) \arrow[r, "\theta^V_W"] \arrow[d, "\pi^p_V"] & \mathscr{G}(W) \arrow[d, "\pi^p_V"] \\
\mathscr{G}(U)/\operatorname{im} F_U \arrow[r, "\theta^U_V"] & \mathscr{G}(V)/\operatorname{im} F_V \arrow[r, "\theta^V_W"] & \mathscr{G}(W)/\operatorname{im} F_W
\end{tikzcd}$$

Erase the middle to obtain:

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[r, "\theta^U_W"] \arrow[d, "\pi^p_U"] & \mathscr{G}(V) \arrow[d, "\pi^p_W"] \\
\mathscr{G}(U)/\operatorname{im} F_U \arrow[r, "\theta^V_W \circ \theta^U_V"] & \mathscr{G}(W)/\operatorname{im} F_W
\end{tikzcd}$$

Then since $\theta_W^V \circ \theta_V^U$ makes this diagram commute, we must have that $\theta_W^V \circ \theta_V^U = \theta_W^U$, so $\operatorname{coker}^p F$ is a presheaf.

To see that this satisfies the universal property of the presheaf cokernel, let $\psi : \mathscr{G} \to \mathscr{H}$ be a morphism of presheaves such that $\psi \circ F = 0$, then we want to find a morphism $\theta : \operatorname{coker}^p F \to \mathscr{H}$ such that for all U:

$$\theta_U \circ \pi_U^p = \psi_U$$

We define θ_U by:

$$\theta_U([g]) = \psi_U(g)$$

and note that this well defined, as if $[h] = [g]$, then we have that $h = g + F_U(s)$ for some $s \in \mathscr{F}(U)$. It follows that since $\psi \circ F = 0$:

$$\psi_U(h) = \psi_U(g + F_U(s)) = \psi_U(g)$$

It is clear that the assignment $U \to \theta_U$ then defines a natural transformation θ, as ψ is a natural transformation. It follows that $(\operatorname{coker}^p F, \pi^p)$ is the cokernel in the category of presheaves, implying that $(\operatorname{coker} F, \pi)$ is the cokernel in the category of sheaves. □

Corollary 1.2.2. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then $(\ker F)_x = \ker F_x$ and $(\operatorname{coker} F)_x \cong \operatorname{coker} F_x$*

Proof. We first that $\iota_x : (\ker F)_x \to \mathscr{F}_x$ is an inclusion map, so $(\ker F)_x \subset \mathscr{F}_x$. It thus suffices to show that $(\ker F)_x = \ker F_x$ as both are subgroups of $\mathscr{F}_x$. Let $s_x \in (\ker F_x)$, then $s_x = [U, s]$ for some $s \in \ker F_U$, it follows that $F_x(s_x) = [U, F_U(s)] = [U, 0] = 0$, so $(\ker F)_x \subset \ker F_x$. Now let $s_x \in \ker F_x$, and let $s_x = [U, s]$ for some $s \in \mathscr{F}(U)$. It follows that $[U, F_U(s)] = [U, 0]$, so there exists a V such that $F_V(s|_V) = 0$, hence $s|_V \in \ker F_V$. We have that $[U, s] = [V, s|_V]$, and $[V, s|_V] \in (\ker F)_x$, hence $s \in (\ker F)_x$, implying equality.

For the other statement, we need only show that $(\operatorname{coker}^p F)_x \cong \operatorname{coker} F_x$, then since sheafification provides an isomorphism $\mathrm{sh}_x : (\operatorname{coker}^p F)_x \to (\operatorname{coker} F)_x$ we will have the claim. Note that we have a map $\pi_x^p : \mathscr{G}_x \to (\operatorname{coker}^p F)_x$, which satisfies $\pi_x^p \circ F_x = 0$, so let $\phi : \mathscr{G}_x \to A$ be an morphism such that $\phi \circ F_x = 0$. Note that for $[U, g] \in \mathscr{G}_x$, we have that:

$$[U, g] \mapsto [U, [g]]$$

We thus define a homomorphism $\theta : (\operatorname{coker}^p F)_x \to A$ by:

$$\theta([U, [g]]) = \psi([U, g])$$

We need to check that this independent of the choice of g, let $[U, [h]] = [U, [g]]$, then there exists an open set $V \subset U$, such that:

$$[h]|_V = [g]|_V \Rightarrow h|_V = g|_V + s$$

where $s \in \operatorname{im} F_V$. Since ψ itself must be well defined, we have that:

$$\theta([U, [h]]) = \psi([U, h]) = \psi([V, h|_V]) = \psi([V, g|_V] + [V, s]) = \psi([V, g|_V]) = \psi([U, g])$$

It follows that $(\operatorname{coker}^p F)_x$ then satisfies the universal property of the cokernel of F_x, hence there is a unique isomorphism $(\operatorname{coker}^p F)_x \cong \operatorname{coker} F_x$, and thus a unique isomorphism $(\operatorname{coker} F)_x \cong \operatorname{coker} F_x$.

□

Now that we know kernels and cokernels exist, we wish to show that the category of sheaves of abelian groups over a topological space X is an abelian category. We need the following terminology:

Definition 1.2.7. A **additive category** is a category with a 0 object[8], finite products and coproducts, and each set $\operatorname{Hom}(A, B)$ for objects A and B has an abelian group structure such that the composition maps are bilinear.

[8]A zero object is one that is both an initial and final object in the category, i.e. for every object A there exist unique morphisms $0 \to A$ (initial), and $A \to 0$ (final)

Lemma 1.2.5. *Let X be a topological space, then the category of sheaves with values in abelian groups is additive.*

Proof. We first note that the trivial sheaf which assigns $\{0\}$ to each open set is easily seen to be a zero object.

We define the product of two sheaves $\mathscr{G}$ and $\mathscr{F}$ to be:

$$(\mathscr{G} \times \mathscr{F})(U) = \mathscr{G}(U) \times \mathscr{F}(U)$$

It is clear that this defines a sheaf, and moreover there clearly exist natural formations $\pi_{\mathscr{G}} : \mathscr{G} \times \mathscr{F} \to \mathscr{G}$ and $\pi_{\mathscr{F}} : \mathscr{G} \times \mathscr{F} \to \mathscr{F}$ such that $(\pi_{\mathscr{G}})_U$ and $(\pi_{\mathscr{F}})_U$ are the natural projections in the category of abelian groups. Let $\mathscr{H}$ be another sheaf with morphisms $\phi_{\mathscr{F}} : \mathscr{H} \to \mathscr{F}$ and $\phi_{\mathscr{G}} : \mathscr{H} \to \mathscr{G}$, then we want to show that there exists a unique $\psi : \mathscr{H} \to \mathscr{G} \times \mathscr{F}$ such that the following diagram commutes:

$$\begin{tikzcd}
& \mathscr{H} \arrow[d, dashed, "\exists!\psi"] \arrow[ddl, bend right, "\phi_{\mathscr{G}}"'] \arrow[ddr, bend left, "\phi_{\mathscr{G}}"] & \\
& \mathscr{G} \times \mathscr{F} \arrow[dl, "\pi_{\mathscr{G}}"'] \arrow[dr, "\pi_{\mathscr{F}}"] & \\
\mathscr{G} & & \mathscr{F}
\end{tikzcd}$$

In the category of abelian groups, we have that ψ_U would be given by:

$$\psi_U(h) = (\phi_{\mathscr{G}}(h), \phi_{\mathscr{F}}(h))$$

so assignment $U \mapsto \psi_U$ is the natural transformation which makes the above diagram commute, demonstrating that $\mathscr{F} \times \mathscr{G}$ is indeed the product. Since the product are coproduct are the same in abelian groups, it follows that the same argument with the arrows reversed shows that $\mathscr{G} \times \mathscr{F}$ is the coproduct in the category of sheaves as well.

We have that $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ is the set of all natural transformations $\mathscr{F} \to \mathscr{G}$. We define addition in this set by:

$$(\phi + \psi)_U = \phi_U + \psi_U \in \mathrm{Hom}(\mathscr{F}(U), \mathscr{G}(U))$$

We see that this is a natural transformation, as:

$$\begin{aligned}
\theta^U_V \circ (\phi + \psi)_U =& \theta^U_V \circ (\phi_U + \psi_U) \\
=& \theta^U_V \circ \phi_U + \theta^U_V \psi_U \\
=& \phi_V \circ \theta^U_V + \psi_V \circ \theta^U_V \\
=& (\phi_V + \psi_V) \circ \theta^U_V \\
=& (\psi + \phi)_V \circ \theta^U_V
\end{aligned}$$

So addition makes sense. Note that natural transformation $U \mapsto 0_U$ [9], which we suggestively denote by 0, is the 0 element in this set. Indeed, we have that for all U:

$$(\phi + 0)_U = \phi_U + 0_U = \phi_U$$

so $\phi + 0 = \phi$. We see that for any $\phi \in \mathrm{Hom}(\mathscr{F}, \mathscr{G})$, we can define $-\phi$ by:

$$(-\phi)_U = -\phi_U$$

which is clearly a natural transformation by the same argument above. It follows that for all U:

$$(\phi - \phi)_U = \phi_U - \phi_U = 0_U$$

[9] 0_U being the trivial morphism in $\mathrm{Hom}(\mathscr{F}(U), \mathscr{G}(U))$

so $\phi - \phi = 0$. We thus see that $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ is indeed an abelian group. Let $\mathscr{H}$ be another sheaf, and consider $\mathrm{Hom}(\mathscr{G}, \mathscr{H})$, we want to show that for all $\theta \in \mathrm{Hom}(\mathscr{G}, \mathscr{H})$ we have that:

$$\theta \circ (\phi + \psi) = \theta \circ \phi + \theta \circ \psi \in \mathrm{Hom}(\mathscr{F}, \mathscr{H})$$

We see that $\theta \circ (\phi + \psi)$ for all U:

$$(\theta \circ (\phi + \psi))_U = \theta_U \circ (\phi + \psi)_U = \theta_U \circ (\phi_U + \psi_U) = \theta_U \circ \phi_U + \theta_U \circ \psi_U = (\theta \circ \phi)_U + (\theta \circ \psi)_U$$

so $\theta \circ (\phi + \psi) = \theta \circ \phi + \theta \circ \psi$. The same argument in the other direction demonstrates that for all $\theta \in \mathrm{Hom}(\mathscr{H}, \mathscr{F})$, we have that:

$$(\phi + \psi) \circ \theta = \phi \circ \theta + \psi \circ \theta \in \mathrm{Hom}(\mathscr{H}, \mathscr{G})$$

so composition is bilinear, implying the claim. $\square$

Definition 1.2.8. Let $\phi : A \to B$ be a morphism in an additive category, then ϕ is **monomorphism** if for all $\theta : Z \to A$, we have that $\phi \circ \theta = 0 \Rightarrow \theta = 0$. A morphism ϕ is an **epimorphism** if for all $\theta : B \to Z$ we have that $\theta \circ \phi = 0 \Rightarrow \theta = 0$. If we are not in an additive category, then ϕ is a monomorphism if for all $\theta_1, \theta_2 : Z \to A$ we have that $\phi \circ \theta_1 = \phi \circ \theta_2 \Rightarrow \theta_1 = \theta_2$. Similarly, ϕ is an epimorphism if for all $\theta_1, \theta_2 : B \to Z$, we have that $\theta_1 \circ \phi = \theta_2 \circ \phi \Rightarrow \theta_1 = \theta_2$. [10]

Lemma 1.2.6. *Let $\phi : A \to B$ a morphism in an additive category with kernels and cokernels. Then, the morphism $\iota : \ker \phi \to A$ is a monomorphism, and the morphism $\pi : B \to \operatorname{coker} \phi$ is an epimorphism.*

Proof. Suppose that $\theta : Z \to \ker \phi$ such that $\iota \circ \theta = 0$, then we have the following commutative diagram:

$$\begin{tikzcd}
Z \arrow[rr, "\iota\circ\theta"] \arrow[dr, "\theta"'] \arrow[rrrr, bend left, "0"] & & A \arrow[rr, "\phi"] & & B \\
& \ker \phi \arrow[ur, "\iota"'] & & &
\end{tikzcd}$$

Where $\theta \circ \iota = 0$ so $\phi \circ \theta \circ \iota = 0$. By the universal property of kernels, it follows that θ is the unique map that makes this commute. However, $\iota \circ \theta = 0$, so $\theta = 0$ also makes this map commute, hence by uniqueness $\theta = 0$, and ϕ is a monomorphism.

Suppose that $\theta : \operatorname{coker} \phi \to Z$ such that $\theta \circ \pi = 0$, then we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[rr, "\phi"] \arrow[rrrr, bend left, "0"] & & B \arrow[rr, "\theta\circ\pi"] \arrow[dr, "\pi"'] & & Z \\
& & & \operatorname{coker} \phi \arrow[ur, "\theta"'] &
\end{tikzcd}$$

Again by the universal property, since θ makes the map we commute we have that it must be unique. However, since $\theta \circ \pi = 0$, clearly $\theta = 0$ makes this map commute as well so by uniqueness $\theta = 0$, and π is an epimorphism. $\square$

Applying Lemma 1.2.6 to the the category sheaves, demonstrates that $(\ker F, \iota)$ and $(\operatorname{coker} F, \pi)$ are monomorphisms and epimorphisms for all natural transformations F.

Proposition 1.2.6. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves with values in abelian groups, then the following are equivalent:*

a) F is a monomorphism.

b) For all $x \in X$, $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective.

c) $F_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all U

Similarly the following are equivalent:

[10] In the category of abelian groups, a morphism is mono if and only if it is injective, and epi if and only if it is surjective.

d) F is an epimorphism.

e) For all $x \in X$, $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is surjective.

We need the following lemma:

Lemma 1.2.7. *Let A be an abelian group, and $x \in X$, then the assignment:*

$$(x_* A)(U) = \begin{cases} A & \textit{if } x \in U \\ 0 & \textit{otherwise} \end{cases}$$

is a sheaf such that $(x_*A)_x = A$ and is 0 for all other points. This is often referred to as the ***skyscraper sheaf****.*

Proof. Define restriction maps by $\theta_V^U = \mathrm{Id}$ if $x \in V$, and U, and 0 otherwise. This is clearly a presheaf, and $x \notin U$ there are no sheaf axioms to check. Suppose $x \in U$, and let U_i be an open cover for U, such that for $s \in A$, we have that $s|_{U_i} = 0$. It follows that $s = 0$, because for at least one i we have that $x \in U_i$ and $\theta_{U_i}^U = \mathrm{Id}$. Now suppose that we have $s_i \in U_i$, such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. If $x \notin U_i$ or U_j, then we have that both restrictions are zero, if $x \in U_i \cap U_j$ then we must have that $s_i = s_j$, hence $s = s_i$ for any U_i containing x restricts to each s_i. It follows that (x_*A) is a sheaf.

If $y \neq x$, then any element $[U, s] \in (x_*A)_y$ is equal to $[V, 0]$ for some smaller V not containing x, so A_y must be the zero group as every element is the zero element. We show that A satisfies the universal property of the direct limit. Let $\phi_U : (x_*A)(U) \to G$ be maps which commute with restriction, and let $\psi_U : (x_*A)(U) \to A$ be the identity, which also commutes with restriction as U contains x. We define $F : A \to G$ by:

$$F(a) = \phi_U(a)$$

which is well defined because for all $\phi_U : (x_*A)(U) \to G$, we must have that $\phi_V = \phi_U$ as the restriction maps are the identity. It follows that A satisfies the universal property of direct limit and is thus the stalk of (x_*A) at x. $\square$

We now prove the proposition:

Proof. Note that $c) \Rightarrow a)$, as if $F_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all U, then F_U is a monomorphism for all U. It follows that if $\theta : \mathscr{H} \to \mathscr{F}$ satisfies $F \circ \theta = 0$, then for all U we have that $\theta_U = 0$, so θ is the trivial morphism.

We now show that $a) \Rightarrow b)$. Take the natural morphism $\iota : \ker F \to \mathscr{F}$, and note that $F \circ \iota = 0$, so $\iota = 0$. However, we have that on each open set $\iota_U(s) = s = 0$, so for all $s \in \ker F_U$, we have that $s = 0$. It follows that $\ker F_U = 0$ so $\ker F$ is the trivial sheaf, and $(\ker F)_x$ is the trivial group, but by Corollary 1.2.2 $(\ker F)_x = \ker F_x$ so $\ker F_x = 0$ and F_x is injective.

We now show that $b) \Rightarrow c)$. Suppose that for all x, $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective, then we have an induced injection:

$$\begin{aligned} \prod_{x \in U} \mathscr{F}_x &\longrightarrow \prod_{x \in U} \mathscr{G}_x \\ (s_x) &\longmapsto (F_x(s_x)) \end{aligned}$$

Suppose that $s, t \in \mathscr{F}(U)$, such that $F_U(s) = F_U(t)$, then we we have that by the definition of the stalk map $F_x(s_x) = F_x(t_x)$ for all $x \in U$. However, the map above is injective so $(s_x) = (t_x)$ implying that $s = t$ by Lemma 1.2.2. We thus have that:

$$a) \Longrightarrow b) \Longrightarrow c) \Longrightarrow a)$$

implying the first part of the claim.

We now show that $d) \Rightarrow e)$. Let $\mathscr{H}$ be the skyscraper sheaf $x_*(\mathscr{G}_x / \operatorname{im} F_x)$, and note that the map $\psi_U : \mathscr{G}(U) \to \mathscr{H}(U)$:

$$\psi_U(g) = \begin{cases} [g_x] & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

trivially commutes with restriction maps, and thus defines a natural transformation. Vacuously we have that $\psi \circ F = 0$, as if $x \in U$, we have that:

$$\psi_U(F(s)) = [F(s)_x] = [F_x(s_x)] = 0$$

and if $x \notin U$, we have that $\psi = 0$ anyways. However, since F is an epimorphism, this implies that $\psi = 0$ so $\psi_x = 0$. Note however, that ψ_x is the map defined by:

$$\begin{aligned} \psi_x : \mathscr{G}_x &\longrightarrow \mathscr{G}_x / \operatorname{im} \mathscr{F}_x \\ g_x &\longmapsto [g_x] \end{aligned}$$

which is clearly a surjection, hence $\mathscr{G}_x / \operatorname{im} F_x = 0$, implying that $\operatorname{im} F_x = \mathscr{G}_x$, and thus the claim.

To show that $e) \Rightarrow d)$, suppose that $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is a surjection for all $x \in X$. Let ϕ be any other an morphism $\mathscr{G} \to \mathscr{H}$ such that $\phi \circ F = 0$. This implies that on stalks:

$$\phi_x \circ F_x = 0$$

But F_x is a surjection, and thus an epimorphism, so on the level of stalks we have that $\phi_x = 0$ for all $x \in U$. Now examine the commutative diagram:

$$\begin{tikzcd} \mathscr{G}(U) \arrow[r, "\phi_U"] \arrow[d] & \mathscr{H}(U) \arrow[d] \\ \prod_{x \in U} \mathscr{G}_x \arrow[r, "0"] & \prod_{x \in U} \mathscr{H}_x \end{tikzcd}$$

If $g \in \mathscr{G}(U)$ we have that:

$$(\phi_U(g)_x) = 0$$

however, the downward maps are injections, hence we must have that $\phi_U(g) = 0$ for all $g \in \mathscr{G}(U)$, and all U, thus $\phi = 0$, so F is an epimorphism.

□

Definition 1.2.9. A category is **abelian**, if it is additive, kernels and cokernels exist, and every monomorphism and epimorphism are the kernel and cokernel of some morphism.[11]

Theorem 1.2.1. *Let X be a topological space, then the category of sheaves with values in abelian groups is an abelian category.*

Proof. We need only show that every monomorphism is the kernel of some morphism, and that every epimorphism is the cokernel of some morphism.

Suppose that $F : \mathscr{F} \to \mathscr{G}$ is a monomorphism, we want to show that $(\mathscr{F}, F)$ is the kernel of some morphism $\psi : \mathscr{G} \to \mathscr{H}$. Well, take $\mathscr{H}$ to be $\operatorname{coker} F$, and ψ to be the projection π. We note that $\pi \circ F = 0$, indeed $\pi = \operatorname{sh} \circ \pi^p$, so for all open sets U, we have that:

$$\pi_U \circ F_U = \operatorname{sh}_U \circ \pi^p_U \circ F_U = \operatorname{sh}_U \circ 0_U = 0$$

so $\pi \circ F$ is the trivial morphism. Now let $\phi : \mathscr{H} \to \mathscr{G}$, such that $\pi \circ \phi = 0$, then we want to obtain the following commutative diagram:

$$\begin{tikzcd} \mathscr{H} \arrow[r, "\phi"] \arrow[rr, bend left, "0"] \arrow[dr, dashed, "\exists! \psi"'] & \mathscr{G} \arrow[r, "\pi"] & \operatorname{coker} F \\ & \mathscr{F} \arrow[u, "F"'] & \end{tikzcd}$$

[11] Often times people refer to the morphisms ι and π as the kernel and cokernel, so when we see every monomorphism (epimorphism) is a kernel (cokernel) of some morphism we are saying every monomorphism (epimorphism) can be written as the inclusion (projection) map ι (π) induced by the kernel (cokernel) of some morphism.

We move to level of stalks, and since $(\operatorname{coker} F)_x \cong \mathscr{G}_x / \operatorname{im} F_x$ we have the following diagram:

$$\begin{tikzcd}
\mathscr{H}_x \arrow[rr, "\phi_x"] \arrow[rrrr, bend left, "0"] \arrow[dr, dashed, "\exists! \psi_x"'] & & \mathscr{G}_x \arrow[rr, "\tilde{\pi}_x"] & & \mathscr{G}_x / \operatorname{im} F_x \\
& \mathscr{F}_x \arrow[ur, "F_x"'] & & &
\end{tikzcd}$$

where $\tilde{\pi}_x$ is π_x composed with the isomorphism $(\operatorname{coker} F)_x \to \mathscr{G}_x / \operatorname{im} F_x$, and by the uniqueness of the quotient map it follows that $\tilde{\pi}_x$ is the quotient map. We see that for all $h_x \in \mathscr{H}_x$:

$$\tilde{\pi}_x \circ \phi_x(h_x) = 0$$

so $\phi_x(h_x) \in \ker \tilde{\pi}_x = \operatorname{im} F_x$, and we have that $\operatorname{im} \phi_x(h_x) \subset \operatorname{im} F_x$. Now consider the following commutative diagram:

$$\begin{tikzcd}
\mathscr{H}(U) \arrow[rr, "\phi_U"] \arrow[ddd] & & \mathscr{G}(U) \arrow[ddd] \\
& \mathscr{F}(U) \arrow[ur, "F_U"] \arrow[d] & \\
& \prod_{x \in U} \mathscr{F}_x \arrow[dr, "(F_x)"] & \\
\prod_{x \in U} \mathscr{H}_x \arrow[rr, "(\phi_x)"] & & \prod_{x \in U} \mathscr{G}_x
\end{tikzcd}$$

Take an $h \in \mathscr{H}(U)$, then we have that:

$$\phi_U(h)_x = \phi_x(h_x) \in \operatorname{im} F_x$$

Since (F_x) is an injection, we see that there exists a unique $(s_x) \in \prod_{x \in U} \mathscr{F}_x$, such that:

$$(F_x(s_x)) = (\phi_U(h)_x)$$

For each $x \in U$, we thus have that there exists an open neighborhood $V_x \subset U$, and a section $s^x \in \mathscr{F}(V_x)$ such that:

$$[V_x, F_{V_x}(s^x)] = [U, \phi_U(h)]$$

implying that there is a $W_x \subset V_x \cap U = V_x$ such that:

$$F_{V_x}(s^x)|_{W_x} = \phi_U(h)|_{W_x}$$

The set of all such W_x's and $s^x|_{W_x}$'s covers U, and we see that for $V_x \cap V_y \neq \emptyset$:

$$F_{W_x \cap W_y}(s^x|_{W_x \cap W_y}) = \phi_U(h)|_{W_x \cap W_y} = F_{W_x \cap W_y}(s^y|_{W_x \cap W_y})$$

however $F_{W_x \cap W_y}$ is injective by Proposition 1.2.6, so we have that:

$$s^x|_{W_x \cap W_y} = (s^x|_{W_x})|_{W_x \cap W_y} = (s^y|_{W_x})|_{W_x \cap W_y} = s^y|_{W_x \cap W_y}$$

The s^x's then glue together to form an $s \in \mathscr{F}(U)$ such that:

$$F_U(s)_x = \phi_U(h)_x$$

for all $x \in U$. Since $F_U(s)$ and $\phi_U(h)$ both lie in $\mathscr{G}(U)$, and they agree on all stalks we must have that $\phi_U(h) = F_U(s)$. It follows that for all U $\operatorname{im} \phi_U \subset \operatorname{im} \mathscr{F}_U$. We now define a morphism $\psi : U \to \psi_U$ by:

$$\psi_U(h) = s$$

where $s \in \mathscr{F}(U)$ is the unique section such that that $\phi_U(h) = F_U(s)$. To see that this commutes with restriction maps, we need to show that $\theta_V^U \circ \psi_U = \psi_V \circ \theta_V^U$. In particular, we need:

$$\psi_V(\theta_V^U(h)) = \theta_V^U(s)$$

Take $h|_V$, then $\psi_V(h|_V) = f \in \mathscr{F}(V)$, where $F_V(f) = \phi_V(h|_V)$. Now note that:

$$F_V(s|_V) = (F_U(s))|_V = (\phi_U(h))|_V = \phi_V(h|_V)$$

so $F_V(s|_V) = F(f)$, and thus $f = s|_V$ implying the claim. To see that this is actually a group homomorphism, let $h, g \in \mathscr{H}(U)$ such that $\psi_U(h) = s$ and $\psi_U(g) = t$. We need to show that:

$$\psi_U(h+g) = s+t$$

Well, let $\psi_U(h+g) = f$ be the unique $f \in \mathscr{F}(U)$ such $\phi_U(h+g) = F_U(f)$. However, we see that $\phi_U(h+g) = \phi_U(h) + \phi_U(g) = F_U(s) + F_U(t)$, hence:

$$F_U(f) = F_U(s) + F_U(t) \implies f = s+t$$

It follows that $\psi_U(h+g) = s+t$, and is thus a group homomorphism. In particular, this implies $(\mathscr{F}, F)$ satisfies the universal property of the kernel of the cokernel of F and is thus the kernel of some morphism.

Now let $F : \mathscr{F} \to \mathscr{G}$ be an epimorphism. We claim that $(\mathscr{G}, F)$ is the cokernel of $\iota : \ker F \to \mathscr{F}$. We first note that clearly:

$$F \circ \iota = 0$$

as $\operatorname{im} \iota_U = \ker F_U$ for all U. Let $\phi : \mathscr{F} \to \mathscr{H}$ be an morphism such that $\phi \circ \iota = 0$, we want to that there exists a unique ψ such that:

$$\begin{tikzcd}
\ker F \arrow[r, "\iota"] \arrow[rr, bend left, "0"] & \mathscr{F} \arrow[r, "\phi"] \arrow[dr, "F"'] & \mathscr{H} \\
& & \mathscr{G} \arrow[u, dashed, "\exists!\psi"']
\end{tikzcd}$$

As before, we exploit the fact that $(\ker F)_x = \ker F_x$, and move to the level of stalks:

$$\begin{tikzcd}
\ker F_x \arrow[r, "\iota_x"] \arrow[rr, bend left, "0"] & \mathscr{F}_x \arrow[r, "\phi_x"] \arrow[dr, "F_x"'] & \mathscr{H}_x \\
& & \mathscr{G}_x \arrow[u, dashed, "\exists!\psi_x"']
\end{tikzcd}$$

Since $\mathscr{F}$ is an epimorphism, we have that F_x is surjective by Proposition 1.2.6. We see that since ι_x is the inclusion map of the kernel of F_x, that $\operatorname{im} \iota_x = \ker F_x$. We also have that $\operatorname{im} \iota_x \subset \ker \phi_x$, hence we define a unique map ψ_x by:

$$\psi_x(g_x) = \phi_x(s_x)$$

where s_x is any element in $F_x^{-1}(g_x)$. This is well defined, as if s_x' is any other element in $F_x^{-1}(g_x)$, we have that:

$$\phi_x(s_x) - \phi_x(s_x') = \phi_x(s_x - s_x')$$

but $s_x, s_x' \in F_x^{-1}(g_x)$, so $s_x - s_x' \in \ker F_x = \operatorname{im} \iota_x$. It follows that $s_x - s_x' \in \ker \phi_x$, so $\phi_x(s_x) = \phi_x(s_x')$ as desired. We now examine the following diagram:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "F"] \arrow[rr, bend left, "\phi_U"] \arrow[d] & \mathscr{G}(U) \arrow[r, dotted, "\exists!\psi_U"] \arrow[d] & \mathscr{H}(U) \arrow[d] \\
\prod_{x\in U}\mathscr{F}_x \arrow[r, "(F_x)"] \arrow[rr, bend right, "(\phi_x)"'] & \prod_{x\in U}\mathscr{G}_x \arrow[r, "(\psi_x)"] & \prod_{x\in U}\mathscr{H}_x
\end{tikzcd}$$

where we want to define ψ_U such that this commutes. Take an element $g \in \mathscr{G}(U)$, then we have a unique corresponding element $(g_x) \in \prod_{x\in U}\mathscr{G}_x$. This then maps to $(\psi_x(g_x)) \in \prod_{x\in U}\mathscr{H}_x$, which is equal to $(\phi_x(s_x))$ for some $(s_x) \in (F_x)^{-1}(g_x)$. We want to find a section $h \in \mathscr{H}(U)$ such that $h_x = \psi_x(g_x)$, as we can then define ψ_U by $\psi_U(g) = h$. For each x we have that:

$$\phi_x(s_x) = \psi_x(g_x) \in \mathscr{H}_x$$

so in particular, there exists an open neighborhood V_x of x, and a section $s^x \in \mathscr{F}(s^x)$ such that:

$$[V_x, \phi_{V_x}(s^x)] = \psi_x(g_x)$$

Cover U with all such V_x, then we want to show that:

$$\phi_{V_x}(s^x)|_{V_x\cap V_y} = \phi_{V_y}(s^y)|_{V_x\cap V_y}$$

However, note that for all $p \in V_x \cap V_y$, we have that:

$$(\phi_{V_x}(s^x)|_{V_x\cap V_y})_p = \phi_{V_x}(s^x)_p = \psi_p(g_p) = \phi_{V_y}(s^y)_p = (\phi_{V_y}(s^y)|_{V_x\cap V_y})_p$$

so we must have that sections agree on overlaps. It follows that that $\phi_{V_x}(s^x)$'s glue together to form a section $h \in \mathscr{H}(U)$ such that $h_x = \psi_x(g_x)$ for all $x \in U$. We thus define ψ_U to be:

$$\psi_U(g) = h$$

It is then clear that h is independent of our choice of (s_x) as ψ_x is independent of that choice, and moreover that it is independent of our choice of cover of U, as any other choice will have to agree on stalks. This is also clearly a group homomorphism, and is compatible with restriction maps; indeed if $g, g' \in \mathscr{G}(U)$, and we have that $\psi_U(g) = h$ and $\psi_U(g') = h'$, then we see that for all $x \in U$:

$$(\psi_U(g) + \psi_U(g'))_x = h_x + h'_x = \psi_x(g_x) + \psi_x(g'_x) = \psi_x((g+g')_x) = \psi_U(g+g')_x$$

Since they agree on stalks we must have that they are equal. Moreover, we want to show that:

$$\psi_U(g)|_V = h|_V$$

However, if we again take stalks, we see that for all $x \in V$,

$$(\psi_U(g)|_V)_x = \psi_U(g)_x = \psi_x(g_x) = h_x = (h|_V)_x$$

so the two must again agree. Finally, we check that $\psi\circ F = \phi$. Let $s \in \mathscr{F}(U)$, then we have $F_U(s) \in \mathscr{G}(U)$, which maps down to sequence $(F_U(s)_x) = (F_x(s_x))$, where each s_x clearly lies in $F_x^{-1}(F_x(s_x))$. We thus have that:

$$\psi_x(F_x(s_x)) = \phi_x(s_x) = \phi_U(s)_x$$

for all x by definition of ψ_x. It follows that from the defining property of ψ_U:

$$\psi_U(F_U(s))_x = \psi_x(F_x(s_x)) = \phi_U(s)_x$$

for all x, hence $\psi_U(F_U(s)) = \phi_U(s)$. We thus have that $(\mathscr{G}, F)$ satisfies the universal property of the cokernel of the kernel of F, and is thus a cokernel as desired. $\square$

We now briefly discuss the image sheaf, so that we can talk of exact sequences of abelian categories.

Definition 1.2.10. let $F : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism. The **image sheaf**, denoted $\operatorname{im} F$ is the sheafification of the presheaf $\operatorname{im}^p F$ defined by:

$$(\operatorname{im}^p F)(U) = \operatorname{im} F_U \subset \mathscr{G}(U)$$

In general, we note that $\operatorname{im}^p F$ is not a sheaf, hence why we take the sheafification. We also have the following definition:

Definition 1.2.11. Let $\mathscr{F}$ be a sheaf over X, then a **subsheaf** $\mathscr{G}$ of $\mathscr{F}$ is a sheaf on X such that $\mathscr{G}(U) \subset \mathscr{F}(U)$ for all U, and the restriction maps on $\mathscr{G}$ are given by the restriction of $\theta_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$ to $\mathscr{G}(U)$ for all $V \subset U$.

Proposition 1.2.7. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. There exists a natural map $\iota : \operatorname{im} F \to \mathscr{G}$, such that $\ker \iota = 0$, and $\iota(\operatorname{im} F) = \operatorname{im}^p \iota$ is a subsheaf of $\mathscr{G}$.*

Proof. First note that we have a clear inclusion morphism $\iota^p : \operatorname{im}^p F \to \mathscr{G}$, which is injective on all U. By the universal property of sheafification, we thus have a unique map $\iota : \operatorname{im} F \to \mathscr{G}$ such that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{im}^p F \arrow[r, "\iota^p"] \arrow[d, "\mathrm{sh}"] & \mathscr{G} \\
\operatorname{im} F \arrow[ur, "\iota"] &
\end{tikzcd}$$

It thus suffices to check that $\ker \iota_U = 0$ for all U. Let $(s_x) \in (\operatorname{im} F)(U)$, and suppose that:

$$\iota((s_x)) = 0$$

Since $(s_x) \in (\operatorname{im} F)(U)$, we have that for each x there exists V_x, and an $s^x \in \mathscr{F}(V_x)$ such that $s_q^x = s_q$ for all $q \in V_x$. Moreover, we have that by our work in Proposition 1.2.3, that:

$$\iota((s_x))|_{V_x} = \iota^p(s^x) = 0$$

However, this implies that $s^x = 0$ for each x, hence $s_p^x = 0 = s_p$ for all $p \in V_x$, and all $x \in V_x$. It follows that $(s_x) = 0$, so the $\ker \iota = 0$.

We have that $\iota(\operatorname{im} F)$ is a sub presheaf, by defining:

$$\iota(\operatorname{im} F)(U) = \iota_U(\operatorname{im} F_U) \subset \mathscr{G}(U)$$

We define restriction maps, $\theta_V^U : \iota(\operatorname{im} F)(U) \to \iota(\operatorname{im} F)(V)$, by restricting $\theta_V^U : \mathscr{G}(U) \to \mathscr{G}(V)$ to the subgroup $\iota(\operatorname{im} F_U)$. It follows if $g \in \iota_U(\operatorname{im} F_U)$, then $g = \iota_U((s_x))$, so $g|_V = \iota_V((s_x)|_V)$, thus θ_V^U has image in $\iota(\operatorname{im} F)(V)$. The restriction maps are then compatible with one another, as they are compatible on $\mathscr{G}$.

To show this is a sheaf, let $\{U_i\}$ be an open cover U, and $g \in \iota(\operatorname{im} F)(U)$, such that $g|_{U_i} = 0$ for all U_i. Well, since $g \in \iota(\operatorname{im} F)(U) \subset \mathscr{G}(U)$, and $\mathscr{G}$ is a sheaf, we must have that $g = 0$. Now suppose that we have $g_i \in \iota(\operatorname{im} F)(U_i)$ such that $g_i|_{U_i \cap U_j} = g_j|_{U_i \cap U_j}$ for all i, j. Then we must have have that there exists a $g \in \mathscr{G}(U)$ such that $g|_{U_i} = g_i$. We need to show that $g \in \iota(\operatorname{im} F)(U)$. For each i write $g_i = \iota_{U_i}((s_x)_i)$, then we have that:

$$\iota_{U_i \cap U_j}((s_x)_i|_{U_i \cap U_j}) = \iota_{U_i \cap U_j}((s_x)_j|_{U_i \cap U_j})$$

which implies that:

$$(s_x)_i|_{U_i \cap U_j} = (s_x)_j|_{U_i \cap U_j}$$

as ι_U is injective for all U. It follows that the $(s_x)_i$ glue together to form a global section $(s_x) \in (\operatorname{im} F)(U)$, such that $(s_x)|_{U_i} = (s_x)_{U_i}$. We see that for all U_i:

$$(g - \iota_U((s_x)))|_{U_i} = g_i - \iota_{U_i}((s_x)_i) = 0$$

hence $g - \iota_U((s_x)) = 0$, implying that $g \in \iota_U(\operatorname{im} F_U)$. It follows that $\iota(\operatorname{im} F)$ is a subsheaf of $\mathscr{G}$ as desired. $\square$

Definition 1.2.12. Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups or rings, then F is **injective** if $\ker F$ is the trivial sheaf, and F is **surjective** if $\iota(\operatorname{im} F) = \mathscr{G}$.

Proposition 1.2.8. *Let $F : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism of abelian groups, then F is surjective if and only if* $\operatorname{coker} F$ *is the trivial sheaf. Moreover, if F is surjective if and only if $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is surjective for all x.*[12]

Proof. Suppose that $\operatorname{coker} F$ is the trivial sheaf, then we have that $\operatorname{im} F_x = \mathscr{G}_x$ for all $x \in X$. Since ι is a monomorphism, we have that $\iota_x : (\operatorname{im} F)_x \to \mathscr{G}_x$ is an injection. Moreover, since $(\operatorname{im}^p F)_x = \operatorname{im} F_x = \mathscr{G}_x$, we have that $\iota_x^p : (\operatorname{im}^p F)_x \to \mathscr{G}_x$ is an isomorphism. Since sh_x is an isomorphism, and:

$$\iota_x \circ \operatorname{sh}_x = \iota_x^p$$

we must have that ι_x is a surjection as well, and thus an isomorphism. Since ι_x is an isomorphism for all, we must have that $\iota(\operatorname{im} F) = \mathscr{G}$ as desired.

Now suppose that F is surjective, that $\iota(\operatorname{im} F) = \mathscr{G}$. Then the stalk maps are isomorphisms, so we once again have that $(\operatorname{im} F)_x \cong \mathscr{G}_x$, implying that $\iota_x((\operatorname{im} F)_x) = \operatorname{im} F_x = \mathscr{G}_x$. The stalks of $\operatorname{coker} F$ are then isomorphic to $\mathscr{G}_x / \operatorname{im} F_x \cong \{0\}$, hence very section of $\operatorname{coker} F$ must be trivial, implying the claim.

Now suppose that $F : \mathscr{F} \to \mathscr{G}$ is surjective, then $\iota(\operatorname{im} F) = \mathscr{G}$. In particular, we have that $\iota_x(\operatorname{im}(F)_x) = \mathscr{G}_x$, however by the commutative diagram in Proposition 1.2.7, this implies that:

$$\iota_x(\operatorname{sh}_x((\operatorname{im}^p F)_x)) = \iota_x^p((\operatorname{im}^p F)_x) = \mathscr{G}_x$$

Since ι^p is an honest to god inclusion map, ι_x^p is an honest to god inclusion map, and it follows that $(\operatorname{im}^p F)_x = \mathscr{G}_x$, hence $\operatorname{im}(F_x) = \mathscr{G}_x$.

Now supposing that F_x is surjective for all $x \in X$. Then the map:

$$\iota^p : \operatorname{im}^p F \to \mathscr{G}$$

is an isomorphism on stalks. It follows that $\iota : \operatorname{im} F \to \mathscr{G}$ is then an isomorphism on stalks, hence ι is an isomorphism so $\operatorname{im} F = \mathscr{G}$. ☐

Definition 1.2.13. A sequence of sheaf morphisms:

$$\cdots \longrightarrow \mathscr{F}_{i-1} \xrightarrow{\quad F_{i-1} \quad} \mathscr{F}_i \xrightarrow{\quad F_i \quad} \mathscr{F}_{i+1} \longrightarrow \cdots$$

is called **exact** if $\ker F_i = \iota(\operatorname{im} F_{i-1})$ for all i.

Proposition 1.2.9. *Let:*

$$0 \longrightarrow \mathscr{F}_{i-1} \xrightarrow{\quad F_{i-1} \quad} \mathscr{F}_i \xrightarrow{\quad F_i \quad} \mathscr{F}_{i+1} \longrightarrow 0$$

be a sequence of sheaf morphisms. Then the sequence is exact if and only if the induced sequence of stalks:

$$\cdots \longrightarrow (\mathscr{F}_{i-1})_x \xrightarrow{\quad (F_{i-1})_x \quad} (\mathscr{F}_i)_x \xrightarrow{\quad (F_i)_x \quad} (\mathscr{F}_{i+1})_x \longrightarrow \cdots$$

is exact for all $x \in X$.

Proof. Suppose that the sequence is exact, then we need to show that for all $x \in X$, $\ker(F_i)_x = \operatorname{im}(F_{i-1})_x$. Let $s_x = [U, s] \in \ker(F_i)_x$, then we have that:

$$[U, (F_i)_U(s)] = [U, 0]$$

It follows that there exists an open neighborhood V_x of x such that:

$$(F_i)_V(s|_V) = 0$$

however, if $(F_i)_V(s|_V) = 0$, we have that by exactness $s|_V \in \iota(\operatorname{im} F_{i-1})(V)$, so there exists an $(s_x) \in (\operatorname{im} F_{i-1})(V)$ such that $\iota((s_x)) = s|_V$. It follows that $s|_V$ is then section such that for each open neighborhood of x, W_x, $s|_{W_x} = \iota^p(s^x)$, implying that $s_x = (s|_V)_x = \iota_x^p(s_x^x) \in \operatorname{im}(F_{i-1})_x$, hence

[12]Our proof of this second fact will hold for sheafs with values in Set, and Ring.

$s_x \in (\operatorname{im} F_{i-1})_x$. Now suppose that $s_x \in \operatorname{im}(F_{i-1})_x$, then we have that there exists an $f_x \in (\mathscr{F}_{i-1})_x$ such that $(F_{i-1})_x(f_x) = s_x$. Hence for some U and V, and some $f \in \mathscr{F}_{i-1}(U)$, $s \in \mathscr{F}_i(V)$ we have that:

$$[U, (F_{i-1})_U(f)] = [V, s]$$

so there exists a open subset $x \in W \subset U \cap V$, such that:

$$(F_{i-1})_W(f|_W) = s|_W$$

It follows that $s|_W \in (\operatorname{im}^p F_{i-1})(W)$, and by the universal property we have that:

$$\iota_W \circ \operatorname{sh}_W(s|_W) = \iota^p_W(s|_W) = s|_W$$

Taking stalks, we find that:

$$\iota_x \circ \operatorname{sh}_x(s_x) = s_x$$

so $s_x \in \iota(\operatorname{im}(F_{i-1}))_x$. We thus see that:

$$(F_{i-1})_x(s_x) = [W, F_W \circ \iota_W \circ \operatorname{sh}_W(s|_W)] = [W, 0] = 0$$

so $s_x \in \ker(F_{i-1})_x$. It follows that that $\ker(F_{i-1})_x = (\operatorname{im} F_{i-1})_x$, so the sequence of stalks is exact.

Now suppose the sequence of stalks is exact, we want to show that $(\ker F_i)(U) = \iota(\operatorname{im} \mathscr{F}_{i-1})(U)$. Note that in the last section, we have implicitly shown that $\iota(\operatorname{im} F_{i-1})_x = \operatorname{im}(F_{i-1})_x$. Let $s \in (\ker F_i)(U) = \ker(F_i)_U$, then we have that for each $x \in U$, $s_x \in \ker(F_i)_x$, hence each $s_x \in \operatorname{im}(F_{i-1})_x = \iota(\operatorname{im} F_{i-1})_x$. It follows that there is an open cover of U, by U_x, such that $s|_{U_x} \in \iota(\operatorname{im} F_{i-1})(U)$, which all vacuously agree on overlaps. We thus have that $s|_{U_x}$ glue together to $s \in \iota(\operatorname{im} F_{i-1})(U)$, implying that $(\ker F_i)(U) \subset \iota(\operatorname{im} F_{i-1}(U))$. Now let $s \in \iota(\operatorname{im} F_{i-1})(U)$, then for all $x \in U$, we have that $s_x \in \iota(\operatorname{im} F_{i-1})_x = \operatorname{im} F_x$, so by exactness each $s_x \in \ker(F_i)_x = \ker(F_i)_x$. It follows by the same argument that $s \in (\ker F_i)(U)$, hence $(\ker F_i)(U) = \iota(\operatorname{im} F_{i-1})$ implying the claim. □

We also have the following result:

Proposition 1.2.10. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups, then F is an isomorphism if and only if it is injective an surjective.*

Proof. Suppose that F is an isomorphism, then in particular, F_U is an isomorphism for all U. It follows that $\ker F_U = (\ker F)(U) = \{0\}$ so $\ker F$ is the trivial sheaf, implying that F is injective. To show that F is surjective, by Proposition 1.2.8 we need only show that $\operatorname{coker} F$ is the trivial sheaf. Since F is an isomorphism, we have that $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism, so $\operatorname{im} \mathscr{F}_x = \mathscr{G}_x$ for all $x \in X$. Since the stalks of $\operatorname{coker} F$ are isomorphic to $\mathscr{G}_x / \operatorname{im} \mathscr{F}_x = 0$ it follows that $(\operatorname{coker} F)(U)$ is trivial for all U, thus F is surjective.

Now suppose that F is injective and surjective. Since F is a sheaf, Proposition 1.2.2 we need only check that F_x is an isomorphism for all x. Since F is injective, we have that $(\ker F)_x = 0$, so by Corollary 1.2.1 we have that $\ker F_x = 0$. Since F is surjective, we have that $\iota(\operatorname{im} F)_x = \mathscr{G}_x$, but $\iota(\operatorname{im} F)_x = \operatorname{im} F_x$, hence F_x is a a surjection, implying that F_x is an isomorphism for all x, hence F is an isomorphism. □

We now discuss the process of ‘gluing together’ sheaves. First some notation, if $\mathscr{F}$ is a sheaf on X, then we can obtain an induced sheaf on any open set $U \subset X$, denote $\mathscr{F}|_U$, by setting $\mathscr{F}_U(V) = \mathscr{F}(V)$ for all open subsets of U. Since any open subset of U is open in X, this assignment makes sense, and clearly determines a sheaf.

Theorem 1.2.2. *Let $\{U_i\}_{i\in I}$ be an open cover for a topological space X, and $\mathscr{F}_i$ be a sheaf on each U_i such that there exist isomorphisms $\phi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$ which satisfy the **cocycle condition** for all i, j, i.e.*

$$\phi_{jk} \circ \phi_{ij} = \phi_{ik} \qquad \text{and } \phi_{ii} = Id$$

on $U_i \cap U_j \cap U_k$. Then there exists a sheaf $\mathscr{F}$ on X such that $\mathscr{F}|_{U_i} \cong \mathscr{F}_i$ for all i.

Proof. Let $V \subset X$ be an open set, we define $\mathscr{F}(V)$ to be the set:

$$\mathscr{F}(V) = \left\{ (s_i) \in \prod_{i \in I} \mathscr{F}_i(V \cap U_i) : \forall i, j \in I, \phi_{ij}(s_i|_{V \cap U_i \cap U_j}) = s_j|_{V \cap U_i \cap U_j} \right\}$$

where it is understood that ϕ_{ij} is the group isomorphism $(\phi_{ij})_{V \cap U_i \cap U_j}$. We check that this is indeed a subgroup of $\prod_i \mathscr{F}(V \cap U_i)$. Clearly, $0 \in \mathscr{F}(V)$, so we need only check that $\mathscr{F}(V)$ is closed under addition and contains inverses. Let $(s_i), (t_i) \in \mathscr{F}(V)$, then we have the sequence $(s_i + t_i) \in \prod_i \mathscr{F}(V \cap U_i)$. We want to show that this sequence lies in $\mathscr{F}(V)$; note that ϕ_{ij} and restriction maps are homomorphisms, so we have that for each i and j:

$$\begin{aligned}\phi_{ij}([s_i + t_i]|_{V \cap U_i \cap U_j}) =& \phi_{ij}(s_i|_{V \cap U_i \cap U_j}) + \phi_{ij}(t_i|_{V \cap U_i \cap U_j}) \\ =& s_j|_{V \cap U_i \cap U_j} + t_j|_{V \cap U_i \cap U_j} \\ =& [s_j + t_j]|_{V \cap U_i \cap U_j}\end{aligned}$$

hence $(s_i + t_i) \in \mathscr{F}(V)$. The same argument demonstrates that $(-s_i) \in \prod_i \mathscr{F}(V \cap U_i)$ is contained in the $\mathscr{F}(V)$, hence $\mathscr{F}(V)$ is a subgroup. Now let $W \subset V$, we define restriction maps θ^V_W by:

$$\theta^V_W((s_i)) = (\theta^{V \cap U_i}_{W \cap U_i}(s_i))$$

where where $\theta^{V \cap U_i}_{W \cap U_i}$ is the restriction map $\mathscr{F}_i(V \cap U_i) \to \mathscr{F}_i(W \cap U_i)$. It is then clear that $\mathscr{F}$ is a presheaf, as the restriction maps clearly satisfy $\theta^V_V = \mathrm{Id}$, and $\theta^W_Z \circ \theta^V_W = \theta^V_Z$. We first verify that $\mathscr{F}|_{U_i} \cong \mathscr{F}_i$. We define a morphism $F_j : \mathscr{F}|_j \to \mathscr{F}|_{U_j}$ on open sets $V \subset U_j$ by:

$$s \longmapsto (\phi_{ji}(s|_{V \cap U_i}))$$

We first check $F_j(s) \in \mathscr{F}|_{U_j}(V) = \mathscr{F}(V)$. We see for all k and l that by the cocycle condition:

$$\begin{aligned}\phi_{kl}(\phi_{jk}(s|_{V \cap U_k})|_{V \cap U_k \cap U_l}) =& \phi_{kl}(\phi_{jk}(s|_{V \cap U_k \cap U_l})) \\ =& \phi_{jl}(s|_{V \cap U_k \cap U_l})\end{aligned}$$

which is the lth component of our element, hence F has image in $\mathscr{F}|_{U_i}$. This map clearly commutes with restriction as ϕ_{ji} commutes restriction, hence F is a natural transformation, and thus indeed a morphism. We define an inverse morphism F_j^{-1} given on open sets by $(s_i) \mapsto s_j$, which is again clearly a natural transformation. We check that this is an inverse, let $s \in \mathscr{F}_j(V) = \mathscr{F}_j(V \cap U_j)$, then we have that:

$$F_j^{-1} \circ F_j(s) = \phi_{jj}(s|_{V \cap U_j}) = s|_{V \cap U_j} = s$$

While:

$$F_j \circ F_j^{-1}((s_i)) = (\phi_{ji}(s_j|_{V \cap U_j}))$$

However, we have that each $s_i \in \mathscr{F}(V \cap U_i)$:

$$s_i = s_i|_{V \cap U_i} = s_i|_{V \cap U_i \cap U_j} = \phi_{ji}(s_j|_{V \cap U_i \cap U_j}) = \phi_{ji}(s_j|_{V \cap U_j})$$

hence $F_j \circ F_j^{-1} = \mathrm{Id}$, and $F_j^{-1} \circ F_j = \mathrm{Id}$. It follows that $\mathscr{F}_j \cong \mathscr{F}|_{U_j}$. We can now show that $\mathscr{F}$ is a sheaf, take the sequence $(s_i) \in \prod_i \mathscr{F}_i(V \cap U_i)$ that satisfies $s_i|_{V_k \cap U_i} = s_i^k$ for each i and k. Moreover we see that if $s \in \mathscr{F}_i|_{U_i \cap U_j}(V)$:

$$\begin{aligned}F_j|_{U_i \cap U_j} \circ \phi_{ij}(s) =& (\phi_{jk}(\phi_{ij}(s)|_{V \cap U_k})) \\ =& (\phi_{jk} \circ \phi_{ij}(s)|_{V \cap U_k}) \\ =& (\phi_{ik}(s)|_{V \cap U_k}) \\ =& F_i(s)\end{aligned}$$

We now check that $\mathscr{F}$ is a sheaf; let V_k be an open cover of V, and $(s_i) \in \mathscr{F}(V)$ such that $(s_i)|_{V_k} = 0$ for all k. We see that $(s_i)|_{V_k} = 0$, implies that for each i we have that:

$$s_i|_{V_k \cap U_i} = 0$$

for all k. Since $\{V_k \cap U_i\}_k$ is an open cover for $V \cap U_i$, it follows that that $s_i = 0$ as $\mathscr{F}_i$ is a sheaf. This holds for all i so $(s_i) = 0$. Now suppose that we have sections $(s_i^k) \in \mathscr{F}(V_k)$ such that:

$$(s_i^k)_{V_k \cap V_m} = (s_i^m)|_{V_k \cap V_m}$$

implying that for all i:

$$s_i^k|_{V_k \cap V_m \cap U_i} = s_i^m|_{V_k \cap V_m \cap U_i}$$

It follows that since $\{V_k \cap U_i\}$ is an open cover of $V \cap U_i$, and $\mathscr{F}_i$ is a sheaf, that there exists a section $s_i \in \mathscr{F}_i(V \cap U_i)$ such that $s_i|_{V_k \cap U_i} = s_i^k$. We thus have a sequence $(s_i) \in \prod_i \mathscr{F}_i(V \cap U_i)$, such that $(s_i)|_{V_k} = (s_i^k)$. We want to show that:

$$\phi_{ij}(s_i|_{V \cap U_i \cap U_j}) = s_j|_{V \cap U_i \cap U_j}$$

However, note that $s_i \in \mathscr{F}_i(V \cap U_i)$, hence we have that:

$$s_i|_{V \cap U_i \cap U_j} = \theta^{V \cap U_i}_{V \cap U_i \cap U_j}(s_i)$$

If we further restrict to $V_k \cap U_i \cap U_j$, then we have that:

$$(s_i|_{V \cap U_i \cap U_j})|_{V_k \cap U_i \cap U_j} = \theta^{V \cap U_i}_{V_k \cap U_i \cap U_j}(s_i) = \theta^{V_k \cap U_i}_{V_k \cap U_i \cap U_j} \circ \theta^{V \cap U_i}_{V_k \cap U_i}(s_i) = \theta^{V_k \cap U_i}_{V_k \cap U_i \cap U_j}(s_i^k) = s_i^k|_{V_k \cap U_i \cap U_j}$$

And we know that for all k:

$$\phi_{ij}(s_i^k|_{V_k \cap U_i \cap U_j}) = s_j^k|_{V_k \cap U_i \cap U_j}$$

Since ϕ_{ij} is a natural transformation, we thus have that:

$$\phi_{ij}(s_i|_{V \cap U_i \cap U_j})|_{V_k \cap U_i \cap U_j} = (s_j|_{V \cap U_i \cap U_j})|_{V_k \cap U_i \cap U_j}$$

Since $V_k \cap U_i \cap U_j$ covers $V \cap U_i \cap U_j$, we have by sheaf axiom one that:

$$\phi_{ij}(s_i|_{V \cap U_i \cap U_j}) = s_j|_{V \cap U_i \cap U_j}$$

hence $(s_i) \in \mathscr{F}(V)$ as desired. ☐

Proposition 1.2.11. *Let U_i be an open cover for the topological space X, $\mathscr{F}_i$ be a sheaf on each U_i, and $\phi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$ isomorphism which satisfy the cocycle condition, then the sheaf $\mathscr{F}$ induced by gluing the of $\mathscr{F}_i$'s satisfies the following universal property: for all sheafs $\mathscr{G}$, and collection of morphisms $\psi_i : \mathscr{F}_i \to \mathscr{G}|_{U_i}$ such that $\psi_j|_{U_i \cap U_j} \circ \phi_{ij} = \psi_i|_{U_i \cap U_j}$, there exists a unique $\psi : \mathscr{F} \to \mathscr{G}$ such that the following diagram commutes for all i:*

$$\begin{tikzcd}
\mathscr{F}|_{U_i} \arrow[r, "\psi|_{U_i}"] \arrow[d, "F_i^{-1}"'] & \mathscr{G}|_{U_i} \\
\mathscr{F}_i \arrow[ur, "\psi_i"'] &
\end{tikzcd}$$

Proof. Let $(s_i) \in \mathscr{F}(V)$, then we see that $(\psi_i(s_i)) \in \prod_i \mathscr{G}(V \cap U_i)$, where we again suppress the notation $(\psi_i)_{V \cap U_i}$. However, note that $\{V \cap U_i\}$ cover V, hence since:

$$\begin{aligned}
\psi_i(s_i)|_{V \cap U_i \cap U_j} =& \psi_i(s_i|_{V \cap U_i \cap U_j}) \\
=& \psi_i|_{U_i \cap U_j}(\phi_{ji}(s_j|_{V \cap U_i \cap U_j})) \\
=& \psi_j|_{U_i \cap U_j}(s_j|_{V \cap U_i \cap U_j}) \\
=& \psi_j(s_j|_{V \cap U_i \cap U_j})
\end{aligned}$$

we have that the sections $\psi_i(s_i) \in \mathscr{G}(V \cap U_i)$ glue to a unique section $g \in \mathscr{G}(V)$, which satisfies $g|_{V \cap U_i} = \psi_i(s_i)$ for all i. We thus define θ on open sets by:

$$\psi_V((s_i)) = g$$

We check that this commutes with restrictions. Let $W \subset V$, then we want to show that:

$$\psi_W((s_i)|_W) = g|_W$$

We see that $(s_i)|_W$ is equal to $(s_i|_{W\cap U_i})$, so $\psi_W((s_i)|_W)$ is the section unique such that:

$$\psi_W((s_i)|_W)|_{W\cap U_i} = \psi_i(s_i|_{W\cap U_i})|_{W\cap U_i} = \psi_i(s_i)|_{W\cap U_i}$$

However, we have that:

$$(g|_W)|_{W\cap U_i} = g|_{W\cap U_i} = (g|_{U_i})|_{W\cap U_i} = (\psi_i(s_i))|_{W\cap U_i}$$

so sheaf axiom one implies that the assignment $V \mapsto \psi_V$ is indeed a natural transformation and thus a morphism as desired. We now show that the diagram commutes; let $V \subset U_j$, and $(s_i) \in \mathscr{F}|_{U_j}(V) = \mathscr{F}(U)$. Then we have that:

$$\psi_j \circ F_j^{-1}(s_i) = \psi_j(s_j)$$

while:

$$(\psi|_{U_j})_V(s) = \psi_V((s_i)) = g$$

where for all i, we have that $g|_{V\cap U_i} = \psi_i(s_i)$. We see that $V \cap U_j = V$, so $g = g|_{V\cap U_j} = \psi_j(s_j)$, so $\mathscr{F}$ satisfies universal property as desired. $\square$

We now show that $\mathscr{F}$ is unique up to unique isomorphism, and that we can always glue a sheaf back together.

Corollary 1.2.3. *Let U_i be an open cover for X, and $\mathscr{F}_i$ sheafs on U_i equipped with isomorphisms $\phi_{ij} : \mathscr{F}_i \to \mathscr{F}_j$ which satisfy the cocycle condition. Then the sheaf $\mathscr{F}$ induced by the gluing of $\mathscr{F}_i$ is unique up to unique isomorphism. In particular, if $\mathscr{F}$ is a sheaf on X, and U_i is any open cover of X, then $\mathscr{F}$ is the sheaf induced by gluing of $\mathscr{F}|_{U_i}$ together.*

Proof. Let $\mathscr{F}$ be the sheaf induced by the gluing of $\mathscr{F}_i$, and $\mathscr{G}$ be any other sheaf which satisfies the universal property outlined in Proposition 1.2.11, i.e. $\mathscr{G}$ is a sheaf with isomorphisms $G_i : \mathscr{F}_i \to \mathscr{G}|_{U_i}$, such that $G_j|_{U_i\cap U_j} \circ \phi_{ij} = G_i|_{U_i\cap U_j}$, and that for any collection of morphisms $\psi_i : \mathscr{F}_i \to \mathscr{H}|_{U_i}$ there exists a unique $\psi : \mathscr{G} \to \mathscr{H}$ that makes the diagram commute for all i. In particular, we have that we get unique maps $\psi_{\mathscr{G}} : \mathscr{G} \to \mathscr{F}$, and $\psi_{\mathscr{F}} : \mathscr{F} \to \mathscr{G}$, such that:

$$G_i \circ F_i^{-1} = \psi_{\mathscr{F}}|_{U_i} \qquad \text{and} \qquad F \circ G^{-1} = \psi_{\mathscr{G}}|_{U_i}$$

On any open set V, we have that $V \cap U_i$ is an open cover. Let $s \in \mathscr{F}(V)$, then $s|_{V\cap U_i} \in \mathscr{F}(V \cap U_i)$, and we have that:

$$\begin{aligned}
(\psi_{\mathscr{G}} \circ \psi_{\mathscr{F}})_V(s)|_{V\cap U_i} =& (\psi_{\mathscr{G}} \circ \psi_{\mathscr{F}})_{V\cap U_i}(s|_{V\cap U_i}) \\
=& (\psi_{\mathscr{G}}|_{U_i} \circ \psi_{\mathscr{F}}|_{U_i})_{V\cap U_i}(s|_{V\cap U_i}) \\
=& (F \circ G \circ G_i^{-1} \circ F_i^{-1})_{V\cap U_i}(s|_{V\cap U_i}) \\
=& s|_{V\cap U_i}
\end{aligned}$$

so by sheaf axiom one we have that $(\psi_{\mathscr{G}} \circ \psi_{\mathscr{F}})_V = \mathrm{Id}$. The same argument shows that $\psi_{\mathscr{F}} \circ \psi_{\mathscr{G}} = \mathrm{Id}$, then $\mathscr{F} \cong \mathscr{G}$, so $\mathscr{F}$ is unique up to unique isomorphism.

Now let $\mathscr{F}$ be a sheaf, and U_i an open cover of X. We see that by setting $\mathscr{F}_i = \mathscr{F}|_{U_i}$ we have natural isomorphisms $\mathscr{F}_i|_{U_i\cap U_j} \to \mathscr{F}|_{U_i\cap U_j}$ given by the identity map $s \mapsto s$ on all open sets. This makes sense as unraveling the notation we have that for any open set $V \subset U_i \cap U_j$:

$$\mathscr{F}_i|_{U_i\cap U_j}(V) = \mathscr{F}_i(V) = \mathscr{F}|_{U_i}(V) = \mathscr{F}(V) = \mathscr{F}|_{U_j}(V) = \mathscr{F}_j(V) = \mathscr{F}_j|_{U_i\cap U_j}(V)$$

It suffices to show that $\mathscr{F}$ satisfies the universal property in Proposition 1.2.11, where the maps F_i^{-1} : $\mathscr{F}|_{U_i} \to \mathscr{F}_i$ are the identity maps. Let $\psi_i : \mathscr{F}_\to \mathscr{G}|_{U_i}$ be any collection of morphisms such that $\psi_i|_{U_i\cap U_j} =$

$\psi_j|_{U_i\cap U_j}$, and take $s \in \mathscr{F}(V)$. We have define $\psi(s)$ to be the unique section $g \in \mathscr{G}(V)$ such that $g|_{V\cap U_i} = \psi_i(s|_{V\cap U_i})$. This section exists as $V \cap U_i$ cover V, and for all i and j we have that:

$$\begin{aligned}\psi_i(s|_{V\cap U_i})|_{V\cap U_i\cap U_j} =& \psi_i|_{U_i\cap U_j}(s|_{V\cap U_i\cap U_j})\\ =&\psi_j|_{U_i\cap U_j}(s|_{V\cap U_i\cap U_j})\\ =&\psi_j(s|_{V\cap U_i\cap U_j})\\ =&\psi_j(s|_{V\cap U_j})|_{V\cap U_i\cap U_j}\end{aligned}$$

hence by sheaf axiom two, the sections glue together to form g. The same argument in Proposition 1.2.11 demonstrates that this a natural transformation, and that $\psi|_{U_i} = \psi_i$, so $\mathscr{F}$ satisfies the universal property as desired. ☐

In the process of proving Corollary 1.2, we have obtained the following corollary as well:

Corollary 1.2.4. *If $\psi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ is a collection of morphisms such that $\psi_i|_{U_i\cap U_j} = \psi_j|_{U_i\cap U_j}$ for all i and j then there exists a unique map $\psi : \mathscr{F} \to \mathscr{G}$ such that $\psi|_{U_i} = \psi_i$. In particular, ψ is an isomorphism if and only if ψ_i is an isomorphism for all i. Moreover, if $\psi : \mathscr{F} \to \mathscr{G}$ is a morphism such that $\psi|_{U_i} : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ is an isomorphism then ψ is an isomorphism.*

Proof. We need only prove the last statement, in particular we need only show that $\psi_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism. However for all $x \in X$, if $x \in U_i$, we have that $\psi_x = (\psi|_{U_i})_x$, as if $[U, s] \in \mathscr{F}_x$, then $[U, s] = [U_i, s|_{U_i}]$, so

$$\psi_x([U, s]) = \psi_x([U_i, s|_{U_i}]) = [U_i, \psi(s|_{U_i})] = [U_i, \psi|_{U_i}(s|_{U_i})] = (\psi|_{U_i})_x([U, s])$$

implying the claim. ☐

1.3 Locally Ringed Spaces

We recall the definition of a local ring:

Definition 1.3.1. A commutative ring R is a **local ring** if there exists a unique maximal ideal. A **local domain** is an integral domain that is local.

Example 1.3.1. Let A be a commutative ring and $\mathfrak{p}$ a prime ideal, then $A_{\mathfrak{p}} = (A \setminus \mathfrak{p})^{-1}A$ is a local ring. Indeed, consider the ideal $\mathfrak{m}$ defined by:

$$\mathfrak{m} = \left\{\frac{p}{a} : p \in \mathfrak{p}\right\}$$

i.e. any element of $\mathfrak{m}$ can be written as the equivalent class $[(p, a)]$ where $p \in \mathfrak{p}$. We check that this is an ideal, clearly $\mathfrak{m}$ is closed under addition, contains inverses, and contains the zero element. It is also clear that $\mathfrak{m}$ swallows multiplication so $\mathfrak{m}$ is an ideal. We check this is maximal, suppose for the sake of contradiction that we have an ideal $J \subset A_{\mathfrak{p}}$ such that $\mathfrak{m} \subset J$. Then there must be some $a/s \in J$ where $a \notin \mathfrak{p}$, but it if $a \notin \mathfrak{p}$, then we have that $a \in A - \mathfrak{p}$, hence:

$$\frac{a}{s} \cdot \frac{s}{a} = 1$$

so $J = A_{\mathfrak{p}}$, so $\mathfrak{m}$ is indeed maximal. Now suppose that J is another maximal ideal not equal to $\mathfrak{m}$, then J contains an element a/s such that $a \notin \mathfrak{p}$, so the same argument shows that $J = A_{\mathfrak{p}}$. It follows that $A_{\mathfrak{p}}$ is a local ring.

We now define locally ringed spaces:

Definition 1.3.2. Let $(X, \mathscr{O}_X)$ be a topological space X, equipped with a sheaf of rings $\mathscr{O}_x$. Then $(X, \mathscr{O}_X)$ is a **locally ringed space** if the stalk of $\mathscr{O}_X$ at x, denoted $(\mathscr{O}_X)_x$ or $\mathscr{O}_{X,x}$, is a local ring for all $x \in X$. We denote the unique maximal ideal of the stalk a locally ringed space as $\mathfrak{m}_x$, and the sheaf $\mathscr{O}_X$ is called the **structure sheaf of** X.

Example 1.3.2. Let (M, C^∞) the data of a smooth manifold M with the sheaf of C^∞ functions on M. The stalk $(C^\infty)_x$ is the set of equivalence classes $[U, f]$, where $x \in U$. Consider the set:

$$\mathfrak{m}_x = \{[U, f] : f(x) = 0\}$$

Note that if $[U,f] \in \mathfrak{m}_x$ then clearly ever $[V,g]=[U,f]$ also satisfies $g(x)=0$, as f and g have to agree on an open set containing x. It follows that $\mathfrak{m}_x$ is well defined. We see that $\mathfrak{m}_x$ is clearly a subgroup of $(\mathbb{C}^\infty_x)$, so we check that $\mathfrak{m}_x$ is an ideal. If $[U,g] \in (\mathbb{C}^\infty)_x$ and $[V,f] \in \mathfrak{m}_x$, then we have that $[U \cap V, f \cdot g]$ satisfies $(f \cdot g)(x) = f(x)g(x) = 0$, so $\mathfrak{m}_x$ is an ideal.

We show that $\mathfrak{m}_x$ is maximal; define a map $\psi : (\mathbb{C}^\infty)_x \to \mathbb{R}$ by:

$$\psi([U,f]) = f(x)$$

This well defined for the same reason that $\mathfrak{m}_x$ is well defined, and satisfies $\ker \psi = \mathfrak{m}_x$ essentially by definition. It is also clearly a ring morphism, and surjective as the constant function maps $f(x)=a$ maps to $[M,f]$ under the map $\mathbb{C}^\infty(M) \to (C^\infty)_x$, which maps to a under ψ. It follows that ψ descends to an isomorphism $(C^\infty)_x/\mathfrak{m}_x \to \mathbb{R}$. Since the quotient space is a field, it follows that $\mathfrak{m}_x$ is maximal.

To see that $\mathfrak{m}_x$ is unique, suppose that J is any other maximal ideal not equal to $\mathfrak{m}_x$. Then there must be some $[U,f] \in J$ such that $f(x) \neq 0$. However, this implies that there exists an open neighborhood V_x of x such that $f(y) \neq 0$ for all $y \in V_x$. The function $g(x) = f(x)^{-1}$ is then smooth on V_x, and we see that:

$$[U,f] \cdot [V_x, g] = [V_x, 1]$$

which is the unit element of $(\mathbb{C}^\infty)_x$, hence $J = (C^\infty)_x$, and $\mathfrak{m}_x$ is unique.

Since every stalk in a locally ringed space has a unique maximal ideal, we can associate to each stalk a unique field as follows:

Definition 1.3.3. Let $(X, \mathscr{O}_X)$ be a locally ringed space, then for all $x \in X$ the **residue field** k_x is given by:

$$k_x = (\mathscr{O}_X)_x/\mathfrak{m}_x$$

For each open $U \subset X$, and all $x \in U$ we have the **evaluation map** $\mathrm{ev}_x : \mathscr{O}_X(U) \to k_x$ given by:

$$s \longmapsto [s_x]$$

where $[s_x]$ is the image of s_x under the projection $(\mathscr{O}_X)_x \to k_x$. We say that an element of s vanishes at s_x if $s \in \ker \mathrm{ev}_x$.

Definition 1.3.4. Let $(X, \mathscr{F})$ be the data of a topological space, and a sheaf on X, and let $f : X \to Y$ be a continuous map. Then $f_*\mathscr{F}$ is the sheaf on Y defined by:

$$(f_*F)(U) = F(f^{-1}(U))$$

We call $f_*\mathscr{F}$ the **pushforward** or **direct image sheaf**.

Proposition 1.3.1. *Let $(X, \mathscr{F})$ be the data of a topological space, and a sheaf on X, and let $f : X \to Y$ be a continuous map. Then $f_*\mathscr{F}$ is indeed a sheaf on Y.*

Proof. We first show that $f_*\mathscr{F}$ is presheaf. Define restriction functions $\theta^U_V : (f_*\mathscr{F})(U) \to (f_*\mathscr{F})(V)$ by:

$$\theta^U_V = \theta^{f^{-1}(U)}_{f^{-1}(V)}$$

Note that this makes sense, as if $V \subset U$, then we have that $f^{-1}(V) \subset f^{-1}(U)$. It follows that for $W \subset V \subset U$:

$$\theta^V_W \circ \theta^U_V = \theta^{f^{-1}(V)}_{f^{-1}(W)} \circ \theta^{f^{-1}(U)}_{f^{-1}(V)} = \theta^{f^{-1}(U)}_{f^{-1}(W)} = \theta^U_W$$

It is clear that $\theta^U_U = \mathrm{Id}$, so $f_*\mathscr{F}$ is a presheaf. Now let $s \in (f_*\mathscr{F})(U)$, and U_i be a cover for U, such that $s|_{U_i} = 0$. Then this we have that $s \in \mathscr{F}(f^{-1}(U))$, and $s|_{f^{-1}(U_i)} = 0$ for all i. We that:

$$f^{-1}(U) = \bigcup_i f^{-1}(U_i)$$

so it follows that $s = 0$, as $\mathscr{F}$ is a sheaf. The same argument demonstrates that $f_*\mathscr{F}$ satisfies sheaf axiom two, so $f_*\mathscr{F}$ is a sheaf. $\square$

Proposition 1.3.2. *Let $(X, \mathscr{F})$ be a sheaf, and $f : X \to Y$ be a continuous map between topological spaces. Then for all $p \in X$, there exists a natural morphism of stalks $(f_*)_p : (f_*\mathscr{F})_{f(p)} \to \mathscr{F}_p$.*

Proof. Let $p \in X$, for all U containing $f(p)$ we define maps $\phi_U : (f_*\mathscr{F})(U) \to \mathscr{F}_p$ by first noting that $(f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}(U))$, hence $p \in f^{-1}(U)$, and it thus makes sense to set:

$$s \mapsto [f^{-1}(U), s]_p$$

Since the restriction maps θ^U_V are $\theta^{f^{-1}(U)}_{f^{-1}(V)}$, it follows that $\phi_V \circ \theta^U_V = \phi_U$, hence by the universal property of the colimit, there exists a unique map:

$$(f_*)_p : (f_*\mathscr{F})_{f(p)} \to \mathscr{F}_p$$

such that:

$$(f_*)_p \circ \psi_U = \phi_U \tag{1.3.1}$$

for all U containing $f(p)$, implying the claim.

□

Note that if $s_{f(p)} \in (f_*\mathscr{F})_{f(p)}$, then by (1.3), we have that:

$$(f_*)_p(s_{f(p)}) = [f^{-1}(U), s]$$

for any $s \in (f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}(U))$, such that $f(p) \in U$, and $s_{f(p)} = [U, s]$.

Let (M, C^∞_M) and (N, C^∞_N) be smooth manifolds equipped with the structure sheaf of smooth functions on M and N respectively. If $F : M \to N$ is a smooth map, then we obtain a map of sheaves $F^\sharp : C^\infty_N \to F_*C^\infty_M$ given on open sets by:

$$\begin{aligned} F^\sharp_U : C^\infty_N(U) &\longrightarrow (F_*C^\infty_M)(U) = C^\infty_M(F^{-1}(U)) \\ f &\longmapsto f \circ F \end{aligned} \tag{1.3.2}$$

When $U = N$, $F^\sharp$ is the standard pull back map $f^* : \mathbb{C}^\infty(N) \to \mathbb{C}^\infty(M)$. In fact, one can show that F is smooth if and only if F induces a morphism on the sheaves of smooth functions. Indeed, if F is smooth then (1.4) is clearly a morphism of sheaves. Now suppose that F is a set map such that $F^\sharp : C^\infty_N \to f_*C^\infty_M$ is a morphism of sheaves. Let (U, ϕ) be a coordinate chart for N, where:

$$\phi = (x^1, \ldots, x^n)$$

It follows that for each i,

$$x^i \circ F : F^{-1}(U) \to \mathbb{R}$$

is a smooth map, hence:

$$\phi \circ F : F^{-1}(U) \to \mathbb{R}^n$$

is smooth. Letting (ψ, V) be any chart contained in $F^{-1}(U)$, we see that the composition:

$$\phi \circ F \circ \psi^{-1} : \psi(V) \to \phi(F(V))$$

is smooth a smooth map $\mathbb{R}^m \to \mathbb{R}^n$, hence F is smooth.

Our next goal is to extend this picture of smooth maps in differential geometry as the data of a continuous map between manifolds, and a sheaf morphisms between the sheaves of C^∞ functions to the general setting of ringed and locally ringed spaces.

Definition 1.3.5. Let $(X, \mathscr{O}_X)$, and $(Y, \mathscr{O}_Y)$ be ringed spaces, a **morphism of ringed spaces** $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is the data of a continuous map $f : X \to Y$, and a sheaf morphism $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$. We generally refer to a morphism of ringed spaces only by the map on the underlying topological spaces.

Let $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$, and $(Z, \mathscr{O}_Z)$ be locally ringed spaces, and consider morphisms $f : X \to Y$ and $g : Y \to Z$, then it makes sense to take the composition of topological maps $g \circ f : X \to Z$. However, we see that $g^\sharp : \mathscr{O}_Z \to g_*\mathscr{O}_Y$, and $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ are morphisms of sheaves over different topological spaces, so it doesn't make sense to compose them. Indeed, $(g \circ f)^\sharp$ should me a morphism of sheaves over Z, $\mathscr{O}_Z \to (g \circ f)_*\mathscr{O}_X$. Well, note that $(g \circ f)_*\mathscr{O}_X = g_*(f_*\mathscr{O}_X)$, and that we obtain an induced map $g_*f^\sharp : g_*\mathscr{O}_Y \to g_*(f_*\mathscr{O}_X)$ given on open sets $V \subset Z$ by:

$$\begin{aligned} g_*f^\sharp : (g_*\mathscr{O}_Y)(V) = \mathscr{O}_Y(g^{-1}(V)) &\longrightarrow \mathscr{O}_X(f^{-1}(g^{-1}(V))) \\ s &\longmapsto f^\sharp_{g^{-1}(V)}(s) \end{aligned}$$

which clearly defines a morphism. We thus define the composition $g \circ f$ to be the data of the topological composition, along with the morphism of sheaves on Z given by $g_*f^\sharp \circ g^\sharp$. Obviously, this makes locally ringed spaces a category.

Lemma 1.3.1. *Let $(X, \mathscr{O}_x)$ and $(Y, \mathscr{O}_Y)$ be ringed spaces, and f a morphism between them. Then for all $x \in X$, there is an induced map on stalks $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$. If $g : Y \to Z$ is another morphism of ringed spaces, then the stalk map $(g \circ f)_x : (\mathscr{O}_Z)_{g(f(x))} \to (\mathscr{O}_X)_x$ is equal to $f_x \circ g_{f(x)}$. In particular, if $f, g : X \to Y$ are two morphisms of ringed spaces, such that the topological maps agree, and such that $f_x = g_x$ for all $x \in X$, then $f^\sharp = g^\sharp$.*

Proof. There is an induced map $f^\sharp_{f(x)} : (\mathscr{O}_Y)_{f(x)} \to (f_*\mathscr{O}_X)_{f(x)}$, and by Proposition 1.3.2, an induced map $(f_*)_x : (f_*\mathscr{O}_X)_{f(x)} \to (\mathscr{O}_X)_x$, hence we define f_x by the composition:

$$f_x = (f_*)_x \circ f^\sharp_{f(x)}$$

which is indeed a map on stalks $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$.

We note that for $s_{f(x)} = [U, s] \in (\mathscr{O}_Y)_{f(x)}$ we have that:

$$f_x(s_{f(x)}) = (f_*)_x([U, f^\sharp_U(s)]) = [f^{-1}(U), f^\sharp_U(s)]$$

Let $g : Y \to Z$ be another morphism of ringed spaces, then we have that:

$$(g \circ f)^\sharp = f_*g^\sharp \circ f^\sharp$$

so:

$$(g \circ f)_x = (g \circ f)_{*x} \circ (g_*f^\sharp \circ g^\sharp)_{g(f(x))}$$

Let $s_{g(f(x))} = [U, s]_{g(f(x))}$, for some open $U \subset Z$ and $s \in \mathscr{O}_Z(U)$, then we have that:

$$\begin{aligned} (g \circ f)_x(s_{f(x)}) =& (g \circ f)_{*x}([U, (g_*f^\sharp)_U \circ g^\sharp_U(s)]) \\ =& (g \circ f)_{*x}([U, f^\sharp_{g^{-1}(U)} \circ g^\sharp_U(s)]_{g(f(x))}) \\ =& [f^{-1}(g^{-1}(U)), f^\sharp_{g^{-1}(U)} \circ g^\sharp_U(s)]_x \end{aligned}$$

Unraveling our definitions, we see that:

$$\begin{aligned} [f^{-1}(g^{-1}(U)), f^\sharp_{g^{-1}(U)} \circ g^\sharp_U(s)]_x =& (f_*)_x([g^{-1}(U), f^\sharp_{g^{-1}(U)} \circ g^\sharp_U(s)]_{f(x)}) \\ =& (f_*)_x \circ (f^\sharp)_{f(x)}([g^{-1}(U), g^\sharp_U(s)]_{f(x)}) \\ =& f_x([g^{-1}(U), g^\sharp_U(s)]_{f(x)}) \\ =& f_x \circ (g_*)_{f(x)}([U, g^\sharp_U(s)]_{g(f(x))}) \\ =& f_x \circ g_{f(x)}([U, s]_{g(f(x))}) \end{aligned}$$

hence:

$$f_x \circ g_{f(x)} = (f \circ g)_x$$

as desired.

The proof of the final statement is left until the introduction the inverse image sheaf. □

We can now adequately define morphisms of locally ringed spaces:

Definition 1.3.6. Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be locally ringed spaces, then a **morphism of locally ringed spaces** is a morphisms of ringed spaces such that for all $x \in X$, the induced map on stalks $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ satisfies:

$$f_x(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$$

where $\mathfrak{m}_{f(x)}$ and $\mathfrak{m}_x$ are the unique maximal ideals of $(\mathscr{O}_Y)_{f(x)}$ and $(\mathscr{O}_X)_x$ respectively. An **isomorphism of locally ringed spaces** is a morphism where f is a homeomorphism and $f^\sharp$ is an isomorphism.[13].

Lemma 1.3.2. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of rings, then $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is an epimorphism if and only if F is an epimorphism..*

Proof. Suppose that F is an epimorphism, and let $\phi_1, \phi_2 : \mathscr{G}_x \to R$ be any two morphisms of rings such that:

$$\phi_1 \circ F_x = \phi_2 \circ F_x$$

Now note there exists maps $\phi^i_U : \mathscr{G}(U) \to R$, given by $\phi_i \circ \psi_U$, where $\psi_U : \mathscr{G}(U) \to \mathscr{G}_x$ is the usual ring homomorphism, such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[rr, "\theta^U_V"] \arrow[dr, "\phi^i_U"'] & & \mathscr{G}(V) \arrow[dl, "\phi^i_V"] \\
& R &
\end{tikzcd}$$

implying that $\phi_i : \mathscr{G}_x \to R$ are the unique maps which make the following diagram commute:

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[rr, "\theta^U_V"] \arrow[dr, "\psi_U"] \arrow[ddr, bend right, "\phi^i_U"'] & & \mathscr{G}(V) \arrow[dl, "\psi_V"'] \arrow[ddl, bend left, "\phi^i_V"] \\
& \mathscr{G}_x \arrow[d, "\phi_i"] & \\
& R &
\end{tikzcd}$$

It thus suffices to check that $\phi_1 \circ \psi_U = \phi_2 \circ \psi_U$ for all U containing x, as then $\phi^1_U = \phi^2_U$ so by uniqueness $\phi_1 = \phi_2$. Note that F is a an epimorphism, and that we have that:

$$F_x \circ \psi_U = \psi_U \circ F_U \tag{1.3.3}$$

where ψ_U on the left hand side is the usual morphism $\mathscr{F}(U) \to \mathscr{F}_x$. Now consider the skyscraper sheaf $x_*(R)$ along with the morphism:

$$\tilde{\phi}^i : \mathscr{G} \longrightarrow x_*(R)$$

defined by:

$$\tilde{\phi}^i_U(s) = \begin{cases} \phi_i \circ \psi_U(s) & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

[13]Note that in category of topological spaces f is a monomorphism (epimorphism) if and only if it is injective (surjective). In the category of sheaves of rings monomorphisms and epimorphisms Proposition 1.2.6 still partially applies; the argument for $a) - c)$ is the same, as well as for $e) \Rightarrow d)$, we will prove a modified version of $d) \Rightarrow e)$ shortly. In particular the kernel sheaf is a sheaf of ideals, while the image sheaf is a sheaf of rings, and the cokernel sheaf is the zero sheaf.

This commutes with restrictions, and thus defines a morphism of sheaves. We see that for all U not containing x we trivially have that $\tilde{\phi}^1_U \circ F_U = \tilde{\phi}^2_U \circ F_U$, while if U contains x then:

$$\begin{aligned}\tilde{\phi}^1_U \circ F_U(s) =& \phi_1 \circ \psi_U \circ F_U(s)\\ =& \phi_1 \circ F_x \circ \psi_U(s)\\ =& \phi_2 \circ F_x \circ \psi_U(s)\\ =& \tilde{\phi}^2_U \circ F_U(s)\end{aligned}$$

hence $\tilde{\phi}^1 \circ F = \tilde{\phi}^2 \circ F$ implying that $\tilde{\phi}^1 = \tilde{\phi}^2$. Thus on opens we must have that:

$$\phi_1 \circ \psi_U = \phi_2 \circ \psi_U$$

for all U containing x, implying the claim.

For the other direction, let F_x be an epimorphism for all x, and suppose that $\phi_i : \mathscr{G} \to \mathscr{H}$ are morphisms of sheaves of rings such that:

$$\phi_1 \circ F = \phi_2 \circ F$$

Then we have that:

$$(\phi_1)_x \circ F_x = (\phi_2)_x \circ F_x$$

however F_x is an epimorphism so $(\phi_1)_x = (\phi_2)_x$ for all $x \in X$. It follows that $\phi_1 = \phi_2$ and so F is an epimorphism. □

We also wish to extend Proposition 1.2.10 to the case of sheaves of rings:

Lemma 1.3.3. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of rings, then F is an isomorphism if and only if it is injective and surjective.*

Proof. If F is injective and surjective then the same argument as in Proposition 1.2.10 holds.

Let F be an isomorphism, then in particular $\ker F_U = 0$ for all 0, so F is injective. Moreover, we have that $\operatorname{im} F_U = \mathscr{G}_U$ for all U, so it follows that $(\operatorname{im}^p F)$ is actually a sheaf. It follows that $\operatorname{sh} : (\operatorname{im}^p F) \to \operatorname{im} F$ and $\iota^p : \operatorname{im}^p F \to \mathscr{G}$ are both isomorphisms, so we have that:

$$\iota \circ \operatorname{sh} = \iota^p \Rightarrow \iota \circ \operatorname{sh} \circ \operatorname{sh}^{-1} = \iota^p \circ \operatorname{sh}^{-1} \Rightarrow \iota = \iota^p \circ \operatorname{sh}^{-1}$$

hence ι must be an isomorphism. It follows that $\iota(\operatorname{im} F) = \mathscr{G}$, implying the claim. □

Lemma 1.3.4. *Let $(X, \mathscr{O}_X)$ be a locally ringed space, and $U \subset X$ be an open set. Then, $(U, \mathscr{O}_X|_U)$ is a locally ringed space equipped with monomorphism $\iota : U \to X$.*

Proof. It is clear that $(U, \mathscr{O}_X|_U)$ is a locally ringed space, and moreover that the inclusion map $\iota : U \to X$ is an injection and thus a monomorphism in the category of topological spaces. We thus need to describe a map $\iota^\sharp : \mathscr{O}_X \to \iota_*(\mathscr{O}_X|_U)$. Let $V \subset X$ be open, and note that:

$$\iota^{-1}(V) = V \cap U$$

as if $x \in V \cap U$, then we have $x \in U$ and $x \in V$, hence $\iota(x) = x \in V$, so $x \in \iota^{-1}(V)$. If $x \in \iota^{-1}(V)$, then we have that $x \in U$, and $\iota(x) = x \in V$, so $x \in V$ and U. We thus define $\iota^\sharp$ on open sets as:

$$\begin{aligned}\iota^\sharp_V : \mathscr{O}_X(V) &\longrightarrow (\iota_* \mathscr{O}_X|_U)(V) = \mathscr{O}_X|_U(\iota^{-1}(V)) = \mathscr{O}_X(U \cap V)\\ s &\longmapsto s|_{U\cap V}\end{aligned}$$

We note that this commutes with restriction maps, as if $W \subset V$ then:

$$\iota^\sharp_W(\theta^V_W(s)) = \theta^W_{W\cap U} \circ \theta^V_W(s) = \theta^V_{W\cap U}(s)$$

while:

$$\theta^V_W \circ \iota^\sharp_V(s) = \theta^{V\cap U}_{W\cap U} \circ \theta^V_{V\cap U}(s) = \theta^V_{W\cap U}(s)$$

so this is a morphism of sheaves. We check that $\iota_x^\sharp$ is a surjection for all x. Let $s_x \in (\iota_*(\mathscr{O}_X|_U))$, then for some $V \subset X$, and some $s \in \mathscr{O}_X(U \cap V)$, we have that $s_x = [V, s]$. However, we have that $[U \cap V, s] \in (\mathscr{O}_X)_x$, so trivially:

$$\iota_x^\sharp([U \cap V, s]) = [U \cap V, s]$$

We want to show that $[V, s] = [U \cap V, s]$. Let $W = U \cap V$, then we have that:

$$\theta^V_{U\cap V}(s) = \theta^{U\cap V}_{U\cap V}(s) = s$$

so $\iota_x^\sharp$ is surjective for all x, and thus an epimorphism. It follows by Lemma 1.3.2 that $\iota^\sharp$ is an epimorphism as well.

Now let $(f, f^\sharp)$ and $(g, g^\sharp)$ be morphisms $Y \to U$, $\mathscr{O}_X|_U \to f_*\mathscr{O}_Y$, such that:

$$\iota \circ f = \iota \circ g$$

and that:

$$(\iota \circ f)^\sharp = (\iota \circ g)^\sharp$$

Clearly since ι is a monomorphism, we have that the topological maps are the same, so we need only show that $f^\sharp = g^\sharp$. By Lemma 1.3.1, we need only show that $f_y = g_y$ for all $y \in Y$. Note that since $(\iota \circ f)^\sharp = (\iota \circ g)^\sharp$, we have that:

$$(\iota \circ f)_y = f_y \circ \iota_{f(y)} = g_y \circ \iota_{g(y)} = (\iota \circ g)_y$$

It follows that:

$$(f_y \circ (\iota_*)_{f(y)}) \circ \iota^\sharp_{\iota(f(y))} = (g_y \circ (\iota_*)_{f(y)}) \circ \iota^\sharp_{\iota(f(y))}$$

however, $\iota^\sharp$ is an epimorphism so we have that:

$$f_y \circ (\iota_*)_{f(y)} = g_y \circ (\iota_*)_{f(y)}$$

It suffices to check that $(\iota_*)_{f(y)} : (\iota_*(\mathscr{O}_X|_U))_{\iota(f(y))} \to (\mathscr{O}_X|_U)_{f(y)}$ is an epimorphism. We show a stronger result, i.e. that $(\iota_*)_{f(y)}$ is surjective. Let $[V, s]_{f(y)} \in (\mathscr{O}_X|_U)_{f(y)}$; then $f(y) \in V$, and $s \in \mathscr{O}_X|_U(V) = \mathscr{O}_X(V)$. It follows that $V \subset U$, so $\iota_*(\mathscr{O}_X|_U)(V) = \mathscr{O}_X(U \cap V) = \mathscr{O}_X(V)$, hence there is an element $[V, s]_{\iota(f(y))}$ in $(\iota_*(\mathscr{O}_X|_U))_{\iota(f(y))}$. We see that:

$$(\iota_*)_{f(y)}([V, s]_{\iota(f(y))}) = [\iota^{-1}(V), s]_{f(y)} = [V, s]_{f(y)}$$

so $(\iota_*)_{f(y)}$ is surjective, and thus an epimorphism. It follows that $f_y = g_y$ for all $y \in Y$, hence $f^\sharp = g^\sharp$ implyingt he claim. $\square$

We then have the obvious corollary:

Corollary 1.3.1. *Let $f : X \to Y$ be a morphism of ringed spaces, then if the topological map f is injective, and for all $x \in X$ the map $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ is an epimorphism then f is a monomorphism.*

Clearly, ι is an isomorphism onto itself as we have that $\iota : U \to U \subset X$ is a homeomorphism, and $\iota^\sharp : \mathscr{O}_X|_U \to (\iota_*\mathscr{O}_X|_U)$ is the identity map. Importantly if $f : X \to Y$ is any morphism, then there exists a restricted map $f|_U : U \to Y$, where the topological map is the standard restriction, and:

$$(f|_U)^\sharp : \mathscr{O}_Y \longrightarrow (f|_U)_*(\mathscr{O}_X|_U)$$

is defined on open sets by:

$$\begin{aligned}(f|_U)^\sharp_V : \mathscr{O}_Y(V) &\longrightarrow ((f|_U)_*(\mathscr{O}_X|_U))(V) = \mathscr{O}_X|_U(f|_U^{-1}(V)) = \mathscr{O}_X(f^{-1}(V) \cap U)\\ s &\longmapsto \theta^{f^{-1}(V)}_{f^{-1}(V)\cap U} \circ f^\sharp_V(s)\end{aligned}$$

where here $\theta^{f^{-1}(V)}_{f^{-1}(V)\cap U}$ is the restriction map on $\mathscr{O}_X$. We see that this commutes with restriction maps as:

$$\begin{aligned}(f|_U)^\sharp_W(\theta^V_W(s)) =&\theta^{f^{-1}(W)}_{f^{-1}(W)\cap U}\circ f^\sharp_W(\theta^V_W(s))\\ =&\theta^{f^{-1}(W)}_{f^{-1}(W)\cap U}\circ \theta^{f^{-1}(V)}_{f^{-1}(W)}\circ f^\sharp_V(s)\\ =&\theta^{f^{-1}(V)}_{f^{-1}(W)\cap U}\circ f^\sharp_V(s)\\ =&\theta^{f^{-1}(V)\cap U}_{f^{-1}(W)\cap U}\circ \theta^{f^{-1}(V)}_{f^{-1}(V)\cap U}\circ f^\sharp_V(s)\\ =&\theta^{f|_U^{-1}(V)}_{f|_U^{-1}(W)}\circ (f|_U)^\sharp_V(s)\\ =&\theta^V_W\circ (f|_U)^\sharp_V(s)\end{aligned}$$

as desired. We can also look at the image restricted analogue, $\tilde{f} : X \to V$, where V is any open set containing $\operatorname{im} f$, and the structure sheaf is $\mathscr{O}_Y|_V$. In this case $\tilde{f}$ is the same as the original topological map, and $\tilde{f}^\sharp$ satisfies:

$$\begin{aligned}\tilde{f}^\sharp_W : \mathscr{O}_Y|_V(W) = \mathscr{O}_Y(W) &\longrightarrow (f_*\mathscr{O}_X(W)) = \mathscr{O}_X(f^{-1}(W))\\ s &\longmapsto f^\sharp_W(s)\end{aligned}$$

We thus have the following definition:

Definition 1.3.7. Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be locally ringed spaces and f a morphism between them. Then f is an **open embedding** if there exists some open $V \subset Y$ such that $\tilde{f} : X \to V$ is an isomorphism of locally ringed spaces.

Now note that Theorem 1.2.2, Proposition 1.2.11, Corollary 1.2, and Corollary 1.2.4 also carry over immediately to the case of sheaves of rings. We want to be able to glue morphisms of ringed spaces together.

Proposition 1.3.3. *Let $(X, \mathscr{O}_X)$ be a locally ringed space, U_i an open cover of X, and $f_i : U_i \to Y$ morphisms which agree on overlaps, i.e. $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$. Then there exists a morphism $f : X \to Y$, such that $f|_{U_i} = f_i$ for all i*

Proof. First note that we can glue together continuous maps by defining $f : X \to Y$ as follows:

$$f(x) = f_i(x)$$

whenever $x \in U_i$. This is well defined as if $x \in U_i \cap U_j$ then we have that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$. Moreover, it is continuous as each f_i is continuous, and the arbitrary union of open sets is open. It is easy to see that $f|_{U_i} = f_i$ for all i.

For each i we have a morphism:

$$f_i^\sharp : \mathscr{O}_Y \longrightarrow f_{i*}(\mathscr{O}_X|_{U_i})$$

Now note that for any $V \subset Y$ we have that:

$$(f_{i*}(\mathscr{O}_X|_{U_i}))(V) = \mathscr{O}_X|_{U_i}(f_i^{-1}(V)) = \mathscr{O}_X(f_i^{-1}(V))$$

We thus define $f^\sharp$ on open sets as:

$$f^\sharp_V(s) = t$$

where t is the unique element in $\mathscr{O}_X(f^{-1}(V))$ such that:

$$t|_{f_i^{-1}(V)} = (f_i^\sharp)_V(s)$$

for all i. First note that:

$$f^{-1}(V) = \bigcup_i U_i \cap f^{-1}(V) = \bigcup_i f|_{U_i}^{-1}(V) = \bigcup_i f_i^{-1}(V)$$

so we need only show that forall i and j:

$$(f_i^\sharp)_V(s)|_{f_i^{-1}(V)\cap f_j^{-1}(V)} = (f_j^\sharp)_V(s)|_{f_i^{-1}(V)\cap f_j^{-1}(V)}$$

However note that by our hypothesis:

$$\begin{aligned}
(f_i^\sharp)_V(s)|_{f_i^{-1}(V)\cap f_j^{-1}(V)} =&\theta^{f_i^{-1}(V)}_{f_i^{-1}(V)\cap f_j^{-1}(V)} \circ (f_i^\sharp)_V(s) \\
=&(f_i^\sharp)_{f_i^{-1}(V)\cap f_j^{-1}(V)}(s|_{f_i^{-1}(V)\cap f_j^{-1}(V)}) \\
=&(f_i^\sharp|_{U_i\cap U_j})_{f_i^{-1}(V)\cap f_j^{-1}(V)}(s|_{f_i^{-1}(V)\cap f_j^{-1}(V)}) \\
=&(f_j^\sharp|_{U_i\cap U_j})_{f_i^{-1}(V)\cap f_j^{-1}(V)}(s|_{f_i^{-1}(V)\cap f_j^{-1}(V)}) \\
=&(f_j^\sharp)_{f_i^{-1}(V)\cap f_j^{-1}(V)}(s|_{f_i^{-1}(V)\cap f_j^{-1}(V)}) \\
=&(f_j^\sharp)_V(s)|_{f_i^{-1}(V)\cap f_j^{-1}(V)}
\end{aligned}$$

hence by sheaf axiom 2 we have that t exists, so $f_V^\sharp$ is well defined. It is clear that this defines a sheaf morphism, so we need only check that $f^\sharp|_{U_i}$ is equal to f_i. We have that the restriction is a morphism:

$$f^\sharp|_{U_i} : \mathscr{O}_Y \longrightarrow (f|_{U_i})_*(\mathscr{O}_X|_{U_i})$$

though $f|_{U_i}$ is equal to f_i hence we have that the restriction is actually a morphism:

$$f^\sharp|_{U_i} : \mathscr{O}_Y \longrightarrow (f_i)_*(\mathscr{O}_X|_{U_i})$$

On an open set $V \subset Y$, we have that:

$$(f_i)_*(\mathscr{O}_X|_{U_i})(V) = \mathscr{O}_X(f_i^{-1}(V)) = \mathscr{O}_X(f^{-1}(V)\cap U)$$

so we have that for $s \in \mathscr{O}_Y(V)$:

$$\begin{aligned}
(f^\sharp|_{U_i})_V(s) =&\theta^{f^{-1}(V)}_{f^{-1}(V)\cap U} \circ f_V^\sharp(s) \\
=&\theta^{f^{-1}(V)}_{f_i^{-1}(V)}(t) \\
=&t|_{f_i^{-1}(V)} \\
=&(f_i^\sharp)_V(s)
\end{aligned}$$

It follows that since f_i is a morphism of locally ringed spaces, and the stalk maps are inherently local, that $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ must be a morphism of local rings, i.e. $f_x(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$, so f is a morphism of locally ringed spaces as desired.

□

Recall our discussion regarding the composition of morphisms of locally ringed spaces; let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on a topological space X, and let $F : \mathscr{F} \to \mathscr{G}$ be a morphism between. Let $f : X \to Y$ be a continuous map, then further recall that $f_*\mathscr{F}$ and $f_*\mathscr{G}$ are sheaves on Y, and we have a morphism between them defined on open sets by $V \subset U$:

$$\begin{aligned}
(f_*F)_V : (f_*\mathscr{F})(V) &\longrightarrow (f_*\mathscr{G})(V) \\
s &\longmapsto F_{f^{-1}(V)}(s)
\end{aligned}$$

It follows easily that this then defines a covariant functor from the category of sheaves on X to the category of sheaves on Y, which we denote $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ respectively.

Definition 1.3.8. Let F be a covariant functor from the category $\mathcal{C}$ to the category $\mathcal{D}$. Then a is covariant functor $G : \mathcal{D} \to \mathcal{C}$, **left adjoint to** F if for all objects $C \in \mathcal{C}$, and all objects $D \in \mathcal{D}$ there exists a natural isomorphism:

$$\mathrm{Hom}_{\mathcal{C}}(G(D), C) \cong \mathrm{Hom}_{\mathcal{D}}(D, F(C))$$

We want to find a way to take sheaf on Y and 'pull it back' to X given a topological map $f : X \to Y$. In light of Definition 1.3.8, the following is probably the most natural construction:

Definition 1.3.9. Let $f : X \to Y$ be a topological map, then the **inverse image functor** from $\mathrm{Sh}(Y)$ to $\mathrm{Sh}(X)$, denoted f^{-1}, is the left adjoint of the direct image functor f_*

While Definition 1.3.9 is elegant enough, we must show that such a functor exists. In particular, we need to *a*) define a sheaf $f^{-1}(\mathscr{F})$ for every sheaf $\mathscr{F}$ on Y, *b*) define a morphism $(f^{-1}F) \in \mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}(\mathscr{F}), f^{-1}(\mathscr{G}))$ for every morphism $F \in \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathscr{F}, \mathscr{G})$, and *c*) show that for every sheaf $\mathscr{G}$ on X, and every sheaf $\mathscr{F}$ on Y there exists a natural isomorphism:

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}(\mathscr{F}), \mathscr{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{F}, f_*(\mathscr{G}))$$

We will prove these statements separately with the following series of results.

Proposition 1.3.4. *Let $f : X \to Y$ be continuous map, and let $\mathscr{F}$ be a sheaf on Y. Then there exists an induced sheaf $f^{-1}(\mathscr{F})$ on X such that for all $x \in X$, $(f^{-1}\mathscr{F})_x$ is uniquely isomorphic to $\mathscr{F}_{f(x)}$.*

Proof. For every open set $U \subset X$, define $f_p^{-1}(\mathscr{F})(U)$ to be:

$$(f_p^{-1}\mathscr{F})(U) = \varinjlim_{V \supset f(U)} \mathscr{F}(V)$$

That is, let I be the partially ordered set:

$$I = \{V \text{ is open in } Y : f(U) \subset V\}$$

where $V_i < V_j$ if $V_j \subset V_i$. Then $f(_p^{-1}\mathscr{F})(U)$ is the unique set/group/ring, equipped with morphisms $\psi_i : \mathscr{F}(V_i) \to (f_p^{-1}\mathscr{F})(U)$ satisfying $\psi_j \circ \theta^{V_i}_{V_j} = \psi_i$, such that for another set/group/ring A, equipped with morphisms $\phi_i : \mathscr{F}(V_i) \to A$ which satisfy the same property, then there exists a unique morphism $\phi : (f_p^{-1}\mathscr{F})(U) \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}(V_i) \arrow[rr, "\theta^{V_i}_{V_j}"] \arrow[dr, "\psi_{V_i}"] \arrow[ddr, "\phi_{V_i}"'] & & \mathscr{F}(V_j) \arrow[dl, "\psi_{V_j}"'] \arrow[ddl, "\phi_{V_j}"] \\
& (f_p^{-1}\mathscr{F})(U) \arrow[d, "\exists!\phi"] & \\
& A &
\end{tikzcd}$$

The same argument as in Proposition 1.2.1 demonstrates that $(f_p^{-1}\mathscr{F})(U)$ must be given by:

$$F = \{(V, s) : V \in I, s \in \mathscr{F}(V)\}$$

modulo the equivalence relation $(V_i, s) \sim (V_j, t)$ if and only there exists a $(W_{ij} \in I) \subset V_i \cap V_j$ such that:

$$s|_W = t|_W$$

We check that this defines a presheaf; suppose that $U_j \subset U_i \subset X$, and let $[V, s]_i \in (f_p^{-1}\mathscr{F})(U_i)$. This implies that $f(U_i) \subset V$, and hence $f(U_j) \subset V$. We thus define $\theta^{U_i}_{U_j}$ by:

$$\theta^{U_i}_{U_j}([V, s]_i) = \psi^j_V(s)$$

where ψ^j_V is the map $\mathscr{F}(V) \to (f_p^{-1}\mathscr{F})(U_j)$. It is clear that $\theta^{U_i}_{U_i} = \mathrm{Id}$, hence we check that:

$$\theta^{U_j}_{U_k} \circ \theta^{U_i}_{U_j} = \theta^{U_i}_{U_k}$$

Let $[V, s] \in (f_p^{-1}\mathscr{F})(U_i)$, then we see that:

$$\begin{aligned}\theta_{U_k}^{U_j} \circ \theta_{U_j}^{U_i}([V,s]_i) =& \theta_{U_k}^{U_j}(\psi_V^j(s)) \\ =& \theta_{U_k}^{U_j}([V,s]_j) \\ =& \psi_V^k(s) \\ =& \theta_{U_k}^{U_i}([V,s]_i)\end{aligned}$$

We thus need only show that $\theta_{U_j}^{U_i}$ is well defined, as if so it is clearly a set/group/ring homomorphism. Let $[W,t]_i = [V,s]_i$, then there exists a subset $Z \subset W \cap V$ such that $f(U_i) \subset Z$ and:

$$t|_W = s|_W$$

We see that:

$$\theta_{U_j}^{U_i}([W,t]) = \psi_W(t) = \psi_Z(t|_Z) = \psi_W(s|_Z) = \psi_V(s) = \theta_{U_j}^{U_i}([V,s])$$

so these are indeed restriction making, making the assignment $U \mapsto f_p^{-1}(U)$ a presheaf.

Note that this not necessarily a sheaf; indeed if $X = \{x_1, x_2\}$, $Y = \{y\}$, both equipped with the discrete topology, and $f : X \to Y$ is the continuous map $x_1 \mapsto y$, $x_2 \mapsto y$, then clearly for every non trivial sheaf $\mathscr{F}$ on Y, $f^{-1}\mathscr{F}$ will fail the gluing axiom as:

$$f_p^{-1}\mathscr{F}(X) = f_p^{-1}\mathscr{F}(\{x_1\}) = f_p^{-1}\mathscr{F}(\{x_2\}) = \mathscr{F}(Y)$$

In particular, $f^{-1}\mathscr{F}$ is the constant presheaf, which we have already shown is not a sheaf.

To complete the proof we simply take:

$$f^{-1}\mathscr{F} = (f_p^{-1}\mathscr{F})^\sharp$$

i.e the sheafification of $f_p^{-1}\mathscr{F}$. The stalks of the sheafification are uniquely isomorphic to the stalks of the presheaf, so we need only show that $(f_p^{-1}\mathscr{F})_x$ is uniquely isomorphic to $\mathscr{F}_{f(x)}$. We first describe a map $\phi_V : \mathscr{F}(V) \to (f_p^{-1}\mathscr{F})_x$, for all V containing $f(x)$. Let $s \in \mathscr{F}(V)$, then we first map s to the equivalence class $[V,s] \in (f_p^{-1}\mathscr{F})(U)$ for any U such that $f(U) \subset V$, and then map $[V,s]$ to the equivalence class $[U,[V,s]] \in (f_p^{-1}\mathscr{F})_x$. We check that this is well defined, i.e. independent of our choice of U. If U' is any other open subset such that $f(U') \subset V$, then we need to show that:

$$[U,[V,s]] = [U',[V,s]']$$

Consider the intersection $W = U \cap U'$, then:

$$[V,s]|_{U\cap U'} = \psi_V(s)$$

where ψ_V is the map $\mathscr{F}(V) \to (f_p^{-1}]\mathscr{F})(U_i \cap U_j)$. We also have that:

$$[V,s]'|_{U_i\cap U_j} = \psi_V(s)$$

hence the map is well defined. We see that if $W \subset V$, then for some U open such that $f(U) \subset W$:

$$\phi_W(s|_W) = [U,[W,s|_W]] = [U,[V,s]]$$

hence the maps commute with restriction. It follows that there is a unique map $\phi : \mathscr{F}_{f(x)} \to (f_p^{-1}\mathscr{F})_x$, such that:

$$\phi([V,s]) = \phi_V(s) = [U,[V,s]]$$

where U is any open set such that $f(U) \subset V$. Suppose that $\phi([V,s]) = 0$, then there exists a $W \subset U$ containing x such that:

$$[V,s]|_W = 0 \Rightarrow \psi_V(s) = 0$$

where ψ_V is the map $\mathscr{F}(V) \to (f_p^{-1}\mathscr{F})(W)$. However, this implies that $[V, s] \in (f_p^{-1}\mathscr{F})(W)$ is zero, hence there exists an open subset $Z_x \subset V$ where $f(W) \subset Z_x$ and:

$$s|_{Z_x} = 0$$

It follows that $f(x) \in V$ and $f(x) \in Z_x$, hence we have that:

$$[V, s] = [Z_x, s|_{V_x}] = [Z_x, 0] = 0 \tag{1.3.4}$$

so ϕ is injective. Now let $[U, [V, s]] \in (f^{-1}\mathscr{F})_x$, we want to find a $[Z, t] \in \mathscr{F}_x$ such that $\phi([Z, t]) = [U, [V, s]]$. Note that $[U, [V, s]] \in (f_p^{-1}\mathscr{F})_x$, implies that $x \in U$, and $f(x) \in f(U) \subset V$. Choose the class $[V, s] \in \mathscr{F}_{f(x)}$, then $\phi([V, s]) = [W, [V, s]]$, for any W such that $f(W) \subset V$. Clearly $W = U$ works, hence $\phi([V, s]) = [U, [V, s]]$ so ϕ is surjective and thus an isomorphism as desired. □

Note that if $\mathscr{F}$ is a locally ringed space, then we clearly have that $f^{-1}\mathscr{F}$ is a locally ringed space from the Proposition 1.3.4. We now proceed with the results:

Proposition 1.3.5. *Let $f : X \to Y$ be a continuos map and $\mathscr{F}$ an object in $\mathrm{Sh}(Y)$. Then the assignment $\mathscr{F} \mapsto f^{-1}\mathscr{F}$ defines a covariant functor $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$.*

Proof. Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on Y; we first define a morphism $f_p^{-1}F : f_p^{-1}\mathscr{F} \to f_p^{-1}\mathscr{G}$. Let $U \subset X$ be an open set, then we define $f_p^{-1}F$ on U by:

$$\begin{aligned} (f_p^{-1}F)_U : (f^{-1}\mathscr{F})(U) &\longrightarrow (f^{-1}\mathscr{G})(U) \\ [V, s] &\longmapsto [V, F_V(s)] \end{aligned}$$

We first check this is well defined, let $[W, t] = [V, s]$, then we want to show that:

$$[V, F_V(s)] = [W, F_V(t)]$$

Note that by assumption there exists a $Z \subset W \cap V$ such that:

$$s|_Z = t|_Z$$

and $f(U) \subset Z$. Note that we have:

$$F_V(s)|_Z = F_Z(s|_Z) = F_Z(t|_Z) = F_V(t)|_Z$$

so $[V, F_V(s)] = [W, F_V(t)]$, and the map is well defined. We check that this commutes with restrictions; let $[V, s] \in (f^{-1}\mathscr{F})(U_i)$, and suppose that $U_j \subset U_i$, then:

$$\begin{aligned} (f_p^{-1}F)_{U_i}([V, s]_i)|_{U_j} =& [V, F_v(s)]_i|_{U_j} \\ =& \psi_V^j(F_V(s)) \\ =& [V, F_V(s)]_j \\ =& (f^{-1}F)_{U_j}([V, s]_j) \\ =& (f^{-1}F)_{U_j}(\psi_V^j(s)) \\ =& (f^{-1}F)_{U_j}([V, s]|_{U_j}) \end{aligned}$$

where ψ_V^j is the map $\mathscr{G}(V) \to (f_p^{-1}\mathscr{G})(U_j)$, and the subscripts describe which image set, $f(U_i)$, or $f(U_j)$ we are taking the colimit over. It follows that $f_p^{-1}F$ is a morphism of presheaves. By the universal property of sheafification we have the following diagram:

$$\begin{tikzcd} f_p^{-1}\mathscr{F} \arrow[r, "f_p^{-1}F"] \arrow[d, "\mathrm{sh}"] \arrow[dr, "\mathrm{sh} \circ f_p^{-1}F"] & f_p^{-1}\mathscr{G} \arrow[d, "\mathrm{sh}"] \\ f^{-1}\mathscr{F} \arrow[r, "\exists! f^{-1}F"] & f^{-1}\mathscr{G} \end{tikzcd}$$

so there exists a unique morphism $f^{-1}F : f^{-1}\mathscr{F} \to f^{-1}\mathscr{G}$. It is clear from the universal property that if $f_p^{-1}(F \circ G) = f_p^{-1}F \circ f_p^{-1}G$, and $f_p^{-1}\mathrm{Id} = \mathrm{Id}$, then the same will be true for $f^{-1}(F \circ G)$ and $f^{-1}\mathrm{Id}$. From our definition of the $f_p^{-1}F$ inverse on open sets however, both the statements in the presheaf case are clear, hence they hold in the sheafification case. It follows that f^{-1} is a functor $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$. □

We can now finally prove the main claim:

Theorem 1.3.1. *Let $f : X \to Y$ be a map of topological spaces, then the functor $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is left adjoint to $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, in the sense that for all object $\mathscr{G}$ of $\mathrm{Sh}(X)$, and all objects $\mathscr{F}$ of $\mathrm{Sh}(Y)$ there is a natural isomorphism:*

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{F}, f_*\mathscr{G})$$

Proof. Let $\mathscr{F} \in \mathrm{Sh}(Y)$, and $\mathscr{G} \in \mathrm{Sh}(X)$, and suppose that $F : f^{-1}\mathscr{F} \to \mathscr{G}$ is a sheaf morphism; we want to define a sheaf morphism $\tilde{F} : \mathscr{F} \to f_*\mathscr{G}$. Let V be an open set of Y, then we want $\tilde{F}$ to be a map on open sets:

$$\tilde{F}_V : \mathscr{F}(V) \longrightarrow \mathscr{G}(f^{-1}(V))$$

Note that $f^{-1}(V) \subset X$, and that $f(f^{-1}(V)) \subset V$, hence there exists a map $\psi_V : \mathscr{F}(V) \to f_p^{-1}\mathscr{F}(f^{-1}(V))$, which takes the section $s \in \mathscr{F}(V)$ to the equivalence class $[V, s] \in f_p^{-1}\mathscr{F}(f^{-1}(V))$. We then have the following chain of maps:

$$\mathscr{F}(V) \xrightarrow{\quad \psi_V \quad} (f_p^{-1}\mathscr{F})(f^{-1}(V)) \xrightarrow{\mathrm{sh}_{f^{-1}(V)}} (f^{-1}\mathscr{F})(f^{-1}(V)) \xrightarrow{\quad F_{f^{-1}(V)} \quad} \mathscr{G}(f^{-1}(V))$$

We define $\tilde{F}_V$ as this composition for all $V \subset Y$. We check that this composition is compatible with restriction maps. Let $V_j \subset V_i \subset Y$, then:

$$\tilde{F}_{V_i}(s)|_{V_j} = \theta^{V_i}_{V_j} \circ F_{f^{-1}(V_i)} \circ \mathrm{sh}_{f^{-1}(V_i)} \circ \psi_{V_i}(s)$$

However, recall that the restriction maps on $f_*\mathscr{G}$ are given by $\theta^{V_i}_{V_j} = \theta^{f^{-1}(V_i)}_{f^{-1}(V_j)}$, hence:

$$\begin{aligned}\tilde{F}_{V_i}(s)|_{V_j} &= \theta^{f^{-1}(V_i)}_{f^{-1}(V_j)} \circ F_{f^{-1}(V_i)} \circ \mathrm{sh}_{f^{-1}(V_i)} \circ \psi_{V_i}(s) \\ &= F_{f^{-1}(V_j)} \circ \mathrm{sh}_{f^{-1}(V_j)} \circ \theta^{f^{-1}(V_i)}_{f^{-1}(V_j)} \circ \psi_{V_i}(s)\end{aligned}$$

where the final $\theta^{f^{-1}(V_i)}_{f^{-1}(V_j)}$ is the restriction map $(f_p^{-1}\mathscr{F})(f^{-1}(V_i)) \to (f_p^{-1}\mathscr{F})(f^{-1}(V_j))$. We see that:

$$\psi_{V_i}(s) = [V_i, s]_i \in (f_p^{-1}\mathscr{F})(f^{-1}(V_i))$$

then:

$$\theta^{f^{-1}(V_i)}_{f^{-1}(V_j)}([V_i, s]_i) = [V_i, s]_j \in (f_p^{-1}\mathscr{F})(f^{-1}(V_j))$$

However, note that we clearly have that $f(f^{-1}(V_j)) \subset V_j$, so:

$$\theta^{f^{-1}(V_i)}_{f^{-1}(V_j)}([V_i, s]_i) = [V_j, s|_{V_j}]_j = \psi_{V_j}(s|_{V_j})$$

where ψ_{V_j} is the map $\mathscr{F}(V_j) \to (f_p^{-1}\mathscr{F})(f^{-1}(V_j))$, hence:

$$\begin{aligned}\tilde{F}_{V_i}(s)|_{V_j} &= F_{f^{-1}(V_j)} \circ \mathrm{sh}_{f^{-1}(V_j)} \circ \psi_{V_j}(s|_{V_j}) \\ &= \tilde{F}_{V_j}(s|_{V_j})\end{aligned}$$

We have thus obtained a set/group/ring homomorphism:

$$\begin{aligned}\Phi : \mathrm{Hom}(f^{-1}\mathscr{F}, \mathscr{G}) &\longrightarrow \mathrm{Hom}(\mathscr{F}, f_*\mathscr{G}) \\ F &\longmapsto \tilde{F}\end{aligned}$$

We will define a set/group/ring homomorphism in the other direction and show that they are inverses of one another. Let $G : \mathscr{F} \to f_*\mathscr{G}$ be a morphism, then we wish to find a $\hat{G} : f^{-1}\mathscr{F} \to \mathscr{G}$. By the universal property of sheafification, it suffices to define a map $\hat{G}_p : f_p^{-1}\mathscr{F} \to \mathscr{G}$. Let $U \subset X$ be open, then on open sets we want $\hat{G}_p$ to be a map:

$$(f_p^{-1}\mathscr{F})(U) \longrightarrow \mathscr{G}(U)$$

For all V_i such that $f(U) \subset V_i$, it thus suffices to define maps $\xi_{V_i} : \mathscr{F}(V_i) \to \mathscr{G}(U)$ which commute with restriction by the universal property of the colimit. Let $s \in \mathscr{F}(V_i)$, and note that we have a map $\mathscr{F}(V_i) \to \mathscr{G}(f^{-1}(V_i))$. Note that $U \subset f^{-1}(f(U)) \subset f^{-1}(V_i)$, hence $U \subset f^{-1}(V_i)$. We thus define ξ_{V_i} by:

$$\xi_{V_i}(s) = \theta_U^{f^{-1}(V_i)} \circ F_{V_i}(s)$$

Suppose that $V_j \subset V_i$, and $f(U) \subset V_j$, then we see that:

$$\begin{aligned}\xi_{V_j}(s|_{V_j}) =&\theta_U^{f^{-1}(V_j)} \circ F_{V_j}(s|_{V_j})\\ =&\theta_U^{f^{-1}(V_j)} \circ \theta_{f^{-1}(V_j)}^{f^{-1}(V_i)} \circ F_{V_i}(s_i)\\ =&\theta_U^{f^{-1}(V_i)} \circ F_{V_i}(s)\\ =&\xi_{V_i}(s)\end{aligned}$$

We thus obtain a unique a map $(f_p^{-1}\mathscr{F})(U) \to \mathscr{G}(U)$ given by:

$$(\hat{G}_p)_U([V,s]) = \xi_V(s)$$

We check that this is actually a presheaf morphism. Let $U_j \subset U_i$, and suppose $[V,s]_i \in (f_p^{-1}\mathscr{F})(U_i)$. Then we have that:

$$(\hat{G}_p)_{U_i}([V,s]_i)|_{U_j} = \theta_{U_j}^{U_i} \circ \xi_V^i(s)$$

where ξ_V^i is the map $\mathscr{F}(V) \to \mathscr{G}(U_i)$. It follows that:

$$\begin{aligned}(\hat{G}_p)_{U_i}([V,s]_i)|_{U_j} =&\theta_{U_j}^{U_i} \circ \theta_{U_i}^{f^{-1}(V)} \circ F_V(s)\\ =&\theta_{U_j}^{f^{-1}(V)} \circ F_V(s)\\ =&\xi_V^j(s)\\ =&(\hat{G}_p)_{U_j}([V,s]_j)\\ =&(\hat{G}_p)_{U_j}([V,s]_i|_{U_j})\end{aligned}$$

so $\hat{G}_p$ is presheaf morphism, and it follows that there exists a unique morphism $\hat{G} : f^{-1}\mathscr{F} \to \mathscr{G}$. We now define the set/group/ring homomorphism:

$$\begin{aligned}\Psi : \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{F}, f_*\mathscr{G}) &\longrightarrow \mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{F}, \mathscr{G})\\ G &\longmapsto \hat{G}\end{aligned}$$

Let $F \in \mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{F}, \mathscr{G})$, then we want to show that $\Psi \circ \Phi(F) = \hat{\tilde{F}} = F$. It suffices to check that the two agree on arbitrary open set. Let $U \subset X$ be open, and take $(s_x) \in (f^{-1}\mathscr{F})(U)$, where (s_x) is a sequence of stalks such that for all x there exists an open neighborhood of x, U_x, and a section $[V_x, f^x] \in (f_p^{-1}\mathscr{F})(U_x)$ such that for all $y \in U_x$, we have:

$$[V_x, f^x]_y = [U_x, [V_x, f^x]] = s_y$$

Now, $\hat{\tilde{F}}_U((s_x))$ is the unique section in $\mathscr{G}(U)$ such that:

$$\hat{\tilde{F}}_U((s_x))|_{U_x} = (\hat{\tilde{F}}_p)_{U_x}([V_x, f^x])$$

We see that by our previous work:

$$\begin{aligned}(\hat{\tilde{F}}_p)_{U_x}([V_x, f^x]) =&\theta_{U_x}^{f^{-1}(V_x)} \circ \tilde{F}_{V_x}(f^x)\\ =&\theta_{U_x}^{f^{-1}(V_x)} \circ F_{f^{-1}(V_x)} \circ \mathrm{sh}_{f^{-1}(V_x)} \circ \psi_{V_x}(f^x)\\ =&F_{U_x} \circ \mathrm{sh}_{U_x} \circ \theta_{U_x}^{f^{-1}(V_x)} \circ \psi_{V_x}(f^x)\end{aligned}$$

Note that $\psi_{V_x}(f^x) = [V_x, f^x] \in (f_p^{-1}\mathscr{F})(f^{-1}(V_x))$, so the restriction to U_x, is equal to $[V_x, f^x] \in (f_p^{-1}\mathscr{F})(U_x)$, we thus have that:

$$\begin{aligned}\hat{\tilde{F}}_U((s_x))|_{U_x} =& F_{U_x} \circ \mathrm{sh}_{U_x}([V_x, f^x]) \\ =& F_{U_x}((s_y)_{y\in U_x}) \\ =& F_{U_x}((s_x)|_{U_x}) \\ =& F_U((s_x))|_{U_x}\end{aligned}$$

Since $\{U_x\}$ is an open cover for U, and we have that:

$$(\hat{\tilde{F}}_U((s_x)) - F_U((s_x)))|_{U_x} = 0$$

for all x, it follows by sheaf axiom one that the two are equal on U, hence:

$$\Psi \circ \Phi = \mathrm{Id}$$

To show the other direction, let $G \in \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{F}, f_*\mathscr{G})$, then we want to show that $\Psi\circ\Phi(G) = \tilde{\hat{G}} = G$. As before it suffices to prove this on open sets. Let $V \subset Y$ be open, and take $s \in \mathscr{F}(V)$, then we that:

$$\tilde{\hat{G}}_V(s) = \hat{G}_{f^{-1}} \circ \mathrm{sh}_{f^{-1}(V)} \circ \psi_V(s)$$

Now note that $\hat{G}_{f^{-1}} \circ \mathrm{sh}_{f^{-1}}(V) = (\hat{G}_p)_{f^{-1}(V)}$, hence:

$$\tilde{G}_V(s) = (\hat{G}_p)_{f^{-1}(V)}([V, s])$$

where $[V, s] \in (f_p^{-1}\mathscr{F})(f^{-1}(V))$. Then by our work defining the map Φ, we have that:

$$\tilde{\hat{G}}_V(s) = \xi_V(s) = \theta^{f^{-1}(V)}_{f^{-1}(V)} \circ G_V(s) = G_V(s)$$

implying that $\tilde{\hat{G}} = G$, and that:

$$\Phi \circ \Psi = \mathrm{Id}$$

It follows that:

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{F}, f_*\mathscr{G})$$

as desired. □

We end this section with the following corollaries:

Corollary 1.3.2. *If $U \subset X$ is open, and $\iota : U \to X$ the inclusion map, then for every sheaf $\mathscr{F}$ on X, we have that $\iota^{-1}\mathscr{F}$ is naturally isomorphic to $\mathscr{F}|_U$.*

Proof. Note that $\iota : U \to X$ is a homeomorphism onto it's image, and it's image is open in X, hence ι is an open map. Let $W \subset U$ be open, then we claim that every element in $(\iota_p^{-1}\mathscr{F})(W)$ can be written as the equivalence class $[W, s]$ for some $s \in \mathscr{F}|_U(W) = \mathscr{F}(W)$. Let $[V, t] \in (\iota_p^{-1}\mathscr{F})(W)$, then $\iota(W) = W \subset V$, hence we have that:

$$[V, t] = [W, t|_W]$$

so without loss of generality we can work with equivalence classes of the form $[W, s]$. We now define a map:

$$\begin{aligned}\phi^p : (\iota_p^{-1}\mathscr{F})(W) &\longrightarrow \mathscr{F}|_U(W) \\ [W, s] &\longmapsto s\end{aligned}$$

Note that this well defined as if $[W, s] = [W, t]$, then there exists some $V \subset W$ such that $\iota(W) = W \subset V$ such that $t|_V = s|_V$. However $V \subset W$ and $W \subset V$ implies that $W = V$, hence $s = t$. This induces a map on stalks given by:

$$\begin{aligned}\phi^p_x : (\iota_p^{-1}\mathscr{F})_x &\longrightarrow (\mathscr{F}|_U)_x \\ [V_x, [V_x, s]] &\longmapsto [V_x, s]\end{aligned}$$

where V_x is some open neighborhood of x. Suppose that $[V_x, [V_x, s]] \mapsto 0$, then this implies that:

$$s|_{Z_x} = 0$$

where Z_x is some open neighborhood of x such that $Z_x \subset V_x$. We want to show that $[V_x, [V_x, s]] = 0$; we note that:

$$[V_x, s]|_{Z_x} = [V_x, s] \in (f_p^{-1}\mathscr{F})(Z_x)$$

which is equal to $[Z_x, s|_{Z_x}] = [Z_x, 0]$ which is the zero section. Hence $[V_x, [V_x, s]] = 0$. The map is clearly surejctive, hence ϕ_x^p is an isomorphism for all x. It follows that induced map on sheaves induces a stalk isomorphism for all x, thus by Lemma 1.2.1 we have the claim. □

Corollary 1.3.3. *A morphism $f : X \to Y$ of locally ringed spaces is equivalent to the data of a continuous map $f : X \to Y$, and a morphism of sheaves $\hat{f} : f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. In particular, there exists natural stalk maps $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ which agree with the direct image counter part, and vice versa.*

Proof. The first statement follows from Theorem 1.3.1. Note that we have map $\hat{f}_x : (f^{-1}\mathscr{O}_Y)_x \to (\mathscr{O}_X)_x$, and moreover that there exists an isomorphism $(f_p^{-1})_x : (\mathscr{O}_Y)_{f(x)} \to (f_p^{-1}\mathscr{O}_Y)_x$, given by:

$$(f_p^{-1})_x([V, s]_{f(x)}) = [U, [V, s]]_x$$

where $[V, s]$ is the equivalence class defined in Proposition 1.3.4, and U is any open set of X such that $f(U) \subset V$. We define the map f_x by:

$$f_x = \hat{f}_x \circ \mathrm{sh}_x \circ (f_p^{-1})_x$$

and note that if $[V, s]_{f(x)} \in (\mathscr{O}_Y)_{f(x)}$, then:

$$f_x([U, s]_{f(x)}) = \hat{f}_x \circ \mathrm{sh}_x([U, [V, s]]_x)$$

Now let $f^\sharp$ be the map induced by by $\hat{f}$ under the isomorphism Φ. It follows that:

$$(f_*)_x \circ f^\sharp_{f(x)}([V, s]_{f(x)}) = [f^{-1}(V), f^\sharp_V(s)]_x$$

However, $f^\sharp_V$ is given by:

$$f^\sharp_V(s) = \hat{f}_{f^{-1}(V)} \circ \mathrm{sh}_{f^{-1}(V)} \circ \psi_V(s)$$

We note that ψ_V is the map $\mathscr{O}_Y(V) \to f_p^{-1}\mathscr{O}_Y(f^{-1}(V))$, given by $s \mapsto [V, s]$. It follows that:

$$\begin{aligned}(f_*)_x \circ f^\sharp_{f(x)}([V, s]_{f(x)}) &= [f^{-1}(V), \hat{f}_{f^{-1}(V)} \circ \mathrm{sh}_{f^{-1}(V)}([f^{-1}(V), [V, s]]_x)] \\ &= \hat{f}_x \circ \mathrm{sh}_x([f^{-1}(V), [V, s]]_x)\end{aligned}$$

We note that $f^{-1}(V)$ is an open set of X such that $f(f^{-1}(V)) \subset V$, so since $(f_p^{-1})_x$ is independent of the choice of U, we can choose $U = f^{-1}(V)$, implying that:

$$f_x = \hat{f}_x \circ \mathrm{sh}_x \circ (f_x^{-1}) = (f_*)_x \circ f^\sharp_{f(x)}$$

Now suppose that we are given the map $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$, and that $\hat{f} = \Psi(f^\sharp)$. We want to show that:

$$f_x = (f_*)_x \circ f^\sharp_{f(x)} = \hat{f}_x \circ \mathrm{sh}_x \circ (f^{-1})_x$$

Note that:

$$\hat{f}_x \circ \mathrm{sh}_x = (\hat{f} \circ \mathrm{sh})_x = \hat{f}_p$$

where $\hat{f}_p$ is the presheaf morphism $f_p^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ given on open sets $U \subset X$:

$$(\hat{f}_p)_U([V, s]) = \theta_U^{f^{-1}(V)} \circ f^\sharp_V(s)$$

Let $[V, s]_{f(x)} \in (\mathscr{O}_Y)_{f(x)}$, then we have that:

$$f_x([V, s]_{f(x)}) = [f^{-1}(V), f^\sharp_V(s)]_x$$

while:

$$\begin{aligned}(\hat{f}_p)_x \circ (f^{-1})_x([V, s]_{f(x)}) =& (\hat{f}_p)_x([f^{-1}(V), [V, s]]_x)\\ =& [f^{-1}(V), \theta^{f^{-1}(V)}_{f^{-1}(V)} \circ f^\sharp_V(s)]_x\\ =& [f^{-1}(V), f^\sharp_V(s)]\end{aligned}$$

implying the claim. □

We can now prove the final statement in Lemma 1.3.1.

Proof. Let $f, g : X \to Y$ be morphisms of (locally) ringed spaces, such that $f = g$, and $f_x = g_x$ for all $x \in X$, then we have that $f^{-1}\mathscr{O}_Y = g^{-1}\mathscr{O}_X$, that $f_p^{-1}\mathscr{O}_Y = g^{-1}\mathscr{O}_X$, and that $(f^{-1})_x = (g^{-1})_x$. It follows by the above proposition that since $f_x = g_x$:

$$\hat{f}_x \circ \mathrm{sh}_x \circ (f^{-1})_x = \hat{g}_x \circ \mathrm{sh}_x \circ (g^{-1})_x$$

where $\hat{f}$ and $\hat{g}$ are the images of $f^\sharp$ and $g^\sharp$ under the isomorphism Ψ. Note that $\mathrm{sh}_x \circ (f^{-1})_x$ is an isomorphism, hence we can apply the inverse map to both sides on the right and obtain that for all $x \in X$:

$$\hat{f}_x = \hat{g}_x$$

for all $x \in X$. It follows that $\hat{f} = \hat{g}$ as maps $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$, so under the isomorphism Φ we have that $f^\sharp = g^\sharp$, as desired. □

We end our section on locally ringed spaces with the following corollary of Theorem 1.3.1:

Corollary 1.3.4. *Let $f : X \to Y$ be a morphism of topological spaces with sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y. Then there are canonical morphisms:*

$$G : \mathscr{G} \longrightarrow f_* f^{-1}\mathscr{G} \qquad \text{and} \qquad F : \mathscr{F} \to f^{-1} f_* \mathscr{F}$$

If f is a closed immersion (in the topological sense) then G is surjective. If f is an open immersion (in the topological sense) then F is an isomorphism.

Proof. Note that by Theorem 1.3.1 if $\mathscr{F}$ is a sheaf on X and $\mathscr{G}$ is a sheaf on Y, we have a natural isomorphism:

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{G}, f_*\mathscr{G})$$

Let $\mathscr{F} = f^{-1}\mathscr{G}$, then we have that the identity morphism $\mathrm{Id} \subset \mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{G}, f^{-1}\mathscr{G})$ corresponds to a unique morphism $\tilde{\mathrm{Id}} : \mathscr{G} \to f_* f^{-1}\mathscr{G}$. Recall from Theorem 1.3.1 that this map is given on open sets $V \subset Y$ by:

$$\mathscr{G}(V) \xrightarrow{\psi_V} (f_p^{-1}\mathscr{G})(f^{-1}(V)) \xrightarrow{\mathrm{sh}_{f^{-1}(V)}} (f^{-1}\mathscr{G})(f^{-1}(V)) \xrightarrow{\mathrm{Id}_{f^{-1}(V)}} (f^{-1}\mathscr{G})(f^{-1}(V))$$

where ψ_V takes a section $s \in \mathscr{G}(V)$ to $[V, s] \in f_p^{-1}\mathscr{G}(f^{-1}(V))$. Now suppose that f is a closed immersion, and let $y \in Y$, if $y \notin f(X)$ then stalk of $(f_* f^{-1}\mathscr{G})_y$ is automatically trivial so $\tilde{\mathrm{Id}}_y$ must be surjective. Now suppose that $y = f(x)$ for some $x \in X$, and $[V, s]_{f(x)} \in (f_* f^{-1}\mathscr{G})_{f(x)}$ where $s \in (f^{-1}\mathscr{G})(f^{-1}(V))$. Since $s \in (f^{-1}\mathscr{G})(f^{-1}(V))$, we have that:

$$s = (t_p) \in \prod_{p \in f^{-1}(V)} (f_p^{-1}\mathscr{G})_x$$

where for each $p \in f^{-1}(V)$ there is a $U_p \subset f^{-1}(V)$ and a section $h \in f_p^{-1}\mathscr{G}(U_p)$ such that $h_q = t_q$ for all $q \in U_p$. Let $p = x$ such that $f(x) = y$ as above, then there exists an open subset $U_x \subset f^{-1}(V)$

and a section $h \in (f_p^{-1}\mathscr{G})(U_x)$ such that $h_x = t_x$. In particular, we have that the isomorphism $\mathrm{sh}_{f(x)} : (f_p^{-1}\mathscr{G})_x \to (f^{-1}\mathscr{G})_x$ sends h_x to s_x. For some $Z \subset Y$ such that $f(U_x) \subset Z$, and some $g \in \mathscr{G}(Z)$, we have that $h = [Z, g]_{U_x}$. Note that there is a smallest subset Z such that $f(U_x) = Z \cap f(X)$, so without loss of generality we can assume that $f(U_x) = Z \cap f(X)$, and that $f^{-1}(Z) = U_x$. We claim that $g \in \mathscr{G}(Z)$ satisfies $\tilde{\mathrm{Id}}_{f(x)}(g_{f(x)}) = [V, s]_{f(x)} \in (f_* f^{-1}\mathscr{G})_{f(x)}$. Indeed, we write that $g_{f(x)} = [Z, g]_{f(x)}$ then:

$$\mathrm{Id}_{f(x)}(g_{f(x)}) = [Z, \tilde{\mathrm{Id}}_Z(g)]_{f(x)}$$

we have that:

$$\begin{aligned}
\mathrm{Id}_Z(g) =& \mathrm{Id}_{f^{-1}(Z)} \circ \mathrm{sh}_{f^{-1}(Z)} \circ \psi_Z(g) \\
=& \,\mathrm{sh}_{f^{-1}(Z)}([Z, g]_{U_x}) \\
=& \,([Z, g]_{U_x, p}) \\
=& (h_p) \\
=& (t_p)
\end{aligned}$$

where $(t_p) = s|_{U_x}$, but $U_x = f^{-1}(Z)$, so we have that:

$$[V, s]_{f(x)} = [Z, s|_{U_x}]_{f(x)} = \mathrm{Id}_{f(x)}(g_{f(x)})$$

It follows that $\tilde{\mathrm{Id}}_{f(x)}$ is surjective implying the first claim.

Now, suppose that f is an open embedding, and note that we again have an identity morphism of sheaves on Y given by:

$$\mathrm{Id} : f_*\mathscr{F} \to f_*\mathscr{F}$$

which induces a unique morphism $\hat{\mathrm{Id}} : f^{-1}f_*\mathscr{F} \to \mathscr{F}$. This map is the one induced by the sheafification of the map $\hat{\mathrm{Id}}_p : f_p^{-1}(f_*\mathscr{F}) \to \mathscr{F}$ given on open subsets of $U \subset X$ by:

$$(\hat{\mathrm{Id}}_p)_U([V, s]) = \xi_V(s) = \theta_U^{f^{-1}(V)} \circ \mathrm{Id}_V(s)$$

Note that if $[V, s] \in f_p^{-1}(f_*\mathscr{F})(U)$, then we have that $s \in f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$, and $f(U) \subset V$. We first claim that $[V, s] = [f(U), s|_{f(U)}] \in f_p^{-1}(f_*\mathscr{F})(U)$. Note that $f(U)$ is open, and that $f(U) \subset f(U) \cap V$, so essentially by definition we have that $[V, s] = [f(U), s|_{f(U)}]$. It follows for any $[V, s] \in f_p^{-1}(f_*\mathscr{F})(U)$ we can write $[V, s]$ as $[f(U), s|_{f(U)}]$. Now we see that:

$$(\hat{\mathrm{Id}}_p)_U([f(U), s]) = \theta_U^U \circ \mathrm{Id}_{f(U)}(s) = s \in \mathscr{F}(U) = f_*\mathscr{F}(f(U))$$

This is then trivially an isomorphism, so we have that $f_p^{-1}(f_*\mathscr{F})$ is actually a sheaf, and that $\mathrm{sh} : f_p^{-1}(f_*\mathscr{F}) \to f^{-1}f_*\mathscr{F}$ is an isomorphism. Since $\hat{\mathrm{Id}} \circ \mathrm{sh} = \hat{\mathrm{Id}}_p$, and both sh and $\hat{\mathrm{Id}}_p$ are isomorphisms, we have tat $\hat{\mathrm{Id}}$ is an isomorphism as desired. □

1.4 The Structure Sheaf of Spec

Let A be a commutative ring; in this section we wish to equip the topological space $\operatorname{Spec} A$ with a sheaf of rings such that $\operatorname{Spec} A$ is a locally ringed space. Note that $A_\mathfrak{p}$ is a local ring by Example 1.3.1, so it would make sense to construct a sheaf on $\operatorname{Spec} A$ such that the stalk at $\mathfrak{p} \in \operatorname{Spec} A$ is $A_\mathfrak{p}$ (our choice of notation for stalks and the localization of a ring is intentionally suggestive). We begin with the following definition:

Definition 1.4.1. Let X be a topological space, and $\mathcal{B}$ be a basis for the topology of X. A **presheaf on a base** is the data of a set/group/ring, $\mathcal{F}(U)$ associated to each open set $U \in \mathcal{B}$, and restriction maps $\theta_V^U : \mathcal{F}(U) \to \mathcal{F}(U)$ whenever $U \subset V$, such that $\theta_W^V \circ \theta_V^U = \theta_V^U$. A **sheaf on a base** is a presheaf on a basis satisfying analogues of sheaf axioms one and two from Definition 1.2.1. Explicitly:

i) Let $\{U_i\} \subset \mathcal{B}$ be an open cover for $U \in \mathcal{B}$, then if $s, t \in \mathcal{F}(U)$ such that $s|_{U_i} = t|_{U_i}$ for all i then $s = t$.

ii) Let $\{U_i\} \subset \mathcal{B}$ be an open cover for $U \in \mathcal{B}$, and $s_i \in \mathcal{F}(U_i)$ sections such that for all basic opens $U_{ij} \subset U_i \cap U_j$:

$$s_i|_{U_{ij}} = s_j|_{U_{ij}}$$

for all i and j, then there exists an $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

We now show that sheaves on a base induce a sheaves on the total space which are unique up to unique isomorphism.

Theorem 1.4.1. *Let X be a topological space, $\mathcal{B}$ a basis for the topology on X, and $\mathcal{F}$ a sheaf on the basis $\mathcal{B}$. Then, there exists a sheaf $\mathscr{F}$ on X induced by $\mathcal{F}$ satisfying the following universal property: for any sheaf $\mathscr{G}$, and any collection of set/group/ring morphisms $\phi_U : \mathcal{F}(U) \to \mathscr{G}(U)$ satisfying $\theta_V^U \circ \phi_U = \phi_V \circ \theta_V^U$ for all $V \subset U \in \mathcal{B}$, there exists a unique sheaf morphism $F : \mathscr{F} \to \mathscr{G}$, such that for all $U \in \mathcal{B}$ the following diagram commutes:*

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "F_U"] \arrow[d, "\psi_U"] & \mathscr{G}(U) \\
\mathcal{F}(U) \arrow[ur, "\phi_U"'] &
\end{tikzcd}$$

where ψ_U is an isomorphism.

Proof. For all $x \in X$, we define the stalk $\mathcal{F}_x$ as:

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$$

where we are clearly taking the colimit over $\mathcal{B}$, partially ordered by $U < V$ if $V \subset U$. The stalk is then the set of equivalence classes satisfying the same equivalence relation as the usual case, just restricted to basic sets. For each $W \subset X$ open, we define the set/group/ring by $\mathscr{F}(W)$:

$$\mathscr{F}(W) = \left\{ (s_x) \in \prod_{x \in W} \mathcal{F}_x : \forall y \in W, \exists U \in \mathcal{B}, y \in U, \text{ and } \exists f \in \mathcal{F}(U), \forall x \in U, f_x = s_x \right\}$$

In other words $(s_x) \in \prod_{x \in W} \mathcal{F}_x$ is an element of $\mathscr{F}(W)$ if for each $y \in W$, we can find an a basic open set U containing y, and a section $f \in \mathcal{F}(U)$ such that the sequence $(f_x) \in \prod_{x \in U} \mathcal{F}_x$ agrees with (s_x) on U. The restriction map θ_Z^W is then given by the restriction of the projection:

$$\prod_{x \in W} \mathcal{F}_x \longrightarrow \prod_{x \in Z} \mathcal{F}_x$$

to the sets/groups/rings $\mathscr{F}(W)$. The same argument as in Proposition 1.2.3 demonstrates that the restriction of the projection to $\mathscr{F}(W)$ has image in $\mathscr{F}(Z)$ when $Z \subset W$, and moreover that $\theta_Y^Z \circ \theta_Z^W = \theta_Z^W$. It follows that the assignment $Z \mapsto \mathscr{F}(Z)$ defines a presheaf on X.

We show that $\mathscr{F}$ is a sheaf. Suppose that $\{W_i\}$ is an open cover of $W \subset X$, and that $(s_x) \in \mathscr{F}(W)$ satisfies $(s_x)|_{W_i} = 0$ for all W_i. It follows that:

$$(s_x)|_{W_i} = (s_{x \in W_i}) = 0$$

implying that s_x is zero for all $x \in W_i$. Since $\{W_i\}$ covers W, it follows that for all $x \in W$ we have $s_x = 0$, hence $(s_x) = 0$.

Now suppose that we have $(s_x^i) \in \mathscr{F}(W_i)$ such that that:

$$(s_x^i)|_{W_i \cap W_j} = (s_x^j)|_{W_i \cap W_j}$$

hence we define an element $(s_x) \in \prod_x \mathcal{F}_x$ by:

$$(s_x) = (s_x^i)$$

whenever $x \in W^i$. This is well defined since $s_x^i = s_x^j$ whenever $x \in U_i \cap U_j$, so we need only show that $(s_x) \in \mathscr{F}(W)$, but this is clear. Indeed, for all $x \in W$, there exists an open set W_i such that $s_x = s_x^i$, however since $(s_x^i) \in \mathscr{F}(W_i)$, there exists a basic open U and a section $f \in \mathcal{F}(U)$ such that $f_y = s_y^i = s_y$ for al $y \in U$. We can do this for all x, hence $(s_x) \in \mathscr{F}(W)$, and clearly restricts to (s_x^i) for all i. It follows that $\mathscr{F}$ is indeed a sheaf.

We define $\psi_U : \mathscr{F}(U) \to \mathcal{F}(U)$ as follows: let $(s_x) \in \mathscr{F}(U)$, and let $U \in \mathcal{B}$, and $\{U_x\}$ be any open cover of U by basic opens such that for each $x \in U$ there is an $f^x \in \mathcal{F}(U_x)$ such that $s_y = f_y^x$ for all $y \in U_x$. We set $\psi_U((s_x))$ to be the unique element in $\mathcal{F}(U)$ satisfying $\psi_U((s_x))|_{U_x} = f^x$ for all U_x. Note that if such an element exists, it is independent of the cover and sections chosen. Indeed if $\{V_x\}$ is an other cover with sections e^x, then we denote the corresponding section $\psi_U^e((s_x))$. It follows that since $\psi_U^e((s_x))|_{V_x} = e^x$, we have for all $y \in U$:

$$\psi_U^e((s_x))_y = e_y^x = s_y = f_y^x = \psi_U((s_x))_y$$

Since the sections agree on stalks, we need only show that the natural map $\mathcal{F}(U) \to \prod_{x\in U} \mathcal{F}_x$ is an injection, but this is clear by the same argument in Lemma 1.2.2. We now show such a section exists. Clearly, we need only show that $f^x|_{U_{xy}} = f^y|_{U_{xy}}$ for all $U_{xy} \subset U_x \cap U_y$, but this is vacuously true, as for any such U_{xy} we have that $f_z^x = s_z = f_z^y$ for all $z \in U_{xy} \subset U_x \cap U_y$, so by the preceding remark it must follows that $f^x|_{U_{xy}} = f^y|_{U_{xy}}$.

We now show that ψ_U is an isomorphism. Note that:

$$\psi_U((s_x))_y = s_y$$

hence if $\psi_U((s_x)) = 0$, we have that $\psi_U((s_x))_y = 0$ for all $y \in U$. It follows that $s_y = 0$ for all $y \in U$, hence $(s_x) = 0$. Moreover, if $s \in \mathcal{F}(U)$, then sequence $(s_x) \in \prod_{x\in U} \mathcal{F}_x$ clearly lies in $\mathscr{F}(U)$, and by definition we have that for all $y \in U$:

$$\psi_U((s_x))_y = s_y$$

for all $y \in U$, hence $\psi_U((s_x)) = s$ implying the claim.

Now suppose that $\mathscr{G}$ is a sheaf, equipped with morphisms $\phi_U : \mathcal{F}(U) \to \mathscr{G}(U)$ for all $U \in \mathcal{B}$, such that $\theta_V^U \circ \phi_U = \phi_V \circ \theta_V^U$. We define a morphism $F : \mathscr{F} \to \mathscr{G}$ on generic open sets $W \subset X$ as follows: let $(s_x) \in \mathscr{F}(W)$, then there exists an open cover $\{W_x\}$ of W by basic opens, along with sections $f^x \in \mathcal{F}(W_x)$ such that $f_y^x = s_y$ for all $y \in W_x$. Then we set $F_W((s_x))$ to be the unique section of $\mathscr{G}(W)$ such that $F_W((s_x))|_{W_x} = \phi_{W_x}(f^x)$. If this section exists, then it is well defined by the same argument as in the ψ_U case. We now show such a section exists; we need only $\phi_{W_x}(f^x)|_{W_x\cap W_y} = \phi_{W_y}(f^y)|_{W_x\cap W_y}$. Cover $W_x \cap W_y$ by basic opens V_i, then we know that $V_i \subset W_x$ and $V_i \subset W_y$ for all i. Since f^x and f^y agree on all open subsets of $W_x \cap W_y$, it follows that for all V_i:

$$(\phi_{W_x}(f^x)|_{W_x\cap W_y} - \phi_{W_y}(f^y)|_{W_x\cap W_y})|_{V_i} = 0$$

hence by sheaf axiom one the two agree on $W_x \cap W_y$. Now let U be a basic open, we want to show that:

$$F_U = \phi_U \circ \psi_U$$

Take $(s_x) \in \mathscr{F}(U)$, then there is a unique section $f \in \mathcal{F}(U)$ such that $\psi_U((s_x)) = f$. In particular, $f_x = s_x$ for all $x \in U$. Since our definition of F_U is independent of our cover and choice of sections, choose the trivial cover $\{U\}$ and the section $s \in \mathcal{F}(U)$. Since there is nothing to glue over, it follows that:

$$F_U((s_x)) = \phi_U(f) = \phi_U \circ \psi_U((s_x))$$

implying the claim. □

Corollary 1.4.1. *Let X be a topological space, $\mathcal{B}$ a basis for its topology, and $\mathcal{F}$ a sheaf on $\mathcal{B}$. Then the induced sheaf $\mathscr{F}$ is unique up to unique isomorphism.*

Proof. Suppose that $\mathscr{G}$ is any other sheaf that satisfies the universal property, i.e. $\mathscr{G}$ is a sheaf on X equipped with isomorphisms $\psi_U^{\mathscr{G}} : \mathscr{G}(U) \to F(U)$ for all $U \in \mathcal{B}$, such that for any other sheaf $\mathscr{H}$ with morphisms $\phi_U : \mathcal{F}(U) \to \mathscr{H}(U)$ which commute with restrictions on basic opens, there is a unique morphism $\phi : \mathscr{G} \to \mathscr{H}$. Now note that $\mathscr{F}$ as constructed in Theorem 1.4.1 comes equipped with isomorphisms $\psi_U^{-1} : \mathcal{F}(U) \to \mathscr{F}(U)$ which trivially commute with restrictions on basic opens. It follows that there exists a unique morphism $F : \mathscr{G} \to \mathscr{F}$; we show that this is an isomorphism.

Let $g_x \in \mathscr{G}_x$, and note that g_x can be written as an equivalence class $[U, g]$ where U is a basic open set. Indeed, if $g_x = [V, g']$, then V is the union of basis open sets, hence there must be some basic open set U containing x. It follows that:

$$[V, g'] = [U, g'|_U]$$

as desired. We see that:

$$F_x(g_x) = [U, F_U(g)] = [U, \psi_U^{-1} \circ \psi_U^{\mathscr{G}}(g)]$$

Suppose this equals zero, then there is an open set $V \subset U$ such that $\psi_U^{-1} \circ \psi_U^{\mathscr{G}}(g)|_V$ is zero. Without loss of generality we can take V to be a basic open by our previous remark. It follows that:

$$\psi_V^{-1} \circ \psi_V^{\mathscr{G}}(g|_V) = 0 \Rightarrow g|_V = 0$$

as $\psi_V^{-1} \circ \psi_V^{\mathscr{G}}$ is an isomorphism. However, we have that:

$$[U, g] = [V, g|_V] = 0$$

hence $g_x = 0$. Now let $s_x \in \mathscr{F}_x$, we can represent s_x as an equivalence class $[U, s]$, where U is a basic open. Since $\psi_U^{-1} \circ \psi_U^{\mathscr{G}}$ is an isomorphism, it follows that there exists a unique $g \in \mathscr{G}(U)$, such that $\psi_U^{-1} \circ \psi_U^{\mathscr{G}}(g) = s$. We thus have that F_x is an isomorphism for all x, hence $\mathscr{G}$ is uniquely isomorphic to $\mathscr{F}$ as desired. $\square$

We now prove two results regarding sheafs on a base as a sanity check that things work as assumed. In particular, it should stand to reason that the stalks of a sheaf on a base are isomorphic, and that restricting a sheaf on to a sheaf on a base yields the same sheaf.

Proposition 1.4.1. *Let X be a topological space, $\mathcal{B}$ a basis for its topology, and $\mathcal{F}$ a sheaf on $\mathcal{B}$. Then the induced sheaf $\mathscr{F}$ satisfies $\mathscr{F}_x \cong \mathcal{F}_x$ for all $x \in X$.*

Proof. Note that we have morphisms $\phi_U : \mathscr{F}(U) \to \mathcal{F}_x$ for all $x \in U \subset X$ given by:

$$\phi_U((s_y)) = s_x$$

These maps trivially commute with restriction. It follows by the universal property of the colimit that there exists a unique morphism $\phi_x : \mathscr{F}_x \to \mathcal{F}_x$ given by:

$$\phi_x([U, (s_y)]) = s_x$$

Suppose that $s_x = 0$, then note that since $(s_y) \in \mathscr{F}(U)$, there exists an open neighborhood V_x of x, and a section $f^x \in \mathcal{F}(U)$ such that $f^x_y = s_y$ for all $y \in V_x$. We can thus write $s_x = [V_x, f^x]$, however this is zero, so there exists another open set such that $x \in Z_x \subset V_x$ such that $f^x|_{Z_x} = 0$. Since stalks commute with restriction, it follows that $(f^x|_{Z_x})_y = s_y$ for all $y \in Z_x$. However, this means that $s_y = 0$ for all $y \in Z_x$, hence:

$$(s_y)|_{Z_x} = 0 \Rightarrow [U, (s_y)] = [Z_x, (s_y)|_{Z_X}] = 0$$

so ϕ_x is injective. Moreover, suppose that $s_x = [U, s] \in \mathcal{F}_x$, then we see that $\psi_U^{-1}(s)$ is a sequence (t_y) in $\mathscr{F}(U)$ which satisfies $t_y = s_y$ for all $y \in U$. It follows that:

$$\phi_x([U, \psi_U^{-1}(s)]) = t_x = s_x$$

hence ϕ_x is surjective and thus an isomorphism as desired. $\square$

Proposition 1.4.2. *Let $\mathscr{F}$ be a sheaf on X, and let $\mathcal{B}$ be a basis for the topology on X. Then for all $U \in \mathcal{B}$, the assignment $U \mapsto \mathcal{F}(U) = \mathscr{F}(U)$ defines a sheaf on a base such that the induced sheaf is uniquely isomorphic to $\mathscr{F}$.*

Proof. First note that since $\mathscr{F}$ is a sheaf, sheaf on a base axiom one is trivially fulfilled. Now let U be a basic open, and $\{U_i\}$ an open cover of U with sections $s_i \in \mathcal{F}(U_i)$ such that for all basic opens $U_{ij} \subset U_i \cap U_j$ we have:

$$s_i|_{U_{ij}} = s_j|_{U_{ij}}$$

Note that we can cover $U_i \cap U_j$ by all such U_{ij}^k indexed by k, and that since $s_i \in \mathcal{F}(U_i) = \mathscr{F}(U_i)$, we can restrict each s_i to $U_i \cap U_j$. It suffices to show that:

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

However, we have that for all $U_{ij}^k \subset U_i \cap U_j$:

$$(s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})|_{U_{ij}^k} = 0$$

hence by sheaf axiom one s_i and s_j agree on $U_i \cap U_j$. It follows that since $\mathscr{F}$ is a sheaf there is a unique section $s \in \mathcal{F}(U) = \mathscr{F}(U)$ such that $s|_{U_i} = s_i$.

We show that $\mathscr{F}$ satisfies the universal property in Theorem 1.4.1, and thus by Corollary 1.4.1 is uniquely isomorphic to the induced sheaf. For all $U \in \mathcal{B}$, let $\phi_U : \mathcal{F}(U) \to \mathscr{G}(U)$ be a collection of morphisms which commute with restriction on a basic open sets, and note that $\psi_U : \mathscr{F}(U) \to \mathcal{F}(U)$ is the identity morphism for all U. We thus need to construct a map $F : \mathscr{F} \to \mathscr{G}$ such that $F_U = \phi_U$. Let W be an arbitrary open set, and $\{W_i\}$ an open cover by basic opens. If $s \in \mathscr{F}(W)$, then we define $F_W(s)$ as the unique element in $\mathscr{G}(W)$ such that $F_W(s)|_{W_i} = \phi_{W_i}(s|_{W_i})$.

Suppose such an element exists, and let $\{V_i\}$ be a different open cover of W by basic opens. Consider $V_i \cap W_j$, and let $Z_{ij} \subset V_i \cap W_j$ be any basic open set, then we have that:

$$\phi_{W_i}(s|_{W_i})|_{Z_{ij}} = \phi_{Z_{ij}}(s|_{Z_{ij}}) = \phi_{V_i}(s|_{V_i})|_{Z_{ij}}$$

It follows that since $\mathscr{G}$ is a sheaf, $\phi_{W_i}(s|_{W_i})$ and $\phi_{V_j}(s|_{V_j})$ agree on overlaps $W_i \cap V_j$. If $F_W(s) = g$ is the element such that $g|_{W_i} = \phi_{W_i}(s|_{W_i})$, and $F_W(s) = h$ is the element such that $h|_{V_i} = \phi_{W_i}(s|_{V_i})$, then we have that for all $V_i \cap W_j$:

$$(h - g)|_{V_i \cap W_j} = 0$$

Since all such intersections form a cover for W, and $\mathscr{G}$ is a sheaf it follows that $g = h$, so F_W is independent of the chosen cover.

Now we show that $F_W(s)$ exists. We need only show that $\phi_{W_i}(s|_{W_i})|_{W_i \cap W_j} = \phi_{W_j}(s_j)|_{W_i \cap W_j}$, however for all basic open sets $W_{ij} \subset W_i \cap W_j$ we have that:

$$\phi_{W_i}(s|_{W_i})|_{W_{ij}} = \phi_{W_{ij}}(s|_{W_{ij}}) = \phi_{W_j}(s|_{W_j})|_{W_{ij}}$$

so since $\mathscr{G}$ is a sheaf we must have that $\phi_{W_i}(s|_{W_i})|_{W_i \cap W_j} = \phi_{W_j}(s_j)|_{W_i \cap W_j}$, so $F_W(s)$ exists.

We need to check that F_W commutes with restrictions. Let $Z \subset W$, then we have an open cover of Z given by $\{Z \cap W_i\}$. For each i, we can cover $Z \cap W_i$ by basic opens Z_{ij} such that $Z_{ij} \subset W_i$ for all j. It follows that:

$$F_Z(s|_Z)|_{Z_{ij}} = \phi_{Z_{ij}}((s|_Z)|_{Z_{ij}}) = \phi_{Z_{ij}}(s|_{Z_{ij}}) = \phi_{W_i}(s|_{W_i})|_{Z_{ij}} = (F_W(s)|_{W_i})|_{Z_{ij}} = F_W(s)|_{Z_{ij}} = (F_W(s)|_Z)|_{Z_{ij}}$$

Since the set of all Z_{ij} cover Z, we must have that $F_Z \circ \theta_Z^W = \theta_Z^W \circ F_W$, hence F defines a sheaf morphism. It is then clear that:

$$F_U = \phi_U$$

whenever U is a basic open, implying the claim. $\square$

Corollary 1.4.2. *Let $\mathscr{F}$ and $\mathscr{G}$ be sheafs on X, and $\mathcal{B}$ a basis for the topology on X. Then, any morphism $F : \mathscr{F} \to \mathscr{G}$ is determined by the morphisms $F_U : \mathscr{F}(U) \to \mathscr{G}(U)$ where U is a basic open. In particular, F is an isomorphism if and only if F_U is an isomorphism for all $U \in \mathcal{B}$.*

Proof. By the preceding proposition, we know that $\mathscr{F}$ satisfies the universal property of the sheaf on a base defined by $\mathcal{F} : U \mapsto \mathscr{F}(U)$ for all $U \in \mathcal{B}$, hence if $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is some collection of morphisms which commute with restrictions on basis opens, then we have a unique map $F : \mathscr{F} \to \mathscr{G}$. Now suppose we are given a map $F : \mathscr{F} \to \mathscr{G}$, and define $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ by $\phi_U = F_U$. Since $\psi_U : \mathscr{F}(U) \to \mathcal{F}(U)$ is the identity, it follows that F trivially satisfies the diagram in Theorem 1.4.1, and is thus the unique morphism determined by $\phi_U = F_U$, implying that F is determined by the morphisms F_U as desired.

Now suppose that F is an isomorphism, then clearly for all $U \in \mathcal{B}$ we have that F_U is an isomorphism. Now suppose that for all $U \in \mathcal{B}$ we have that F_U is an isomorphism. If we can show that $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism for all $x \in X$ then we are done. Let $[U, s] \in \mathscr{F}_x$, and suppose that $F_x([U, s]) = [U, F_U(s)] = 0$. Then this implies that $F_U(s)|_V = 0$ for some $V \subset U$. Since V is the union of basic opens, we can further restrict to a basis open W to obtain that $F_U(s)|_W = 0$, which implies that $s|_W = 0$, as F_W is an isomorphism. It follows that $[U, s] = [W, s|_W] = [W, 0] = 0$, so F_x is injective. Now let $[U, t] \in \mathscr{G}_x$, and let $W \subset U$ be any basic open set, then $[U, t] = [W, t|_W]$, and there is a unique element $s \in \mathscr{F}(W)$ such that $F_W(s) = t|_W$. It follows that $[W, s] \in \mathscr{F}_x$ satisfies $[W, F_W(s)] = [W, t|_W] = [U, t]$ hence F_x is surjective and thus an isomorphism, implying the claim. □

Now let A be a commutative ring; consider $\operatorname{Spec} A$ with the Zariski topology, then the set $\mathcal{B} = \{U_f\}_{f \in A}$ of distinguished opens forms a basis for the topology on $\operatorname{Spec} A$ by Lemma 1.1.2. We define a sheaf on $\mathcal{B}$ via the assignment:

$$U_f \longmapsto A_f \tag{1.4.1}$$

where A_f is the localization of A at f. By Lemma 1.1.3 we also have that $U_f = U_g$ if and only if $\sqrt{\langle f \rangle} = \sqrt{\langle g \rangle}$, and by Lemma 1.1.4 we then have that $A_f \cong A_g$, so the assignment is well defined. We need the following lemma:

Lemma 1.4.1. *Let A be a commutative ring, then every open cover of* $\operatorname{Spec} A$ *has a finite subcover. In particular, every distinguished open can be written as the finite union of distinguished opens.*

Proof. Let $\{V_i\}$ be an open cover of A, then we have that:

$$\operatorname{Spec} A = \bigcup_i V_i$$

For each i we have that:

$$V_i = \bigcup_j U_{f_{j_i}}$$

hence:

$$\begin{aligned}
\operatorname{Spec} A &= \bigcup_i \left(\bigcup_{j_i} U_{f_{j_i}} \right) \\
&= \bigcup_{i, j_i} U_{f_{j_i}} \\
&= \bigcup_{i, j_i} \mathbb{V}(\langle f_{j_i} \rangle)^c \\
&= \left(\bigcap_{i, j_i} \mathbb{V}(\langle f_{j_i} \rangle) \right)^c \\
&= \left(\mathbb{V} \left(\sum_{i, j_i} \langle f_{j_i} \rangle \right) \right)^c
\end{aligned}$$

It follows that since $\mathbb{V}(\langle 1 \rangle) = \operatorname{Spec} A$, we have that 1 can be written as a finite linear combination:

$$1 = \sum_{k=1}^{n} a_k f_k$$

where $f_k = f_{j_i}$ for some j_i. We thus have that:

$$\operatorname{Spec} A = \bigcup_{k=1}^{n} U_{f_k}$$

By hypothesis we have that each U_{f_k} is contained in a V_k, hence:

$$\operatorname{Spec} A = \bigcup_{k=1}^{n} V_k$$

so the cover $\{V_i\}$, admits a finite subcover $\{V_k\}_{k=1}^n$.

Let U_f be a distinguished open, and $\{U_{g_i}\}$ an open covering of U_f. We see that:

$$U_f = (\mathbb{V}(\langle f \rangle))^c = \left(\mathbb{V} \left(\sum_i \langle g_i \rangle \right) \right)^c$$

It follows by Lemma 1.1.1 that:

$$\sqrt{\langle f \rangle} = \sqrt{\sum_i \langle g_i \rangle}$$

It follows that there exists an m such that $f^m \in \sum_i \langle g_i \rangle$ hence

$$f^m = \sum_{j=1}^{p} a_j g_j$$

for some $a_j \in A$ and g_i. We want to show that:

$$\sqrt{\langle f \rangle} = \sqrt{\sum_{j=1}^{p} \langle g_j \rangle}$$

Let $a \in \sqrt{\langle f \rangle}$, then $a^n = f^k \cdot b$ for some $b \in A$, for some $n \in \mathbb{Z}^+$; we see that:

$$\begin{aligned} a^{nm} =& f^{mk} \cdot b^m \\ =& \left(\sum_{j=1}^{p} a_j g_j \right)^k b^m \in \sqrt{\sum_{j=1}^{n} \langle g_j \rangle} \end{aligned}$$

so $\sqrt{\langle f \rangle} \subset \sqrt{\sum_{j=1}^{n} \langle g_j \rangle}$. Now let $b \in \sqrt{\sum_{j=1}^{p} \langle g_j \rangle}$, then there exists an $n \in \mathbb{Z}^+$ such that:

$$b^n = \sum_{j=1}^{p} c_j g_j$$

for some $c_j \in A$. Now we note that $b^n \in \sqrt{\sum_i \langle g_i \rangle}$, hence $b^n \in \sqrt{\langle f \rangle}$ implying the claim. □

We also have the following:

Lemma 1.4.2. *Let X be a topological space and $\mathcal{F}$ a presheaf on a basis for its topology $\mathcal{B}$, such that every cover of a basic set by basic sets admits a finite subcover. If the sheaf on a base axioms hold for all such finite covers, then they hold in generality.*

Proof. We begin with sheaf axiom one; let U be a basic open, $\{U_i\}$ an open covering of U, and $s \in \mathcal{F}(U)$ such that $s|_{U_i} = 0$ for all i. Let $\{U_j\}_{j=1}^k$ be a finite subcover, then we have $s|_{U_j} = 0$ for $1 \leq j \leq k$, so by the hypothesis it follows that $s = 0$ and sheaf axiom one is satisfied.

Now let $\{U_i\}$ be an cover of U, and $s_i \in \mathcal{F}(U_i)$ such that for all basic sets $U_{ij} \subset U_i \cap U_j$ we have:

$$s_i|_{U_{ij}} = s_j|_{U_{ij}}$$

Then there exists a finite subcover $\{U_j\}_{j=1}^k$ such that for all basic open sets $U_{jl} \subset U_j \cap U_l$:

$$s_j|_{U_{jl}} = s_l|_{U_{jl}}$$

It follows by the hypothesis that there exists an $s \in \mathcal{F}(U)$ such that $s|_{U_j} = s_j$ for all $1 \leq j \leq k$. We need only show that for $U_i \notin \{U_j\}_{j=1}^k$ we have $s|_{U_i} = s_i$. We have a finite cover of U_i by $\{U_i \cap U_j\}_{j=1}^k$, each $U_i \cap U_j$ has a cover of basic opens by $\{U_{ijm}\}_m$, hence we obtain a cover of U_i by basic opens $\{U_{ijm}\}$ such that $U_{ijm} \subset U_i \cap U_j \subset U_j$. We see that:

$$\theta^{U_i}_{U_{ijm}}(s|_{U_i}) = (s|_{U_i})|_{U_{ijm}} = s|_{U_{ijm}} = \theta^{U_j}_{U_{ijm}} \circ \theta^{U}_{U_j}(s) = s_j|_{U_{ijm}} = s_i|_{U_{jm}}$$

It follows that:

$$\theta^{U_i}_{U_{ijm}}(s|_{U_i} - s_i) = 0$$

for all j and m, hence $s|_{U_i} = s_i$, implying the claim. □

Proposition 1.4.3. *Let A be commutative ring, and $\mathcal{B}$ be the basis of distinguished opens for the Zariski topology on* $\operatorname{Spec} A$*. Then the assignment* (1.4.1) *defines a sheaf $\mathcal{F}$ on $\mathcal{B}$.*

Proof. We first define restriction maps; by Lemma 1.1.3 we have that if $U_f \subset U_g$, then there exists an $m \in \mathbb{Z}^+$ and $a \in A$ such that $f^m = a \cdot g$. Note that we have maps $\pi_f : A \to A_f$ and $\pi_g : A \to A_g$, and that the image of g is a unit in A_f. Indeed, we have that:

$$\frac{g}{1} \cdot \frac{r}{f^m} = \frac{g \cdot r}{f^m} = \frac{f^m}{f^m} = 1$$

It follows that there exists a unique map $\theta^g_f : A_g \to A_f$ given by:

$$\theta^g_f\left(\frac{b}{g^k}\right) = \frac{b \cdot a^k}{f^{mk}}$$

Now suppose that $U_g \subset U_h$, then we have that there exists a $c \in A$, and an $n \in \mathbb{Z}^+$, such that $g^n = h \cdot c$. By the same argument we obtain a ring homomorphism:

$$\begin{aligned} \theta^h_g : A_h &\longrightarrow A_g \\ \frac{b}{h^k} &\longmapsto \frac{b \cdot c^k}{g^{nk}} \end{aligned}$$

We want to show that $\theta^g_f \circ \theta^h_g = \theta^h_f$. First note that we have:

$$f^m = a \cdot g \Rightarrow f^{mn} = a^n \cdot g^n = a^n \cdot c \cdot h$$

so the map θ^h_f is given by:

$$\frac{b}{h^k} \longmapsto \frac{b \cdot a^{nk} \cdot c^k}{f^{mnk}}$$

Now note that:

$$\begin{aligned} \theta^g_f \circ \theta^h_g\left(\frac{b}{h^k}\right) &= \theta^g_f\left(\frac{b \cdot c^k}{g^{nk}}\right) \\ &= \frac{b \cdot c^k \cdot a^{nk}}{f^{mnk}} \\ &= \theta^h_g\left(\frac{b}{h^k}\right) \end{aligned}$$

It is clear that $\theta^h_h = \mathrm{Id}_{A_h}$, hence $\mathcal{F}(U_f) = A_f$ defines a presheaf on $\mathcal{B}$.

We now check sheaf axiom one. Suppose that U_f is a distinguished open set, $\{U_{g_i}\}$ an open covering of U_f, and $s \in A_f$ such that $s|_{g_i} = 0$ for all i. By Lemma 1.4.1, and Lemma 1.4.2 it suffices to check this on all finite subcoverings of U_f, so without loss of generality we suppose that $\{U_{g_i}\}$ is finite. Since $U_{g_i} \subset U_f$, we have that for each i there exists an $m_i \in \mathbb{Z}^+$, and a $c_i \in A$ such that:

$$g_i^{m_i} = c_i \cdot f$$

Now we note that since f is a unit in A_f, so $s = 0$ if and only if $f^k s = 0$. Indeed, if $s = 0$ then clearly $f^k s = 0$, while if $f^k s = 0$, we have that $(f^k)^{-1} f^k s = s = 0$. If $s = a/f^k$, it thus suffices to show that $f^k s = a/1 = 0$. We have that $f^k s \in \ker \theta^f_{g_i}$, hence there exists an $l_i \in \mathbb{Z}^+$ such that:

$$g^{l_i} \cdot a = 0$$

Now note that:

$$\sqrt{\langle f \rangle} = \sqrt{\sum_i \langle g_i \rangle} = \sqrt{\sum_i \left\langle g_i^{l_i} \right\rangle}$$

hence there exists a $k \in \mathbb{Z}^+$ such that:

$$f^k = \sum_i g_i^{l_i} c_i$$

for some $c_i \in A$. Since each $g^{l_i} \cdot a = 0$, we have that:

$$0 = \sum_i g^{l_i} a = \sum_i g^{l_i} c_i a = a \cdot f^k$$

hence $a/1$ is zero in A_f.

To check sheaf axiom two, it again suffices to assume that $\{U_{g_i}\}$ is a finite open cover of U_f. Let $s_i \in A_{g_i}$ be sections such that:

$$s_i|_{U_{ij}} = s_j|_{U_{ij}}$$

for all $U_{ij} \subset U_{g_i} \cap U_{g_j}$. Then since $U_{g_i} \cap U_{g_j} = U_{g_i g_j}$, we have that:

$$s_i|_{U_{g_i} \cap U_{g_j}} = s_j|_{U_{g_i} \cap U_{g_j}}$$

Since $U_{g_i g_j} \subset U_{g_i}, U_{g_j}$, we have that there exists $k_i \in \mathbb{Z}^+$ and $c_i \in A$ such that:

$$(g_i g_j)^{k_i} = g_i \cdot c_i \qquad \text{and} \qquad (g_i g_j)^{k_j} = g_j c_j$$

Clearly $k_i = 1$ with $c_i = g_j$ fit the bill, hence our restriction maps are given by:

$$\theta^{g_i}_{g_i g_j}\left(\frac{a_i}{g_i^{k_i}}\right) = \frac{a_i \cdot g_j^{k_i}}{g_i^{k_i} g_j^{k_i}}$$

hence on overlaps we have that:

$$\frac{a_i \cdot g_j^{k_i}}{g_i^{k_i} g_j^{k_i}} = \frac{a_j \cdot g_i^{k_j}}{g_i^{k_j} g_j^{k_j}}$$

Since $\{U_{g_i}\}$ is finite, there exists some K such that for all i and j:

$$(g_i g_j)^K \left((g_i g_j)^{k_j} \cdot a_i \cdot g_j^{k_i} - (g_i g_j)^{k_i} \cdot a_j \cdot g_i^{k_j}\right) = 0$$

We multiply by $g_i^{k_i} g_j^{k_j}$ to obtain:

$$\begin{aligned} 0 =& (g_i g_j)^K \left(g_i^{k_i} g_j^{k_j} (g_i g_j)^{k_j} a_i g_j^{k_i} - g_i^{k_i} g_j^{k_j} (g_i g_j)^{k_i} a_j g_i^{k_j}\right) \\ =& (g_i g_j)^{K + k_i + k_j} \left(a_i g_j^{k_j} - a_j g_i^{k_i}\right) \end{aligned} \tag{1.4.2}$$

Set K' to be large enough such that expression above holds for all i, j, and define:

$$h_i = a_i g_i^{K'}$$

Now since:

$$\sqrt{\langle f\rangle} = \sqrt{\sum_i \langle g_i\rangle} = \sqrt{\sum_i \left\langle g_i^{K'+k_i}\right\rangle}$$

we have that for some M, there exist c_i such that:

$$f^M = \sum_i c_i g_i^{K'+k_i}$$

We define s to be:

$$s = \sum_i \frac{c_i h_i}{f^M}$$

Now note that for each j, we have that $g_j^{n_j} = f \cdot b$, then the restriction is given by:

$$s|_{U_{g_j}} = \sum_i \frac{c_i h_i b^M}{g_j^{n_j \cdot M}}$$

We claim this equal to $s_j = a_j / g_j^{k_j}$; examine the expression:

$$g_j^{K'}\left(\sum_i c_i \cdot h_i \cdot b^M \cdot g_j^{k_j} - a_j \cdot g_j^{n_j M}\right)$$

Examine the first term,

$$\sum_i c_i \cdot h_i \cdot b^M \cdot g_j^{k_j} \cdot g_j^{K'} = \sum_i c_i \cdot a_i \cdot g_i^{K'} \cdot b^M \cdot g_j^{k_j} \cdot g_j^{K'}$$

for each i we have that by (1.4.2):

$$g_i^{K'} \cdot g_j^{K'} \cdot g_j^{k_j} a_i = g_i^{K'} \cdot g_j^{K'} \cdot g_i^{k_i} a_j$$

hence we have that:

$$\begin{aligned}\sum_i c_i \cdot h_i \cdot b^M \cdot g_j^{k_j} \cdot g_j^{K'} &= \left(g_j^{K'} a_j b^M\right) \sum_i g_i^{K'} \cdot c_i \cdot g_i^{k_i} \\ &= \left(g_j^{K'} a_j b^M\right) f^M \\ &= g_j^{K'} a_j g_j^{n_j M}\end{aligned}$$

implying the claim. ☐

Definition 1.4.2. Let A be a commutative ring, the the **structure sheaf of** $\operatorname{Spec} \mathbf{A}$, denoted $\mathscr{O}_A$, is the sheaf induced by the sheaf on the base of distinguished opens given by the assignment:

$$U_f \longmapsto A_f$$

The pair $(\operatorname{Spec} A, \mathscr{O}_A)$ is called an **affine scheme**[14].

Proposition 1.4.4. *Let A be a commutative ring, then $(\operatorname{Spec} A, \mathscr{O}_A)$ is a locally ringed space. In particular, the stalk $(\mathscr{O}_A)_{\mathfrak{p}}$ is uniquely isomorphic to $A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} A$.*

Proof. By Proposition 1.4.1, it is sufficient to show that $\mathcal{F}_{\mathfrak{p}}$ is a local ring for all $\mathfrak{p} \in \operatorname{Spec} A$, where $\mathcal{F}$ is the sheaf on a base discussed defined by $U_f \mapsto A_f$. Let $\mathfrak{p} \in \operatorname{Spec} A$, then note that if $\mathfrak{p} \in U_f$, we have that $f \notin \mathfrak{p}$, hence $f \in A - \mathfrak{p}$. It follows that f is a unit in $A_{\mathfrak{p}}$, thus there exists a unique map $\phi_f : A_f \to A_{\mathfrak{p}}$ given by:

$$\phi_f : \frac{a}{f^k} \longmapsto \frac{a}{f^k}$$

[14]This is a tentative definition of an affine scheme, but will be easily seen to be compatible with our future one.

Now suppose that $U_f \subset U_g$, then we have that $f^m = b \cdot g$, so the restriction map θ_f^g is given by:

$$\theta_f^g : \frac{a}{g^k} \longmapsto \frac{a \cdot b^k}{f^{mk}}$$

We want to show that $\phi_g = \phi_f \circ \theta_f^g$, i.e. that:

$$\frac{a \cdot b^k}{f^{mk}} = \frac{a}{g^k}$$

in $A_{\mathfrak{p}}$. We want to show that there exists a $u \in A - \mathfrak{p}$ such that:

$$u \cdot \left(a \cdot b^k \cdot g^k - a \cdot f^{mk}\right) = 0$$

However, $b^k \cdot g^k = f^{mk}$, so this statement is vacuously true. By the universal property of the colimit there thus exists a unique map $\phi : \mathcal{F}_{\mathfrak{p}} \to A_{\mathfrak{p}}$. Suppose we have $[U_f, s] \in \mathcal{F}_{\mathfrak{p}}$ such that:

$$\phi([U_f, s]) = 0$$

Since $s \in A_f$, we have that s is of the form a/f^k, so we must have that:

$$\frac{a}{f^k} = 0$$

in $A_{\mathfrak{p}}$, thus there exists a $u \in A - \mathfrak{p}$ such that:

$$u \cdot a = 0$$

We claim that $[U_f, a/f^k] = 0$; well since $u \in A - \mathfrak{p}$, we have that $u \notin \mathfrak{p}$, hence $\mathfrak{p} \in U_u$. Note that:

$$[U_f, a/f^k] = [U_{fu}, a/f^k|_{U_u}]$$

Since $U_{fu} \subset U_f$, we have that there exists some $n \in \mathbb{Z}^+$ and some $c \in A$ such that:

$$(u \cdot f)^n = c \cdot f$$

however, $n = 1$, and $c = u$ fits the bill, hence our restriction map is given by:

$$a/f^k \longmapsto \frac{a \cdot u^k}{(u \cdot f)^k}$$

however $a \cdot u = 0$, hence we have that the above expression is 0, implying ϕ is injective. Now suppose that $a/r \in A_{\mathfrak{p}}$, then $r \in A - \mathfrak{p}$, hence $\mathfrak{p} \in U_r$. It follows that:

$$\phi([U_r, a/r]) = a/r$$

implying that ϕ is surjective and thus an isomorphism as desired. □

We also have the following facts:

Lemma 1.4.3. *Let* $(\operatorname{Spec} A, \mathscr{O}_A)$ *be an affine scheme, then there are unique isomorphisms* $\mathscr{O}_A(U_f) \cong A_f$*, and* $\mathscr{O}_A(\operatorname{Spec} A) \cong A$.

Proof. By Theorem 1.4.1 we have that $\mathscr{O}_A(U_f) \cong A_f$ as $\mathcal{F}(U_f) = A_f$. Moreover, note that:

$$U_1 = \{\mathfrak{p} \in \operatorname{Spec} A : 1 \notin \mathfrak{p}\}$$

which is equal to all of $\operatorname{Spec} A$, because no prime ideal contains 1. We easily see that $A_1 \cong A$, implying the claim. □

We now determine some topological properties of affine schemes.

Definition 1.4.3. A topological space X is **irreducible** if it is non empty, and cannot be written as the union of two proper closed subsets. A subspace $Z \subset X$ of a topological space is called an **irreducible subspace** if it is irreducible in the subspace topology. A **irreducible component** of a topological space is a maximal irreducible subspace.

Lemma 1.4.4. *Let X be a topological space which is irreducible. Then X is connected and every open subset of X is dense.*

Proof. Suppose that X is disconnected, then $X = U \cup V$ for some disjoint open sets U and V. It follows that since taking the closure over binary unions distributes that:

$$X = \bar{U} \cup \bar{V}$$

so X is reducible. The claim follows by the contrapositive.

Now let $U \subset X$ be any open set, and suppose that U is not dense. It follows that $\bar{U} \neq X$, and that the compliment U^c is closed. We claim that:

$$X = \bar{U} \cup U^c$$

However this is vacuously true, as:

$$X = U \cup U^c \Rightarrow X = \bar{U} \cup \bar{U^c} = \bar{U} \cup U^c$$

hence X is reducible and the claim again follows by the contrapositive. □

Lemma 1.4.5. *Let A be an integral domain, then* $\operatorname{Spec} A$ *is irreducible. In particular, every open set is dense, and* $\operatorname{Spec} A$ *is connected.*

Proof. Recall that if A is an integral domain then we have that $a \cdot b = 0$ if and only if a or b is zero. This then implies that that $\langle 0 \rangle$ is a prime ideal of A, and is thus a point in $\operatorname{Spec} A$. Now suppose that $\operatorname{Spec} A$ is reducible, then we have that by Proposition 1.1.1:

$$\operatorname{Spec} A = \mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cap J) = \mathbb{V}(\langle 0 \rangle)$$

for some ideals I and J. Lemma 1.1.1 then implies that:

$$\sqrt{I \cap J} = \sqrt{\langle 0 \rangle}$$

Since A is an integral domain, we must have that $\sqrt{\langle 0 \rangle} = \langle 0 \rangle$, so

$$\sqrt{I \cap J} = \langle 0 \rangle$$

However, we note that $\sqrt{I \cap J} = \sqrt{IJ}$, then we have that:

$$\sqrt{IJ} = \langle 0 \rangle$$

However, this implies that $IJ \subset \sqrt{IJ} = \langle 0 \rangle$, hence $IJ = \langle 0 \rangle$. It follows that every finite sum of the form:

$$\sum_k i_k j_k$$

where $i_k \in I$ and $j_k \in J$ is zero, hence either I or J is the zero ideal. The claim then follows from the contrapositive, and Lemma 1.4.4. □

Note that when A is an integral domain, we have that every nonempty open set contains $\langle 0 \rangle$. Indeed, note that U_0 is the empty set, and that if $f \neq 0$, then $\langle 0 \rangle \in U_f$ as $f \notin \langle 0 \rangle$. In particular this implies that $\operatorname{Spec} A$ is not Hausdorff, as if $\mathfrak{p}$ and $\langle 0 \rangle$ are both contained in some open set U, then any open set containing $\mathfrak{p}$ will also contain $\langle 0 \rangle$. In general the Zariski topology will be non Hausdorff.

Example 1.4.1. Let k be a field, then we set $\mathbb{A}^n_k$ to be affine scheme:

$$\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$$

Note that in this case $k[x_1, \ldots, x_n]$ is an integral domain, so in particular $\mathbb{A}^n_k$ is irreducible, and connected. The singleton set consisting of the zero ideal is then clearly dense, and so not closed, nor is it open.

The remainder of this section will be dedicated to demonstrating that the category of affine schemes is (anti) equivalent to the category of commutative rings. A morphisms of affine schemes $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is simply a morphism of locally ringed spaces. Let $\phi : B \to A$ be a homomorphism, then we have an induced topological map $f : \operatorname{Spec} A \to \operatorname{Spec} B$ given by $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$. We want to define a morphism $f^\sharp : \mathscr{O}_B \to f_*\mathscr{O}_A$. By Theorem 1.4.1 it suffices to define morphisms $\psi_g : \mathcal{F}_B(U_g) \to (f_*\mathscr{O}_A)(U_g)$ for each $g \in B$ which commute with the restriction maps on distinguished opens. This means we need a morphism:

$$\psi_g : B_f \longrightarrow \mathscr{O}_A(f^{-1}(U_f))$$

We see that:

$$f^{-1}(U_g) = U_{\phi(g)}$$

so it suffices to define a map:

$$\phi_g : B_g \to A_{\phi(g)}$$

and compose it with the isomorphism $A_{\phi(f)} \to \mathscr{O}_A(U_{\phi(f)})$. We define a morphism $B \to A_{\phi(g)}$ by $b \mapsto \phi(b)/1$. Note that the image of g is a unit under this morphism, hence there exists a unique morphism $\phi_g : B_g \to A_{\phi(g)}$ by

$$\frac{b}{g^k} \longmapsto \frac{\phi(b)}{\phi(g^k)}$$

Note that the isomorphism $A_{\phi(g)} \to \mathscr{O}_A(U_{\phi(f)})$ is given by:

$$\frac{a}{\phi(g)^k} \longmapsto \left(\frac{a}{\phi(g^k)}_{\mathfrak{p}}\right) \in \prod_{\mathfrak{p}:\phi(g)\notin\mathfrak{p}} A_{\mathfrak{p}}$$

so ψ_g is the map:

$$\begin{aligned} \psi_g : \mathcal{F}_B(U_g) &\longrightarrow (f_*\mathscr{O}_A)(U_g) \\ b/g^k &\longmapsto ((\phi(b)/\phi(g^k))_{\mathfrak{p}}) \end{aligned} \tag{1.4.3}$$

It is clear that this map commutes with restrictions on a base, so we have morphism of sheaves:

$$f^\sharp : \mathscr{O}_B \longrightarrow f_*\mathscr{O}_A$$

We now need to check that for all $\mathfrak{p} \in \operatorname{Spec} A$ we have that:

$$f_{\mathfrak{p}} : (\mathscr{O}_B)_{f(\mathfrak{p})} \longrightarrow (\mathscr{O}_A)_{\mathfrak{p}}$$

is a local ring homomorphism. Let $s_{f(\mathfrak{p})} \in (\mathscr{O}_B)_{f(\mathfrak{p})}$, then without loss of generality, we can take $s_{f(\mathfrak{p})} = [U_g, (s_{\mathfrak{q}})]$ where $g \in B$ satisfies $g \notin f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$, and $(s_{\mathfrak{q}}) \in \mathscr{O}_B(U_g) \cong B_g$. We than thus write $(s_{\mathfrak{q}}) = \psi_{U_g}^{-1}(b/g^k)$ for some $b/g^k \in B_g$. We thus have that:

$$\begin{aligned} f_{\mathfrak{p}}([U_g, \psi_{U_g}^{-1}(b/g^k)]_{f(\mathfrak{p})}) &= (f_{\mathfrak{p}}^*)([U_g, f_{U_g}^\sharp \circ \psi_{U_g}^{-1}(s_{\mathfrak{q}})]_{f(\mathfrak{p})}) \\ &= (f_{\mathfrak{p}}^*)([U_g, \psi_g(b/g^k)]_{f(\mathfrak{p})}) \\ &= [U_{\phi(g)}, (((\phi(b))/\phi(g^k))_{\mathfrak{p}})]_{\mathfrak{p}} \end{aligned}$$

We then have the following chain of isomorphisms:

$$\begin{aligned} (\mathscr{O}_A)_{\mathfrak{p}} &\longrightarrow \mathcal{F}_{\mathfrak{p}}^A \longrightarrow A_{\mathfrak{p}} \\ \left[U_{\phi(g)}, \left(\frac{\phi(b)}{\phi(g^k)}_{\mathfrak{p}}\right)\right]_{\mathfrak{p}} &\longmapsto \frac{\phi(b)}{\phi(g^k)}_{\mathfrak{p}} \longmapsto \frac{\phi(b)}{\phi(g^k)} \end{aligned}$$

and the same chain of isomorphisms in the opposite direction maps $b/g^k \in B_{f(\mathfrak{p})}$ to $[U_g, \psi_{U_g}^{-1}(b/g^k)]_{f(\mathfrak{p})}$. It thus suffices to check that if $b/g^k \in \mathfrak{m}_{f(\mathfrak{p})} \subset B_{f(\mathfrak{p})}$, then $\phi(b)/\phi(g^k) \in m_{\mathfrak{p}} \subset A_{\mathfrak{p}}$. However, this is clear, as if $b/g^k \in m_{f(\mathfrak{p})}$, we have that $b \in f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$, so $\phi(b) \in \mathfrak{p}$, implying that $\phi(b)/\phi(g^k) \in m_{\mathfrak{p}}$. It follows that $f^\sharp$ is a morphism of local rings as desired.

Note that if $\phi : B \to A$ is a ring homomorphism, then both the morphism of sheaves and the maps on stalks are fully determined by the induced maps $B_{\phi^{-1}(\mathfrak{p})} \to A_{\mathfrak{p}}$, and $B_g \to A_{\phi(g)}$. Moreover note that the induced morphism $f^\sharp$ satisfies $\psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B} = \phi$, where $\psi_{\operatorname{Spec} A/B}$ is the isomorphism $\mathscr{O}_{A/B}(\operatorname{Spec} A/B) \to A/B$. We now wish to prove the following:

Proposition 1.4.5. *If $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of affine schemes, then f and $f^\sharp$ are induced by a unique ring homomorphism $\phi : B \to A$.*

Proof. Let $\phi = \psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B}$; we first want to show that the topological map f satisfies:

$$f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$$

for all $\mathfrak{p} \in \operatorname{Spec} A$. Consider the stalk $(\mathscr{O}_A)_{\mathfrak{p}} \cong A_{\mathfrak{p}}$, and the unique maximal ideal of $A_{\mathfrak{p}}$, $\mathfrak{m}_{\mathfrak{p}}$. We obtain a field $k'_{\mathfrak{p}}$ by taking the quotient $A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$, where $\mathfrak{m}_{\mathfrak{p}}$ is the unique maximal ideal of $A_{\mathfrak{p}}$. Note that we now have a unique map:

$$\begin{aligned} \mathscr{O}_{\operatorname{Spec} A}(A) \cong A \longrightarrow A_{\mathfrak{p}} &\longrightarrow k'_{\mathfrak{p}} \\ a \longmapsto a/1 &\longmapsto [a/1] \end{aligned}$$

Denote this map by $\operatorname{ev}'_{\mathfrak{p}}$, then it is clear that $\psi^{-1}_{\operatorname{Spec} A}(\mathfrak{p}) \subset \ker \operatorname{ev}'_{\mathfrak{p}}$. Moreover, if $a \in \psi_{\operatorname{Spec} A}(\ker \operatorname{ev}'_{\mathfrak{p}})$, then we have that $a/1 \in \mathfrak{m}_{\mathfrak{p}}$, hence there must exist some $p \in \mathfrak{p}$, and some $c \in A - \mathfrak{p}$ such that:

$$\frac{a}{1} = \frac{p}{c} \Rightarrow a \cdot c - p = 0$$

This implies that either $a \in \mathfrak{p}$, or $c \in \mathfrak{p}$, however c can't lie in $\mathfrak{p}$ by construction, hence $a \in \mathfrak{p}$. It follows that $\psi^{-1}_{\operatorname{Spec} A}(\mathfrak{p}) = \ker \operatorname{ev}'_{\mathfrak{p}}$. Similarly, we have that $f(\mathfrak{p})$ is a prime ideal of B, so $\psi^{-1}_{\operatorname{Spec} B}(f(\mathfrak{p})) = \ker \operatorname{ev}'_{f(\mathfrak{p})}$. Via the unique isomorphism of stalks with localizations, and the isomorphism of global sections with the rings A and B, $\mathfrak{p}$ (and $f(\mathfrak{p})$) can be identified with global sections which vanish at $\mathfrak{p}$ (and $f(\mathfrak{p})$).

Now note that:

$$\begin{aligned} \phi^{-1}(\mathfrak{p}) =& (\psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B})^{-1}(\mathfrak{p}) \\ =& (f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B})^{-1} \left(\psi^{-1}_A(\mathfrak{p})\right) \\ =& (f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B})^{-1} \left(\ker \operatorname{ev}'_{\mathfrak{p}}\right) \end{aligned}$$

It thus suffices to show that:

$$(f^\sharp_{\operatorname{Spec} B})^{-1}(\ker \operatorname{ev}'_{\mathfrak{p}}) = \ker \operatorname{ev}'_{f(\mathfrak{p})}$$

as then we will have that:

$$\phi^{-1}(\mathfrak{p}) = \psi_{\operatorname{Spec} B}(\ker \operatorname{ev}'_{f(\mathfrak{p})}) = f(\mathfrak{p})$$

Let $s \in \ker \operatorname{ev}'_{f(\mathfrak{p})}$, then we want to show that $f^\sharp_{\operatorname{Spec} B}(s) \in \ker \operatorname{ev}'_{\mathfrak{p}}$. Let $b \in B$ be the unique element satisfying $\psi_{\operatorname{Spec} B}(s) = b$, then, under the isomorphism $B_{f(\mathfrak{p})} \cong (\mathscr{O}_{\operatorname{Spec} B})_{f(\mathfrak{p})}$, we see that the stalk $s_{f(\mathfrak{p})}$ gets mapped to $b/1 \in B_{f(\mathfrak{p})}$. Since $s \in \ker \operatorname{ev}_{f(\mathfrak{p})}$, it follows that $b/1 \in \mathfrak{m}_{f(\mathfrak{p})}$, so $s_{f(\mathfrak{p})} \in \mathfrak{m}'_{f(\mathfrak{p})} \subset (\mathscr{O}_{\operatorname{Spec} B})_{f(\mathfrak{p})}$. We thus have that $f_{\mathfrak{p}}(s_{f(\mathfrak{p})}) \in \mathfrak{m}'_{\mathfrak{p}} \subset (\mathscr{O}_{\operatorname{Spec} A})_{\mathfrak{p}}$[15], since f is a morphism of local rings. Hence:

$$f_{\mathfrak{p}}(s_{f_{\mathfrak{p}}}) = [\operatorname{Spec} A, f^\sharp_{\operatorname{Spec} B}(s)]_{\mathfrak{p}} \in \mathfrak{m}'_{\mathfrak{p}}$$

and under the isomorphism $(\mathscr{O}_{\operatorname{Spec} A})_{\mathfrak{p}} \cong A_{\mathfrak{p}}$, this gets mapped to $(\psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B}(s))/1 = \phi(b)/1$, which must lie in $\mathfrak{m}_{\mathfrak{p}}$. It follows from our previous argument that $f^\sharp_{\operatorname{Spec} B}(s) \in \ker \operatorname{ev}'_{\mathfrak{p}}$, implying one inclusion.

Now let $s \in (f^\sharp_{\operatorname{Spec} B})^{-1}(\ker \operatorname{ev}_{\mathfrak{p}})$, then we have that $f^\sharp_{\operatorname{Spec} B}(s) \in \ker \operatorname{ev}_{\mathfrak{p}}$, so the stalk $f^\sharp_{\operatorname{Spec} B}(s)_{\mathfrak{p}}$ lies in $\mathfrak{m}'_{\mathfrak{p}}$, hence $f_{\mathfrak{p}}(s_{f(\mathfrak{p})}) \in \mathfrak{m}'_{\mathfrak{p}}$. Now suppose that $s_{f(\mathfrak{p})} \notin \mathfrak{m}'_{f(\mathfrak{p})}$, then we have that the corresponding element

[15] The primed maximal ideals are the unique maximal ideal in the stalk.

$b/1 \in B_{f(\mathfrak{p})}$ does not lie in $\mathfrak{m}_{f(\mathfrak{p})}$, but the element $\phi(b)/1 \in A_{\mathfrak{p}}$ corresponding to $f_{\mathfrak{p}}(s_{f(\mathfrak{p})})$ lies in $\mathfrak{m}_{\mathfrak{p}}$. Since $b/1 \notin m_{f(\mathfrak{p})}$, we know that $b \notin f(\mathfrak{p})$, hence we have that $b/1$ is a unit in $B_{f(\mathfrak{p})}$, and so $\phi(b)/1$ must be a unit in $A_{\mathfrak{p}}$ as well. It follows that $\mathfrak{m}_{\mathfrak{p}}$ contains the identity, so it is not maximal yielding a contradiction. We thus have that $s_{f(\mathfrak{p})} \in \mathfrak{m}'_{f(\mathfrak{p})}$, hence $s \in \ker \mathrm{ev}'_{f(\mathfrak{p})}$ as desired, implying the claim.

We now need to check that ϕ induces the same map sheaves as $f^\sharp : \mathscr{O}_{\mathrm{Spec}\, B} \to f_* \mathscr{O}_{\mathrm{Spec}\, A}$. First note that we have morphisms:

$$f^\sharp_{U_g} \circ \psi^{-1}_{U_g} : \mathcal{F}_B(U_g) \longrightarrow f_*\mathscr{O}_{\mathrm{Spec}\, A}(U_g) = \mathscr{O}_{\mathrm{Spec}\, A}(U_{\phi(g)})$$

for each distinguished $U_g \subset \mathrm{Spec}\, B$, which trivially make the diagram in Theorem 1.4.1 commute. Note that the equality follows from the fact the topological map is equal to taking preimages by ϕ. It follows that $f^\sharp$ is the unique morphism induced by these morphisms on a base, hence we need only show that ϕ induces the same map on distinguished opens. Note that for each $g \in B$, we have a map $\phi_g : B_g \to A_{\phi(g)}$ by:

$$\frac{b}{g^k} \longmapsto \frac{\phi(b)}{\phi(g^k)} = \frac{\psi_{\mathrm{Spec}\, A} \circ f^\sharp_{\mathrm{Spec}\, B} \circ \psi^{-1}_{\mathrm{Spec}\, B}(b)}{\psi_{\mathrm{Spec}\, A} \circ f^\sharp_{\mathrm{Spec}\, B} \circ \psi^{-1}_{\mathrm{Spec}\, B}(g^k)}$$

Since $\psi_{U_{\phi(g)}} : \mathscr{O}_{\mathrm{Spec}\, A}(U_{\phi(g)}) \to A_{\phi(g)}$ is an isomorphism, it thus suffices to show that:

$$\psi_{U_{\phi(g)}} \circ f^\sharp_{U_g} \circ \psi^{-1}_{U_g}(b/g^k) = \frac{\psi_{\mathrm{Spec}\, A} \circ f^\sharp_{\mathrm{Spec}\, B} \circ \psi^{-1}_{\mathrm{Spec}\, B}(b)}{\psi_{\mathrm{Spec}\, A} \circ f^\sharp_{\mathrm{Spec}\, B} \circ \psi^{-1}_{\mathrm{Spec}\, B}(g^k)}$$

Note that ϕ_g is the unique map which satisfies $\phi_g \circ \theta^B_g = \phi'$, where ϕ' is the map $b \mapsto \phi(b)/1$, and θ^B_g is the restriction map, which is simply localization. By the universal property of localization, it then suffices to check that

$$(\psi_{U_{\phi(g)}} \circ f^\sharp_{U_g} \circ \psi^{-1}_{U_g}) \circ \theta^B_g = \phi'$$

Well, note that isomorphisms ψ_U, and their inverses trivially commute with restrictions of a sheaf on a base, hence we have that:

$$\begin{aligned}(\psi_{U_{\phi(g)}} \circ f^\sharp_{U_g} \circ \psi^{-1}_{U_g}) \circ \theta^B_g(b) &= \theta^A_{\phi(g)} \circ (\psi_{\mathrm{Spec}\, A} \circ f^\sharp_{\mathrm{Spec}\, B} \circ \psi^{-1}_{\mathrm{Spec}\, B})(b) \\ &= \frac{\psi_{\mathrm{Spec}\, A} \circ f^\sharp_{\mathrm{Spec}\, B} \circ \psi^{-1}_{\mathrm{Spec}\, B}(b)}{1} \\ &= \phi'(b)\end{aligned}$$

implying the claim. □

We briefly mention the definition of an (anti)-equivalence of categories.

Definition 1.4.4. Let C and D be categories, then C is **(anti) equivalent** to D if there is a a (contravariant) covariant functor $\mathcal{F} : C \to D$, such that that for every objects X and Y of C there is a bijection induced by $\mathcal{F}$[16]:

$$\mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_D(\mathcal{F}(X), \mathcal{F}(Y))$$

and for every object Z of D there exists an object X of C such that $F(X)$ is isomorphic to Z, i.e. $\mathcal{F}$ is **essentially surjective.**

We end with the following corollary:

Corollary 1.4.3. *The category of commutative rings is anti equivalent to the category of affine schemes.*

Proof. Note that we have a contravariant functor $\mathrm{Spec} : \mathrm{Ring} \to \mathrm{AffS}$ given by $A \mapsto (\mathrm{Spec}\, A, \mathscr{O}_{\mathrm{Spec}\, A})$. We see that this if $(X, \mathscr{O}_X)$ is an affine scheme, then $(X, \mathscr{O}_X)$ is isomorphic to $(\mathrm{Spec}\, A, \mathscr{O}_{\mathrm{Spec}\, A})$ for some commutative ring A, implying that Spec is essentially surjective.

We need to show that the induced map:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ring}}(A, B) &\longrightarrow \mathrm{Hom}_{\mathrm{AffS}}(\mathrm{Spec}\, B, \mathrm{Spec}\, A) \\ \phi &\longmapsto (f, f^\sharp)\end{aligned}$$

[16] If $\mathcal{F}$ is contravariant then clearly the order switches.

where $f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$, and $f^\sharp$ is the morphism of sheaves induced by the sheaf on a base morphisms given by (1.9)[17], is a bijection. Note that it is clearly injective, as if $\phi = \psi$, then $\phi^{-1}(\mathfrak{p}) = \psi^{-1}(\mathfrak{p})$, and for all $a/g^k \in \mathcal{F}_A(U_g)$ we have that:

$$\frac{\phi(a)}{\phi(g^k)} = \frac{\psi(a)}{\psi(g^k)}$$

so the induced morphisms are equivalent as well. It follows that the morphisms of affine schemes are then equal so the map is injective.

Suppose that $(f, f^\sharp) : \operatorname{Spec} B \to \operatorname{Spec} A$ is morphism of affine schemes. Then it follows from Proposition 1.4.5 that the $\phi = \psi_{\operatorname{Spec} B} \circ f^\sharp_{\operatorname{Spec} A} \circ \psi^{-1}_{\operatorname{Spec} A}$ is a ring homomorphism that maps to $(f, f^\sharp)$, so Spec is anti equivalence of categories as desired. $\square$

Note that clearly if $\phi : A \to B$ is an isomorphism then $(f, f^\sharp)$ is an isomorphism and vice versa.

[17]The domains and codomains have swithced, but this is just a result of our choice of ϕ.

Schemes

2.1 Definition and Examples

We are now in a position to define a scheme in full generality.

Definition 2.1.1. Let $(X, \mathscr{O}_X)$ be a locally ringed space, then $(X, \mathscr{O}_X)$ is an **affine scheme** if $(X, \mathscr{O}_X)$ is isomorphic to $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ for some commutative ring A. $(X, \mathscr{O}_X)$ is a scheme if every point $x \in X$ has an open neighborhood U of x such that $(U, \mathscr{O}_X|_U)$ is an affine scheme. We call such open sets **affine opens**, and the topology on X is called the **Zariski topology**.

Note that a morphism of schemes is simply a morphism of locally ringed spaces, and hence an isomorphism of schemes is an isomorphism of locally ringed spaces.

Example 2.1.1. We wish to show that affine schemes are schemes. Let A be a commutative ring, then for every element $\mathfrak{p} \in \operatorname{Spec} A$, we need to show that there is open neighborhood U of $\mathfrak{p}$ such that $(U, \mathscr{O}_{\operatorname{Spec} A}|_U)$ is an affine scheme. Let $g \notin \mathfrak{p}$, then U_g is an open set containing $\mathfrak{p}$; we claim that $(U_g, \mathscr{O}_{\operatorname{Spec} A}|_U)$ is isomorphic to $(\operatorname{Spec} A_g, \mathscr{O}_{\operatorname{Spec} A_g})$. We already have from Proposition 1.1.3 that there exists a homeomorphism $\eta : U_g \to \operatorname{Spec} A_g$ given by:

$$\eta(\mathfrak{p}) = \left\{ \frac{p}{g^k} \in A_g : p \in \mathfrak{p}, k \geq 0 \right\}$$

so we want to describe an isomorphism:

$$\eta^\sharp : \mathscr{O}_{\operatorname{Spec} A_g} \longrightarrow \eta_*(\mathscr{O}_{\operatorname{Spec} A}|_{U_g})$$

First recall that every distinguished open $U_{f/g^k} \subset \operatorname{Spec} A_g$ is equal to $U_{f/1} \subset \operatorname{Spec} A_f$, so it suffices to define a morphism on the set of distinguished opens of the form $U_{f/1}$ for some $f/1 \in A_g$. We see that:

$$\eta_*(\mathscr{O}_{\operatorname{Spec} A}|_{U_g})(U_{f/1}) = \mathscr{O}_{\operatorname{Spec} A}(\eta^{-1}(U_{f/1})) = \mathscr{O}_{\operatorname{Spec} A}(U_{fg})$$

So it suffices to prescribe maps $\phi_{f/1} : (A_g)_{f/1} \to A_{fg}$ and compose with the isomorphism $A_{fg} \to \mathscr{O}_{\operatorname{Spec} A}(U_{fg})$. We set $\phi_{f/1}$ to be the unique isomorphism $(A_g)_{f/1} \to A_{fg}$, so we then obtain a set of isomorphisms $\psi_{f/1} : (A_f)_{f/1} \to \mathscr{O}_{\operatorname{Spec} A}(U_{fg})$. By Theorem 1.4.1 there thus exists a unique morphism:

$$\eta^\sharp : \mathscr{O}_{\operatorname{Spec} A_g} \longrightarrow \eta_*(\mathscr{O}_{\operatorname{Spec} A}|_{U_g})$$

We need to show that $\eta^\sharp$ is an isomorphism (and thus a morphism of locally ringed spaces), and it suffices to check by Corollary 1.4.2 that $\eta^\sharp$ is an isomorphism on distinguished open sets of $\operatorname{Spec} A_g$. In particular, since $\eta_*(\mathscr{O}_{\operatorname{Spec} A}|_{U_g}(U_{f/1})) = \mathscr{O}_{\operatorname{Spec} A}(U_{fg})$, we have the following diagram:

$$\begin{tikzcd}
\mathscr{O}_{\operatorname{Spec} A_g}(U_{f/1}) \arrow[r, "\eta^\sharp_{U_{f/1}}"] \arrow[d, "\psi_{U_{f/1}}"] & \mathscr{O}_{\operatorname{Spec} A}(U_{fg}) \\
(A_g)_{f/1} \arrow[r, "\phi_{f/1}"] & A_{fg} \arrow[u, "\psi^{-1}_{U_{fg}}"]
\end{tikzcd}$$

so $\eta^\sharp_{U_{f/1}}$ is the composition of isomorphisms, and is thus an isomorphism, implying that $\eta^\sharp$ is indeed a natural isomorphism.

The following lemmas will prove useful in the future:

Lemma 2.1.1. *Let $(X, \mathscr{O}_X)$ be a scheme, then the following hold:*

a) *The set of open affines form a basis for the topology on X.*

b) *The sheaf on a base defined by $\mathcal{F}^X(U) = \mathscr{O}_X(U)$ induces a structure sheaf on X which is isomorphic to $\mathscr{O}_X$.*

c) *For every two open affines U, V, we $U \cap V$ can be covered by open affines which are simultaneously distinguished opens in both U and V.*

Proof. We begin with a). It is clear that the set of affine opens cover X, so we need only check that any open set U can be written as the union of affine opens. For each $x \in U$ we have an affine open V_x, the collection $\{V_x\}$ then defines a cover of U by:

$$\{V_x \cap U\}$$

Equipped with the subspace topology, we have that $V_x \cap U$ is open in $V_x \cong \operatorname{Spec} A_x$ for some ring A_x. It follows that since $\operatorname{Spec} A_x$ can be covered by distinguished opens $U_{g_x^i}$, and each $U_{g_x^i}$ is itself an affine open of $\operatorname{Spec} A_x$, that $V_x \cap U$ can be covered by affine opens V_{x^i} isomorphic to $(U_{g_x^i}, \operatorname{Spec} A_x|_{U_{g_x^i}})$. We thus have that:

$$U = \bigcup_{x \in U} V_x \cap U = \bigcup_{x \in U} \bigcup_i V_x^i$$

so U can be covered by affine opens, as desired.

We see that b) follows from a) by Proposition 1.4.2.

For c), Let $U \cong \operatorname{Spec} A$, $V \cong \operatorname{Spec} B$, and $x \in U \cap V$. The isomorphisms $f : U \to \operatorname{Spec} A$ and $g : V \to \operatorname{Spec} B$ induce an isomorphism $h : f(U \cap V) \subset \operatorname{Spec} A \to g(U \cap V) \subset \operatorname{Spec} B$ such that $h \circ f|_{U \cap V} = g|_{U \cap V}$. So when we say that $U \cap V$ can be covered by open affines which are simultaneously distinguished opens in $\operatorname{Spec} A$ and $\operatorname{Spec} B$, we mean that that there exists an open cover $\{W_i\}$ of $U \cap V$, such that $f(W_i)$ are distinguished opens in $\operatorname{Spec} A$, and $g(W_i)$ is a distinguished open in $\operatorname{Spec} B$.[18]

It suffices to show that every $x \in U \cap V$ has such a neighborhood. Since $x \in U \cap V$, we have that $f(x) = \mathfrak{p} \in \operatorname{Spec} A$, $g(x) = \mathfrak{q} \in \operatorname{Spec} B$, and $h(\mathfrak{p}) = \mathfrak{q}$. Since $f(U \cap V)$ is an open set in $\operatorname{Spec} A$, there is a distinguished open $U_a = \operatorname{Spec} A_a$ such that $U_a \subset f(U \cap V)$, and $\mathfrak{p} \in U_f$. We see that $h(U_a)$ is an affine open subscheme of $g(U \cap V) \subset \operatorname{Spec} B$, hence there is an open embedding $\iota_a : \operatorname{Spec} A_a \hookrightarrow \operatorname{Spec} B$, such that $\iota_a = h|_{U_a}$. In particular, there is a distinguished open set $U_b \subset \iota_a(\operatorname{Spec} A_a)$ determined by some $b \in B$. Let

$$\phi : \mathscr{O}_{\operatorname{Spec} B}(\iota_a(\operatorname{Spec} A_a)) \to \mathscr{O}_{\operatorname{Spec} A_a}(\operatorname{Spec} A_a) \cong A_a$$

Let $U_{\phi(b)} \subset \operatorname{Spec} A_a$ be the distinguished open associated to $\phi(b)$, then we claim that:

$$\iota_a(U_{\phi(b)}) = U_b \tag{2.1.1}$$

Indeed for any $b \in B$ we have:

$$\begin{aligned}
\iota_a(U_{\phi(b)}) =& \{\iota_a(\mathfrak{p}) : \mathfrak{p} \in U_{\phi(b)}\} \\
=& \{\iota_a(\mathfrak{p}) : \phi(b) \notin \mathfrak{p}, \mathfrak{p} \in \operatorname{Spec} A_a\} \\
=& \{\iota_a(\mathfrak{p}) : b \notin \phi^{-1}(p), \mathfrak{p} \in \operatorname{Spec} A_a\} \\
=& U_b \cap \iota_a(\operatorname{Spec} A_a)
\end{aligned}$$

however since $U_b \subset \iota_a(\operatorname{Spec} A_a)$ we have obtained the equality (2.1.1) as desired.

Let $\phi(b) = c/a^n$, then by Lemma 1.1.4, $U_{\phi(b)}$ identified as a subset of $\operatorname{Spec} A$ is the distinguished open set $U_{c \cdot a} \subset \operatorname{Spec} A$. Since $\iota_a = h|_{h(U_a)}$, it follows that:

$$h(U_{c \cdot a}) = \iota_a(U_{\phi(b)}) = U_b$$

hence setting $W = f^{-1}(U_{c \cdot a}) \subset U \cap V$ is an affine which can simultaneously be identified with distinguished opens in both $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ as desired.

□

[18] In the future, as we get more comfortable with the local nature of schemes, we will gradually suppress these isomorphisms for ease of notation, and simply work with affine opens as $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$.

Lemma 2.1.2. *Let $(X, \mathscr{O}_X)$ be a scheme, then $(U, \mathscr{O}_X|_U)$ is scheme equipped with an open embedding $\iota : U \to X$.*

Proof. If we can show that the locally ringed space $(U, \mathscr{O}_X|_U)$ is a scheme, then the claim follows from Lemma 1.3.4. We need only show that for each $x \in U$ there is an open neighborhood V_x of x such that:

$$(V_x, \mathscr{O}_X|_U|_{V_x}) = (V_x, \mathscr{O}_X|_{V_x}) \cong (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$$

However, we have that by Lemma 2.1.1 U can be written as the union of affine opens, all of which will be open in the subspace topology of U. It follows that every $x \in U$ must lie in one of these affine opens, so by the definition of an affine open the claim follows. □

Note that by Lemma 2.1.1 and Corollary 1.4.2, it suffices to define morphisms between schemes on the basis affine opens.

Definition 2.1.2. Let X be a scheme, and U an open subset of X. The induced scheme $(U, \mathscr{O}_X|_U)$ is then called a **open subscheme**.

Lemma 2.1.3. *Let $f : X \to Y$ be a morphism of locally ringed spaces, then $(f, f^\sharp)$ is an isomorphism if and only if f is a homeomorphism and the stalk map $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ is an isomorphism for all $x \in X$.*

Proof. Suppose that $(f, f^\sharp)$ is an isomorphism of locally ringed spaces, then by definition f is a homeomorphism, and $f^\sharp$ is an isomorphism of sheaves. It follows that:

$$f^\sharp_{f(x)} : (\mathscr{O}_Y)_y \to (f_*\mathscr{O}_X)_y$$

is an isomorphism for all $y \in Y$. Since f is a homeomorphism, it suffices to check that:

$$(f_*)_x : (f_*\mathscr{O}_X)_{f(x)} \to (\mathscr{O}_X)_x$$

is an isomorphism for all $x \in X$. We first show that $(f_*)_x$ is injective, suppose that $[U, s]_{f(x)}$ satisfies $[f^{-1}(U), s]_x = 0$, then there exists some open neighborhood of x $V \subset f^{-1}(U)$ such that $s|_V = 0$. Since f is a homeomorphism, we have that $f(V)$ is an open subset of U, and satisfies:

$$s|_{f(V)} = s|_{f^{-1}(f(V))} = s|_V = 0$$

so $[U, s]_{f(x)} = 0$. Now suppose that $[V, s]_x \in (\mathscr{O}_X)_x$, then we see that $f(V)$ is an open subset of Y, and thus $[f(V), s]_{f(x)} \in (f_*\mathscr{O}_X)_x$. It is then clear that $(f_*)_x([f(V), s]_{f(x)}) = [V, s]_x$ so $(f_*)_x$ is an isomorphism as desired. Since $f_x = (f_*)_x \circ f^\sharp_{f(x)}$, we have that f_x must be an isomorphism for all $x \in X$.

Now suppose that f is a homeomorphism, and $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ is an isomorphism for all $x \in X$. It suffices to check that $f^\sharp_y$ is an isomorphism for all $y \in Y$. Let $y \in Y$, then since f is a homeomorphism, there is a unique element $x \in X$ such that $x = f^{-1}(y)$. We have that $f_{f^{-1}(y)}$ is an isomorphism, and is equal to $(f_*)_{f^{-1}(y)} \circ f^\sharp_y$, however by the proceeding paragraph, $(f_*)_{f^{-1}(y)}$ is an isomorphism, so we see that $f^\sharp_y = (f_*)^{-1}_{f^{-1}(y)} \circ f_{f^{-1}(y)}$, hence $f^\sharp_y$ is an isomorphism for al $y \in Y$. It follows that $f^\sharp$ is an isomorphism of sheaves so $(f, f^\sharp)$ is an isomorphism of locally ringed spaces. □

With the lemma above, we now have the following result, which is important for sanity reasons.

Corollary 2.1.1. *Let $f : X \to Y$ be a homeomorphism, $\mathscr{F}$ a sheaf on Y, and $\mathscr{G}$ a sheaf on X. Then a morphism $F : \mathscr{F} \to f_*\mathscr{G}$ is an isomorphism if and only if the unique map $\hat{F} : f^{-1}\mathscr{F} \to \mathscr{G}$ is an isomorphism. In other words, the isomorphism in Theorem 1.3.1 preserves isomorphisms.*

Proof. Suppose that $F : \mathscr{F} \to f_*\mathscr{G}$ is an isomorphism, then we have that the stalk map $F_y : \mathscr{F}_y \to (f_*\mathscr{G})_y$ is an isomorphism for all $y \in Y$. It suffices to check that the stalk map $\hat{F}_x : (f^{-1}\mathscr{F})_x \to \mathscr{G}_x$ is an isomorphism. By Corollary 1.3.3 we have that:

$$\hat{F}_x \circ \operatorname{sh}_x \circ (f_p^{-1})_x = (f_*)_x \circ F_{f(x)} \tag{2.1.2}$$

and by the preceding we have that $(f_*)_x$ is an isomorphism, and by hypothesis $F_{f(x)}$ is an isomorphism for all $f(x)$. It follows that $\hat{F}_x$ must be an isomorphism for all $x \in X$, as both sh_x and $(f_p^{-1})_x$ are isomorphisms for all $x \in X$, hence $\hat{F}$ is an isomorphism.

Conversely, suppose that $\hat{F}$ is an isomorphism, then $\hat{F}_x$ is an isomorphism for all $x \in X$. Then (2.1) implies that $F_{f(x)}$ is an isomorphism for all $f(x)$, as $(f_*)_x$ is an isomorphism and the composition on the left is a composition of isomorphisms. Since f is a homeomorphism and thus surjective, it follows that F_y is an isomorphism for all $y \in Y$, hence F is an isomorphism of sheaves as desired. $\square$

As of this moment, we have two example of schemes, namely given a commutative ring A, we can construct an affine scheme, and given a scheme X we can take any open subset of X and obtain an open subscheme. We would like to be able to construct more examples, hence the following gluing proposition:

Theorem 2.1.1. *Let $\{X_i\}$ be a family of schemes, and suppose for each $i \neq j$ there exists an open subscheme $U_{ij} \subset X_i$. Suppose also that for each $i \neq j$ an isomorphism of schemes $\phi_{ij} : U_{ij} \to U_{ji}$ satisfying $\phi_{ij}^{-1} = \phi_{ij}$, $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$ for all i, j and k. Then, there exists a scheme X, together with morphisms $\psi_i : X_i \to X$ such that each ψ_i is an open embedding, $\psi_i(X_i)$ cover X, $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij}.*

Proof. We first begin by constructing the topological space X. As a set define X to be:

$$X = \left(\coprod_i X_i\right) / \sim$$

where the equivalence relation $\sim$ is given by $x_i \in X_i$ and $x_j \in X_j$ are equivalent if and and only if $x_i \in U_{ij}$, $x_j \in U_{ji}$, and $\phi_{ij}(x_i) = x_j$. We check that this is an equivalence relation. Note that we have $x_i \sim x_i$, as $x_i \in U_{ii} = X_i$, and $\phi_{ii} = \mathrm{Id}_{X_i}$. The relation is symmetric, as if $x_i \sim x_j$, then we have $\phi_{ij}(x_i) = x_j$, so $\phi_{ji}(x_j) = x_i$, hence $x_j \sim x_i$. Now suppose that $x_i \sim x_j$ and $x_j \sim x_k$, then $x_i \in U_{ij}$, $x_j \in U_{ji}$, $x_j \in U_{jk}$, and $x_k \in U_{kj}$. It follows that $\phi_{ij}(x_i) = x_j \in U_{ji} \cap U_{jk}$, so $x_i \in U_ij \cap U_{ik}$, and moreover that $x_k \in U_{kj} \cap U_{ki}$. We also see that $\phi_{jk} \circ \phi_{ij}(x_i) = \phi_{jk}(x_j) = x_k$, so we have that $\phi_{ik}(x_i) = x_k$, implying that $x_i \sim x_k$ as desired.

Note that since $\phi_{ii} = \mathrm{Id}$, no two elements $x_i, y_i \in X_i$ such that $x_i \neq y_i$ can be equivalent to one another. We thus have natural injections $\psi_i : X_i \to X$ given by $x_i \mapsto [x_i] \in X$, and thus ψ_i is a bijection onto it's image. We define a topology on X by $U \subset X$ is open if and only $\psi_i^{-1}(U) \subset X_i$ is open for all i. We check that this is a topology; note that the empty set is vacuously open, and that X is open as $\psi_i^{-1}(X) = X_i$. Moreover, arbitrary unions of open sets are open as:

$$\psi_i^{-1}\left(\bigcup_j U_j\right) = \bigcup_j \psi_i^{-1}(U_j)$$

which is the union of open sets in X_i by hypothesis, and so the original set is open in X. For finite intersections we have that:

$$\psi_i^{-1}(U \cap V) = \psi_i^{-1}(U) \cap \psi_i^{-1}(V)$$

which is open in X_i, so $U \cap V$ is open in X, so this assignment defines a topology on X.

Clearly, by the construction of the topology on X, we have that each $\psi_i : X_i \to X$ is a continuous map. We want to show that $\psi_i(X_i)$ cover X, and that each ψ_i satisfies $\psi_i(U_{ij}) = \psi(X_i) \cap \psi_j(X_j)$ and that $\psi_i = \psi_j \circ \phi_{ij}$. The first statement is clear, indeed let $x \in X$, then by the definition of X, x is an equivalence class with a class representative $x_i \in X_i$, so $\psi_i(x_i) = [x_i] = x$. Now let $U_{ij} \subset X_i$, and suppose that $x \in \psi_i(U_{ij})$, then x is an equivalence class with class representative $x_i \in X_i$. Since $x_i \in U_{ij}$, and $\phi_{ij} : U_{ij} \to U_{ji}$ is a homeomorphism, there must be a unique element $x_j \in U_{ji}$ such that $\phi_{ij}(x_i) = x_j$, hence $[x_i] = x = [x_j]$. It follows that $x \in \psi_j(X_j)$ as well, so $\psi_i(U_{ij}) \subset \psi_i(X_i) \cap \psi_j(X_j)$. Now suppose that $x \in \psi_i(X_i) \cap \psi_j(X_j)$, then $x = [x_i] = [x_j]$ for some $x_i \in X_i$ and $x_j \in X_j$. It follows that $x_i \sim x_j$, so $x_i \in U_{ij}$ and $x_j \in U_{ji}$ such that $\phi_{ij}(x_i) = x_j$, hence $[x_i] \in \psi_i(U_{ij})$, and we have that $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ as desired. Finally, let $x_i \in U_{ij}$, then $\psi_i(x_i) = [x_i]$, and $\psi_j \circ \phi_{ij}(x_i) = [\phi_{ij}(x_i)]$, however $\phi_{ij}(x_i) \in U_{ji}$, and we vacuously have that $\phi_{ij}(x_i) = \phi_{ij}(x_i)$, hence $\psi_i = \psi_j \circ \phi_{ij}$.

To show that $\psi_i : X_i \to \psi_i(X_i)$ is a homeomorphism, we first note that ψ_i is an injective open map. Indeed, let $x_i, y_i \in X_i$, such that $[x_i] = [y_i]$, implying that $x_i \sim y_i$, but x_i and y_i both lie in X_i, so we must have that $\phi_{ii}(x_i) = y_i$ implying that $x_i = y_i$. Now let $U \subset X_i$ be an open set, we want to show that $\psi_i(U)$ is open in X. It is clear that since ψ_i is injective we have that $\psi_i^{-1}(\psi_i(U)) = U$. Let $j \neq i$, then we want to show that $\psi_j^{-1}(\psi_i(U))$ is open in X_j. If $U \cap U_{ij}$ is empty then we see that there is no

$x_j \in X_j$ such that $\psi_j(x_j) \in \psi_i(U)$, hence $\psi_j^{-1}(\psi_i(U)) = \emptyset$ and is thus open. Suppose that $U \cap U_{ij}$ is not empty, then we claim that:

$$\psi_j^{-1}(\psi_i(U)) = \phi_{ij}(U \cap U_{ij}) \tag{2.1.3}$$

which is an open subset of X_j as $U \cap U_{ij} \subset U_{ij}$ is open in the subspace topology, and ϕ_{ij} is a homeomorphism of open subspaces, so $\phi_{ij}(U \cap U_{ij}) \subset U_{ji}$ is open in the subspace topology, and thus open in X_j. Let $x_j \in \psi_j^{-1}(\psi_i(U))$, then we have that $\psi_j(x_j) = [x_j] \in \psi_i(U)$, hence there exists an $x_i \in U$ such that $[x_j] = [x_i]$ implying that $x_i \in U_{ij}$, $x_j \in U_{ji}$ and $\phi_{ij}(x_i) = x_j$. It follows that $x_i \in U \cap U_{ij}$, and that $x_j = \phi_{ij}(x_i)$ so $x_j \in \phi_{ij}(U \cap U_{ij})$. Now suppose that $x_j \in \phi_{ij}(U \cap U_{ij}) \subset U_{ji}$, then there exists a unique $x_i \in U \cap U_{ij}$ such that $\phi_{ij}(x_i) = x_j$. We see that $\psi_j(x_j) = [x_j]$, and that $[x_j] = [x_i]$ as $x_j \in U_{ji}$, $x_i \in U_{ij}$ and $\phi_{ij}(x_i) = x_j$. Since $x_i \in U \cap U_{ij} \subset U$, we have that $\psi_j(x_j) = \psi_i(x_i) \in \psi_i(U)$, hence $x_j \in \psi_j^{-1}(\psi_i(U))$, so (2.1.2) holds. It follows that $\psi_i(U)$ is thus open in X, and thus ψ_i is an open injective map, and is a bijection onto it's image, and thus a homeomorphism.

Now, denote $\psi_i(X_i)$ by $\mathscr{X}_i$, we want to put the structure of a scheme on $\mathscr{X}_i$. Note that each X_i is a scheme, hence comes equipped with a sheaf of local rings $\mathscr{O}_{X_i}$; we define the sheaf $\mathscr{O}_{\mathscr{X}_i}$ by:

$$\mathscr{O}_{\mathscr{X}_i} = \psi_{i*}\mathscr{O}_{X_i}$$

Since the $\psi_i : X_i \to \mathscr{X}_i$ is a homeomorphism, note that $(\psi_{i*})_x : (\mathscr{O}_{\mathscr{X}_i})_{\psi_i(x)} \to (\mathscr{O}_{\mathscr{X}_i})_x$ is an isomorphism for $x_i \in X_i$. It follows that the stalk of $\mathscr{O}_{\mathscr{X}_i}$ is a local ring, so $\mathscr{O}_{\mathscr{X}_i}$ is a locally ringed space. We now check that $(\mathscr{X}_i, \mathscr{O}_{\mathscr{X}_i})$ is a scheme, let $x \in \mathscr{X}_i$, then there exists an open neighborhood U of $\psi_i^{-1}(x_i) \in X_i$ such that $(U, \mathscr{O}_{X_i}|_U) \cong (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ for some ring A. It thus suffices to check that $(\psi_i(U), \mathscr{O}_{\mathscr{X}_i}|_{\psi_i(U)})$ is isomorphism to $(U, \mathscr{O}_{X_i})$. We first note, that since ψ_i is a homeomorphism, that $\psi_i^{-1} : \psi_i(U) \to U$ is a homeomorphism. So we need only define a morphism $(\psi_i^{-1})^\sharp : \mathscr{O}_{X_i}|_U \to (\psi_i^{-1})_*(\mathscr{O}_{\mathscr{X}_i}|_{\psi_i(U)})$. Let $V \subset U$ be an open set, then:

$$\mathscr{O}_{X_i}|_U(V) = \mathscr{O}_{X_i}(V)$$

while:

$$(\psi_i^{-1})_*(\mathscr{O}_{\mathscr{X}_i}|_{\psi_i(U)})(V) = (\mathscr{O}_{\mathscr{X}_i}|_{\psi_i(U)})(\psi_i(V))$$

since $\psi_i(V) \subset \psi_i(U)$ we have that:

$$\begin{aligned}(\psi_i^{-1})_*(\mathscr{O}_{\mathscr{X}_i}|_{\psi_i(U)})(V) &= (\mathscr{O}_{\mathscr{X}_i})(\psi_i(V)) \\ &= (\psi_{i*}\mathscr{O}_{X_i})(\psi_i(V)) \\ &= \mathscr{O}_{X_i}(\psi_i^{-1}(\psi_i(V))) \\ &= \mathscr{O}_{X_i}(V)\end{aligned}$$

We thus define $(\psi_i^{-1})_V^\sharp$ to be the identity map; it is clear that this commutes with restrictions, hence this assignment defines a natural transformation, and since $(\psi_i^{-1})_V^\sharp$ is the identity for all $V \subset U$ we have that $(\psi_i^{-1})^\sharp$ is an isomorphism as desired. It follows that $(\mathscr{X}_i, \mathscr{O}_{\mathscr{X}_i})$ is a scheme as desired.

Now we have that $\{\mathscr{X}_i\}$ is an open cover of of X, and moreover that $\psi_i : X_i \to \mathscr{X}_i$ is an isomorphism of schemes for each i, by applying the the same argument above to $U = X_i$. If we can show that there exist isomorphisms $\beta_{ij} : \mathscr{O}_{\mathscr{X}_i}|_{\mathscr{X}_i \cap \mathscr{X}_j} \to \mathscr{O}_{\mathscr{X}_j}|_{\mathscr{X}_i \cap \mathscr{X}_j}$, which satisfy the cocycle condition then we will obtain a sheaf on X such that $\mathscr{O}_X|_{\mathscr{X}_i} \cong \mathscr{O}_{\mathscr{X}_i}$ by Theorem 1.2.2. Note that we have:

$$\mathscr{X}_i \cap \mathscr{X}_j = \psi_i(X_i) \cap \psi_j(X_j) = \psi_i(U_{ij}) = \psi_j(U_{ji})$$

Furthermore, since $(X_i, \mathscr{O}_{X_i}) \cong (\mathscr{X}_i, \mathscr{O}_{\mathscr{X}_i})$ via $(\psi_i^{-1}, (\psi_i^{-1})^\sharp)$, we have an inverse map given by $(\psi_i, \psi_i^\sharp)$, where $\psi_i^\sharp : \mathscr{O}_{\mathscr{X}_i} \to \psi_{i*}\mathscr{O}_{X_i}$, so we have map $(\psi_i^\sharp)_V : \mathscr{O}_{\mathscr{X}_i}(V) \to \mathscr{O}_{X_i}(\psi_i^{-1}(V))$. Now note that since $V \subset \psi_i(U_{ij})$, $\psi_i^{-1}(V) \subset U_{ij}$, so:

$$\mathscr{O}_{X_i}(\psi_i^{-1}(V)) = \mathscr{O}_{U_{ij}}(\psi_i^{-1}(V))$$

and we have an isomorphism $\phi_{ji}^\sharp : \mathscr{O}_{U_{ij}} \to \phi_{ji*}\mathscr{O}_{U_{ji}}$. We have that:

$$\phi_{ji}^{-1}(\psi_i^{-1}(V)) = (\psi_i \circ \phi_{ji})^{-1}(V) = \psi_j^{-1}(V)$$

so we have an isomorphism:

$$(\phi_{ji}^\sharp)_{\psi_i^{-1}(V)} : \mathscr{O}_{U_{ij}}(\psi_i^{-1}(V)) \longrightarrow \mathscr{O}_{U_{ji}}(\psi_j^{-1}(V)) = \mathscr{O}_{X_j}(\psi_j^{-1}(V))$$

Finally we have our isomorphism $(\psi_j^{-1})^\sharp : \mathscr{O}_{X_j} \to (\psi_j^{-1})_* \mathscr{O}_{\mathscr{X}_j}$, and note that:

$$(\psi_j^{-1})_* \mathscr{O}_{\mathscr{X}_j}(\psi_j^{-1}(V)) = \mathscr{O}_{\mathscr{X}_j}(V)$$

thus we have that the composition:

$$(\psi_j^{-1})^\sharp_{\psi_j^{-1}(V)} \circ (\phi_{ji})^\sharp_{\psi_i^{-1}(V)} \circ (\psi_i^\sharp)_V$$

is an isomorphism:

$$\mathscr{O}_{\mathscr{X}_i}|_{\mathscr{X}_i \cap \mathscr{X}_j}(V) \to \mathscr{O}_{\mathscr{X}_j}|_{\mathscr{X}_i \cap \mathscr{X}_j}(V)$$

We define β_{ij} on open sets $V \subset \mathscr{X}_i \cap \mathscr{X}_j$ as this composition. We check that this commutes with restriction maps, let $W \subset V$, then we see that:

$$(\psi_i^\sharp)_W \circ \theta_W^V = \theta_W^V \circ (\psi_i^\sharp)_W$$

On $\psi_* \mathscr{O}_{X_i}$, the restriction maps are given by $\theta_W^V = \theta_{\psi_i^{-1}(W)}^{\psi_i^{-1}(V)}$, so we see that:

$$(\phi_{ji}^\sharp)_{\psi_i^{-1}(W)} \circ \theta_W^V = \theta_{\psi_i^{-1}(W)}^{\psi_i^{-1}(V)} \circ (\phi_{ji}^\sharp)_{\psi_i^{-1}(V)}$$

Now on $\phi_{ji*}\mathscr{O}_{U_{ji}}$ the restriction maps are given by:

$$\theta_{\psi_i^{-1}(W)}^{\psi_i^{-1}(V)} = \theta_{\psi_j^{-1}(W)}^{\psi_j^{-1}(V)}$$

as $\psi_i \circ \phi_{ji} = \psi_j$ on U_{ji}, hence:

$$\begin{aligned}(\psi_j^{-1})^\sharp_{\psi_j^{-1}(W)} \circ \theta_{\psi_i^{-1}(W)}^{\psi_i^{-1}(V)} =& (\psi_j^{-1})^\sharp_{\psi_j^{-1}(W)} \circ \theta_{\psi_j^{-1}(W)}^{\psi_j^{-1}(V)} \\ =& \theta_{\psi_j^{-1}(W)}^{\psi_j^{-1}(V)} \circ (\psi_j^{-1})^\sharp_{\psi_j^{-1}(V)}\end{aligned}$$

Finally, on $(\psi_j^{-1})_* \mathscr{O}_{\mathscr{X}_j}$, the restriction maps are given by:

$$\theta_{\psi_j^{-1}(W)}^{\psi_j^{-1}(V)} = \theta_W^V$$

so since $W \subset V \subset \mathscr{X}_i \cap \mathscr{X}_j$, we have that:

$$(\beta_{ij})_W \circ \theta_W^V = \theta_W^V \circ (\beta_{ij})_V$$

so $\beta_{ij} : \mathscr{O}_{\mathscr{X}_i}|_{\mathscr{X}_i \cap \mathscr{X}_j} \to \mathscr{O}_{\mathscr{X}_j}|_{\mathscr{X}_i \cap \mathscr{X}_j}$ is an isomorphism of sheaves as desired. It is clear that $\beta_{ii} = \mathrm{Id}$, so we want to check that $\beta_{ik} = \beta_{jk} \circ \beta ij$ on $\mathscr{X}_i \cap \mathscr{X}_j \cap \mathscr{X}_k$. However this is essentially a tautology, as on all open set $V \subset \mathscr{X}_i \cap \mathscr{X}_j \cap \mathscr{X}_k$:

$$\begin{aligned}(\beta_{jk} \circ \beta_{ij})_V =& (\psi_k^{-1})^\sharp_{\psi_k^{-1}(V)} \circ (\phi_{kj}^\sharp)_{\psi_j^{-1}(V)} \circ (\psi_j^\sharp)_V \circ (\psi_j^{-1})^\sharp_{\psi_j^{-1}(V)} \circ (\phi_{ji})^\sharp_{\psi_i^{-1}(V)} \circ (\psi_i^\sharp)_V \\ =& (\psi_k^{-1})^\sharp_{\psi_k^{-1}(V)} \circ (\phi_{kj}^\sharp)_{\psi_j^{-1}(V)} \circ (\phi_{ji})^\sharp_{\psi_i^{-1}(V)} \circ (\psi_i^\sharp)_V \\ =& (\psi_k^{-1})^\sharp_{\psi_k^{-1}(V)} \circ (\phi_{ki}^\sharp)_{\psi_i^{-1}(V)} \circ (\psi_i^\sharp)_V \\ =& \beta_{ik}\end{aligned}$$

Note that this chain of equality hinges on two statements. First the fact that:

$$(\psi_j^\sharp)_V \circ (\psi_j^{-1})^\sharp_{\psi_j^{-1}(V)} = \mathrm{Id}$$

However this is trivial, as:

$$(\psi_j^{-1})^\sharp_{\psi_j^{-1}(V)} : \mathscr{O}_{X_j}(\psi_j^{-1}(V)) \longrightarrow \mathscr{O}_{\mathscr{X}_j}(V)$$

is the identity map, and:

$$(\psi_j^\sharp)_V : \mathscr{O}_{\mathscr{X}_j}(V) \longrightarrow \mathscr{O}_{X_j}(\psi_j^{-1}(V))$$

is also the identity. The more challenging statement is the following:

$$(\phi_{kj})^\sharp_{\psi_j^{-1}(V)} \circ (\phi_{ji})^\sharp_{\psi_i^{-1}(V)} = (\phi_{ki}^\sharp)_{\psi^{-1}(V)}$$

This follows from the fact that $\phi_{ki} = \phi_{ji} \circ \phi_{kj}$, so:

$$\phi_{ki}^\sharp = \phi_{ji*}\phi_{kj}^\sharp \circ \phi_{ji}^\sharp$$

so we have that:

$$\begin{aligned}(\phi_{ki}^\sharp)_{\psi_i^{-1}(V)} =&(\phi_{ji*}\phi_{kj}^\sharp)_{\psi_i^{-1}(V)} \circ (\phi_{ji}^\sharp)_{\psi_i^{-1}(V)}\\ =&(\phi_{kj}^\sharp)_{\phi_{ji}^{-1}(\psi_i^{-1}(V))} \circ (\phi_{ji}^\sharp)_{\psi_i^{-1}(V)}\\ =&(\phi_{kj}^\sharp)_{\psi_j^{-1}(V)} \circ (\phi_{ji}^\sharp)_{\psi_i^{-1}(V)}\end{aligned}$$

implying the claim. It follows that the $(\mathscr{X}_i, \mathscr{O}_{\mathscr{X}_i})$ glue together to form a sheaf $X, \mathscr{O}_X$ such that $\mathscr{O}_X|_{\mathscr{X}_i} \cong \mathscr{O}_{\mathscr{X}_i}$, implying that each $\psi_i : X_i \to X$ is an open embedding. It is also clear that as morphisms of locally ringed space $\psi_i = \psi_j \circ \phi_{ij}$, essentially by the construction of our maps $\psi_i^\sharp$.

All that remains to show is that $(X, \mathscr{O}_X)$ is a scheme. Let $x \in X$, then $x \in \mathscr{X}_i$ for some i. There is then an isomorphism $(\mathscr{X}_i, \mathscr{O}_{X_i}|_{\mathscr{X}_i})$ to $(\mathscr{X}_i, \mathscr{O}_{\mathscr{X}_i})$, the latter of which is a scheme as it is isomorphic to $(X_i, \mathscr{O}_{X_i})$. Examine the image of $x \in X_i$ under this composition of isomorphisms, and denote it by x_i. Since X_i is a scheme, it follows that there is an open neighborhood V_{x_i} of X_i such that $(V_{x_i}, \mathscr{O}_{X_i}|_{V_i})$ is isomorphic to an affine scheme. Take the preimage of V_{x_i} under this composition of isomorphism, and we obtain an open neighborhood of x whose image under the composition of isomorphisms is isomorphic to an affine scheme. It follows that x has an open neighborhood W_x such that $(W_x, \mathscr{O}_X|_W)$ is isomorphic to an affine scheme implying the claim. □

We have the obvious corollary:

Corollary 2.1.2. *Let $\{X_i\}$ be a family of schemes satisfying the criteria of Theorem 2.1.1, then the scheme X is unique up to unique isomorphism.*

Proof. This follows from Theorem 1.2.2, and the uniqueness of gluing topological spaces together, i.e. uniqueness of the quotient topology and the natural topology on the disjoint union of topological spaces. □

We now show some easy examples of non affine schemes:

Example 2.1.2. Let $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$, and note that for any $(z_1, z_2) \in \mathbb{C}$, the ideal $\langle x - z_1, y - z_2\rangle$ is prime. It suffices to check that $\mathbb{C}[x, y]/\langle x - z_1, y - z_2\rangle$ is an integral domain; in fact, we claim that $\mathbb{C}[x, y]/\langle x - z_1, y - z_2\rangle \cong \mathbb{C}$.

We define a map $\phi : \mathbb{C}[x, y] \to \mathbb{C}$ by $p \mapsto p(z_1, z_2)$. This clearly a surjective morphism as the constant polynomial $p(x) = w$ maps to $w \in \mathbb{C}$. We thus see that $\mathbb{C} \cong \mathbb{C}[x, y]/\ker\phi$. Clearly $\langle x - z_1, y - z_2\rangle \subset \ker\phi$, suppose $p \in \ker\phi$, and write:

$$p = \sum_{ij} a_{ij} x^i y^j$$

Note that $x^n = (x - z_1 + z_1)^n$, and $y^n = (y - z_2 + z_2)^n$, hence there exists a P such that:

$$p(x, y) = P(x - z_1, y - z_2)$$

so $p(z_1, z_2) = P(0,0) = 0$, hence P as 0 constant term. Every term is then divisible by x or y, and we thus have that there exist polynomials Q and R such that:

$$P(x,y) = xQ(x,y) + yR(x,y)$$

so:

$$p(x,y) = (x - z_1)Q(x - z_1, y - z_2) + (y - z_2)R(x - z_1, y - z_2)$$

and so $p(x,y) \in \langle x - z_1, y - z_1 \rangle$ implying that $\mathbb{C} \cong \mathbb{C}[x,y]/\langle x - z_1, y - z_2 \rangle$.

We define a map $\psi : \mathbb{C}[x,y] \to \mathbb{C}$ by $p \longmapsto p(0,0)$, i.e. we evaluate the polynomial p in two variables at the point $(0,0$. Note that this is clearly a ring homomorphism, and that if $p \in \langle x,y \rangle$, that $p(0,0) = 0$, so $\langle x,y \rangle \subset \ker \psi$. We also see that if $p \in \ker \psi$, then the leading coefficient of p must be 0. It follows that:

$$p = \sum_{ij} w_{ij} x^i y^j$$

where if $i = j$, then $j \neq 0$, so:

$$\begin{aligned} p =& x \sum_{i>0,j} w_{ij} x^{i-1} y^j + \sum_{i=0,j} w_{ij} x^i y^j \\ =& x \sum \sum_{i>0,j} w_{ij} x^{i-1} y^j + y \sum_{i=0,j} w_{ij} x^i y^{j-1} \in \langle x,y \rangle \end{aligned}$$

so $\ker \psi = \langle x,y \rangle$. Moreover, this map is clearly surjective, as if $z \in \mathbb{C}$, the constant polynomial $z \in \mathbb{C}[x,y]$ maps to z as well. We thus get a unique isomorphism $\psi' : \mathbb{C}[x,y]/\langle x,y \rangle \to \mathbb{C}$ by the universal property of quotient rings. It follows that $\mathbb{C}[x,y]/\langle x - z_1, y - z_2 \rangle \cong \mathbb{C}$, so every ideal of the form $\langle x - z_1, x - z_2 \rangle$ is maximal[19], and thus prime.

It follows that we can identify $\mathbb{A}^2_{\mathbb{C}}$ with $\mathbb{C}^2$ along with some extra points (such as the zero ideal $\langle 0 \rangle$). We thus denote the ideal $\langle x - z_1, x - z_2 \rangle$ by (z_1, z_2), and claim that $\mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$ is an open subscheme of $\mathbb{A}^2_{\mathbb{C}}$ which is not affine. First note that:

$$\mathbb{A}^2_{\mathbb{C}} \setminus (0,0) = U_x \cup U_y$$

Indeed, if $\mathfrak{p} \in U_x \cup U_y$, then we have $x \notin \mathfrak{p}$ or $y \notin \mathfrak{p}$, hence $\mathfrak{p} \neq (0,0)$ implying that $p \in \mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$. Now suppose that $\mathfrak{p} \in \mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$, then $\mathfrak{p} \neq (0,0)$, in particular, since $(0,0)$ is a maximal ideal we have that $(0,0) \not\subset \mathfrak{p}$. Now suppose that $x \in \mathfrak{p}$ and $y \in \mathfrak{p}$, then we clearly have that $(0,0) \subset \mathfrak{p}$, so either $x \notin \mathfrak{p}$, or $y \notin \mathfrak{p}$, implying that $\mathfrak{p} \in U_x \cup U_y$.

We know that there is a unique scheme structure on $\mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$, and we can further deduce that this must be the one obtained by gluing the sheaf $\mathscr{O}_{U_x}$ to $\mathscr{O}_{U_y}$. Since there are only two sets which cover the space, we need only check that $\mathscr{O}_{U_x}|_{U_x \cap U_y} \cong \mathscr{O}_{U_y}|_{U_x \cap U_y}$. Let $V \subset U_x \cap U_y$, then we have that $V \subset U_x$ and $V \subset U_y$, hence:

$$\mathscr{O}_{U_x}|_{U_x \cap U_y}(V) = \mathscr{O}_{U_x}(V) = \mathscr{O}_X(V)$$

and similarly for O_{U_y}, hence we get a sheaf of rings on $\mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$, and since every element $x \in \mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$ lies in either U_x or U_y, and U_x and U_y are both affine schemes, it follows that with this structure sheaf, $\mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$ is the scheme isomorphic to the open subscheme $U_x \cup U_y$. We want to show that $\mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$ is not affine; denote by X the open subscheme $\mathbb{A}^1_{\mathbb{C}} \setminus (0,0)$, we want to calculate $\mathscr{O}_X(X)$. Note that by our work in Theorem 1.2.2 we have that:

$$\mathscr{O}_X(X) = \{(s_x, s_y) \in \mathscr{O}_{U_x}(U_x) \times \mathscr{O}_{U_y}(U_y) : s_x|_{U_x \cap U_y} = s_y|_{U_x \cap U_y}\}$$

as the morphism $\mathscr{O}_{U_x}|_{U_x \cap U_y} \to \mathscr{O}_{U_x}|_{U_x \cap U_y}$ is the identity morphism. Now note that since U_x and U_y are distinguished open sets, we have that:

$$\mathscr{O}_{U_x}(U_x) \cong (\mathbb{C}[x,y])_x = \mathbb{C}[x,y,1/x] \qquad \mathscr{O}_{U_y}(U_y) \cong (\mathbb{C}[x,y])_y = \mathbb{C}[x,y,1/y]$$

[19]In particular, it is a standard fact that any maximal ideal of a polynomial ring over $k = \bar{k}$ is of this form.

while we have that:

$$\mathscr{O}_{U_x}|_{U_x\cap U_y} = \mathscr{O}_{U_y}|_{U_x\cap U_y} \cong \mathbb{C}[x,y,1/x,1/y]$$

By our earlier work on affine schemes, we know that the restriction maps (up to isomorphism) here are just the obvious inclusions. It follows that if $s_x|_{U_x\cap U_y} = s_y|_{U_x\cap U_y}$, then s_x and s_y are in the image of the injections $\mathbb{C}[x,y] \to \mathbb{C}[x,y,1/x]$, and $\mathbb{C}[x,y] \to \mathbb{C}[x,y,1/y]$, as the must be polynomials with no $1/x$ or $1/y$ terms. It also follows that the preimages of s_x and s_y under these injections must be equal as well, hence:

$$\mathscr{O}_X(X) \cong \{(p,q) \in \mathbb{C}[x,y]\times\mathbb{C}[x,y] : p = q\} \cong \mathbb{C}[x,y]$$

Now suppose that X is affine, then we have that there is an isomorphism $(X,\mathscr{O}_X) \cong (\operatorname{Spec} A, \mathscr{O}_A)$ for some commutative ring A. We thus have that $\mathscr{O}_X(X) \cong A$, but we have just shown that $\mathscr{O}_X(X) \cong \mathbb{C}[x,y]$, implying that $X \cong \operatorname{Spec}\mathbb{C}[x,y]$ as topological spaces. Now in an affine scheme there is a bijection between the points of $\mathbb{A}^2_{\mathbb{C}}$ and the prime ideals of $\mathbb{C}[x,y]$, however $\mathbb{A}^2_{\mathbb{C}} \smallsetminus (0,0)$ is missing the prime ideal $(0,0)$, so it cannot be affine.

Example 2.1.3. Let $\{X_i\}$ be a family of affine schemes, and then we claim that:

$$X = \coprod_i X_i$$

equipped with the natural disjoint union topology is a scheme which is affine if and only if the the family is finite. We can prove one direction immediately, suppose that X is an affine scheme, then we need to show that the family is finite. We prove this by the contrapositive, i.e. if the family is infinite then X is not affine, and we prove the contrapositive by contradiction. Assume that X is affine, then every open cover of X has a finite subcover by Lemma 1.4.1, however this is clearly not true as the infinite disjoint union of any family of topological spaces cannot be quasi-compact[20]. It follows that if the family is infinite then X is not affine, hence if X is affine then the family is finite.

Now suppose that the family is finite, by induction, and the associativity of the disjoint operation on topological spaces, it suffices to check that check that:

$$\operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2$$

is affine for any two rings A and B. Indeed, we claim that:

$$\operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2 \cong \operatorname{Spec}(A_1 \times A_2)$$

In particular, we claim that $\operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2$ is the coproduct in the category of affine schemes. Set $X = \operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2$, then we want to first show that X is a scheme. Let $U \subset X$ be open, then we have that $\psi_1^{-1}(U) \subset \operatorname{Spec} A_1$ and $\psi_2^{-1}(U) \subset \operatorname{Spec} A_2$ are both open, where ψ_1 and ψ_2 are the canonical injections. We define:

$$\mathscr{O}_X(U) = \mathscr{O}_{\operatorname{Spec} A_1}(\psi_1^{-1}(U)) \times \mathscr{O}_{\operatorname{Spec} A_2}(\psi_2^{-1}(U))$$

and restriction maps to be $\theta^U_V = (\theta^{\psi_1^{-1}(U)}_{\psi_1^{-1}(V)}, \theta^{\psi_2^{-1}(U)}_{\psi_2^{-1}(V)})$ We check that this is a sheaf, let $s \in \mathscr{O}_X(U)$, and U_i an open cover of U such that $s|_{U_i} = 0$ for all U_i, we want to show that $s = 0$. First note that we can write $s = (s_1, s_2) \in \mathscr{O}_{\operatorname{Spec} A_1}(\psi_1^{-1}(U)) \times \mathscr{O}_{\operatorname{Spec} B}(\psi_2^{-1}(U))$, and that

$$\psi_1^{-1}(U) = \bigcup_i \psi_1^{-1}(U_i)$$

and similarly for A_2. It follows that

$$s|_{U_i} = \left(s_1|_{\psi_1^{-1}(U_i)}, s_2|_{\psi_2^{-1}(U_i)}\right)$$

implying that $s_1|_{\psi_1^{-1}(U_i)} = 0$ for all U_i, hence $s_1 = 0$, and similarly for A_2 implying sheaf axiom one. The same argument adapted to sheaf axiom two implies that this indeed a sheaf.

[20]A topological space is quasi-compact if every open cover has a finite subcover. Note that this is often taken as the definition of compactness, but for some reason algebraic geometer's prefer this nomenclature.

Now note that as a set:

$$X = \bigcup_i \{(\mathfrak{p}, i) : \mathfrak{p} \in A_i\}$$

so we define a map:

$$\eta : X \longrightarrow \mathrm{Spec}(A \times B)$$

by:

$$\eta((\mathfrak{p}, i)) = \begin{cases} \mathfrak{p} \times A_2 & \text{if } i = 1 \\ A_1 \times \mathfrak{p} & \text{if } i = 2 \end{cases}$$

We note that if $\mathfrak{p} \subset A_1$ is prime, then $\mathfrak{p} \times A_2$ is prime. Indeed, let (a, b) and (c, d) lie in $A_1 \times A_2$ such that $(ac, cd) \in \mathfrak{p} \times A_2$, then it follows that $ac \in \mathfrak{p}$, hence $a \in \mathfrak{p}$ or $c \in \mathfrak{p}$, implying that (a, b) or (c, d) in $\mathfrak{p} \times A_2$ so $\mathfrak{p} \times A_2$ is prime. It follows that this map is well defined. It is clearly injective, as we can't have $\mathfrak{p} \times A_2 = A_1 \times \mathfrak{q}$, so if $\mathfrak{p} \times A_2 = \mathfrak{q} \times A_2$, then this implies that $\mathfrak{p} = \mathfrak{q}$ hence $(\mathfrak{p}, 1) = (\mathfrak{q}, 1)$. To check that this map is surjective, first note that $\mathfrak{p} \times \mathfrak{q}$ is not prime for any ideals (not necessarily prime) $\mathfrak{p} \subset A_1$ and $\mathfrak{q} \subset A_2$ in $A_1 \times A_2$, where $\mathfrak{p}$ and $\mathfrak{q}$ are not the whole ring. Indeed, if $a \in \mathfrak{p}$, $b \in A_2$, $c \in A_1$, $d \in \mathfrak{q}$, then $(a, b), (c, d) \notin \mathfrak{p} \times \mathfrak{q}$, but $(ac, cd) \in \mathfrak{p} \times \mathfrak{q}$. Now let $\mathfrak{q} \subset A_1 \times A_2$ be a prime ideal, then it follows that $\mathfrak{q}_1 = \pi_2(\mathfrak{q})$ is an ideal and $\mathfrak{q}_2 = \pi_2(\mathfrak{p})$ are ideals of A and B respectively as the surjective image of an ideal is an ideal. We claim that:

$$\mathfrak{q} = \mathfrak{q}_1 \times \mathfrak{q}_2$$

It is clear that $\mathfrak{q} \subset \mathfrak{q}_1 \times \mathfrak{q}_2$, so now let $(a, b) \in \mathfrak{q}_1 \times \mathfrak{q}_2$. This implies that $(a, s_2) \in \mathfrak{q}$ and a $(s_1, b) \in \mathfrak{q}$ for some $s_i \in A_i$. Note that since $\mathfrak{q}$ is an ideal, we thus have that $(a, s_1) \cdot (1, 0) = (a, 0) \in \mathfrak{q}$, and similarly for $(0, b)$. Since $\mathfrak{q}$ is closed under addition it follows that $(a, b) \in \mathfrak{q}$. Since $\mathfrak{q}$ is prime however, we must have that $\mathfrak{q}_i = A_i$ for $i = 1$ or 2. Without loss of generality suppose that $\mathfrak{q}_2 = A_2$, then $\mathfrak{q}_1$ must be prime, as if $a \cdot c \in \mathfrak{q}_1$, then we must have that $(a, b) \cdot (c, d) \in \mathfrak{q} = \mathfrak{q}_1 \times A_2$, hence either $(a, b) \in \mathfrak{q}$ or $(c, d) \in \mathfrak{q}$, implying that either $a \in \mathfrak{q}_1$ or $c \in \mathfrak{q}_1$. Now let $\mathfrak{q} \subset A \times B$ be prime, then $\mathfrak{q} = \mathfrak{q}_1 \times A_2$ or $\mathfrak{q} = A_1 \times \mathfrak{q}_2$ where $\mathfrak{q}_i$ is prime, so it follows that $(\mathfrak{q}_i, i) \in X$, and satisfies $\eta((\mathfrak{q}_i, i)) = \mathfrak{q}$, so η is surjective.

We check that the map is continuos, by noting that η is continuous if and only if $\eta \circ \psi_i$ is continuous for each i. It suffices to check this on distinguished open sets. Let $U_{(a,b)} \subset \mathrm{Spec}\, A_1 \times A_2$ be a distinguished open, then:

$$\begin{aligned} \eta^{-1}(U_{(a,b)}) =& \{(\mathfrak{p}, i) \in X : \eta((\mathfrak{p}, i)) \in U_{(a,b)}\} \\ =& \{(\mathfrak{p}, i) : (a, b) \notin \eta((\mathfrak{p}, i))\} \end{aligned}$$

Then for $i = 1$:

$$\begin{aligned} \psi_1^{-1}(\eta^{-1}(U_{(a,b)})) =& \{\mathfrak{p} \in \mathrm{Spec}\, A_1 : \psi_1(\mathfrak{p}) \in \eta^{-1}(U_{(a,b)})\} \\ =& \{\mathfrak{p} \in \mathrm{Spec}\, A_1 : (\mathfrak{p}, 1) \in \eta^{-1}(U_{(a,b)})\} \\ =& \{\mathfrak{p} \in \mathrm{Spec}\, A_1 : \mathfrak{p} \times A_2 \in U_{(a,b)}\} \\ =& \{\mathfrak{p} \in \mathrm{Spec}\, A_1 : (a, b) \notin \mathfrak{p} \times \mathbb{A}_2\} \\ =& \{\mathfrak{p} \in \mathrm{Spec}\, A_1 : a \notin \mathfrak{p}\} \\ =& U_a \end{aligned}$$

similarly:

$$\psi_2^{-1}(\eta^{-1}(U_{(a,b)})) = U_b$$

so η is continuous. We want to show that η is an open map. Let $U \subset X$ be open, then:

$$U = \psi_1(\psi_1^{-1}(U)) \cup \psi_2(\psi_2^{-1}(U))$$

We can write $\psi_i^{-1}(U)$ as union of distinguished opens, hence:

$$U = \bigcup_j \psi_1(U_{a_j}) \cup \bigcup_k \psi_2(U_{b_k})$$

Taking the image of this under η we find that:

$$\eta(U) = \bigcup_j \eta(\psi_1(U_{a_j})) \cup \bigcup_k \eta(\psi_2(U_{b_k}))$$

so it suffices to check that $\eta \circ \psi_i$ is an open map. Let $U_a \subset \operatorname{Spec} A_1$, then:

$$\psi_1(U_a) = \{(\mathfrak{p}, 1) \in X : a \notin \mathfrak{p}\}$$

so:

$$\eta(\psi_1(U_a)) = \{\mathfrak{p} \times A_1 : a \notin \mathfrak{p}\}$$

We claim that:

$$\eta(\psi_1(U_a)) = U_{(a,0)} = \{\mathfrak{q} \in \operatorname{Spec}(A_1 \times A_2) : (a, 0) \notin \mathfrak{q}\}$$

Let $\mathfrak{q}$ in $U_{(a,0)}$, then $\mathfrak{q} \neq A_1 \times \mathfrak{p}$ for some $\mathfrak{p} \subset A_2$ as $a \in A_1$ and $0 \in \mathfrak{p} \subset A_2$. It follows that $\mathfrak{q} = \mathfrak{p} \times A_2$ for some $\mathfrak{p} \subset A_1$, and that $a \notin \mathfrak{p}$, hence $q \in \eta(\psi_1(U_a))$. Now suppose that $p \times A_1 \in \eta(\psi_1(U_a))$, then $a \notin \mathfrak{p}$, hence $(a, 0) \notin \mathfrak{p} \times A_1$, so $\mathfrak{p} \times A_1 \in U_{(a,0)}$. A similar proof follows for ψ_2, hence $\eta(U)$ is the union of open sets and is thus open. It follows that η is a homeomorphism as it is an open continuous bijection.

We now want to define sheaf isomorphism:

$$\eta^\sharp : \mathscr{O}_{\operatorname{Spec}(A \times B)} \longrightarrow \eta_* \mathscr{O}_X$$

It suffices to define the sheaf morphism on basic open sets of $\operatorname{Spec}(A \times B)$. Let $U_{(a,b)}$ be a basic open, and let $V = \eta^{-1}(U_{(a,b)}) \subset X$, then note that:

$$V = \psi_1(\psi_1^{-1}(V)) \cup \psi_2(\psi_2^{-1}(V))$$

hence:

$$\begin{aligned} (\eta_* \mathscr{O}_X)(U_{(a,b)}) =& \mathscr{O}_{\operatorname{Spec} A_1}(\psi_1^{-1}(V)) \times \mathscr{O}_{\operatorname{Spec} A_2}(\psi_2^{-1}(V)) \\ =& \mathscr{O}_{\operatorname{Spec} A_1}(U_a) \times \mathscr{O}_{\operatorname{Spec} A_1}(U_b) \\ =& (A_1)_a \times (A_2)_b \end{aligned}$$

It thus suffices to show by Corollary 1.4.2 that:

$$(A_1 \times A_2)_{(a,b)} \cong (A_1)_a \times (A_2)_b$$

and that the isomorphisms commute with restrictions on a base. We define a map $A_1 \times A_2 \to (A_1)_a \times (A_2)_b$ by:

$$(s, t) \longmapsto \left(\frac{s}{1}, \frac{t}{1}\right)$$

and note that the image (a, b) is a unit with inverse given by $(1/a, 1/b)$ so we obtain a unique map:

$$\begin{aligned} \phi : (A_1 \times A_2)_{(a,b)} &\longrightarrow (A_1)_a \times (A_2)_b \\ \frac{(s,t)}{(a,b)^k} &\longmapsto \left(\frac{s}{a^k}, \frac{t}{b^k}\right) \end{aligned}$$

It is clear that this map is surjective. If $\phi((s,t)/(a,b)^k) = 0$, then we have that:

$$\frac{s}{a^k} = 0 \qquad \text{and} \qquad \frac{t}{b^k} = 0$$

implying that there exists some m and some n such that:

$$a^m s = 0 \qquad \text{and} \qquad b^n t = 0$$

Let $K > \max\{m, n\}$ then:

$$a^K s = 0 \qquad \text{and} \qquad b^K t = 0$$

so:

$$(a,b)^K(s,t) = (a^K s, b^K t) = (0,0)$$

implying that $(s,t)/(a,b)^k = 0$ as well, so ϕ is injective and an isomorphism. It is clear (albeit a little messy to check explicitly) that the isomorphisms commute with restrictions on a base, hence $X \cong \operatorname{Spec}(A_1 \times A_2)$ and is thus an affine scheme.

To see that X is a coproduct, it suffices to check that $\operatorname{Spec}(A_1 \times A_2)$ satisfies the properties of the coproduct. Note that we have natural morphisms $\operatorname{Spec} A_i \to \operatorname{Spec}(A_1 \times A_2)$ given by the map $\pi_i^{-1} : \operatorname{Spec} A_i \to \operatorname{Spec}(A_1 \times A_2)$, and the induced map $\pi_i^\sharp : \mathscr{O}_{\operatorname{Spec}(A_1 \times A_2)} \to \operatorname{Spec}(A_i)$. Since there is an isomorphism:

$$\operatorname{Hom}(A,B) \cong \operatorname{Hom}(\operatorname{Spec} B, \operatorname{Spec} A)$$

and $A_1 \times A_2$ satisfies the universal property of the product in the category of rings, it follows that $\operatorname{Spec}(A_1 \times A_2)$ must satisfy the universal property of the coproduct, hence so must X. In particular, the isomorphism $X \cong \operatorname{Spec}(A_1 \times A_2)$ is unique.

As the preceding example states, the infinite disjoint unions of schemes (affine or not) is not affine. We will show later in this section that the disjoint union of schemes is the coproduct in the category of schemes. Funnily enough however, the product of schemes is not in general a scheme, so schemes are a category without products.

Example 2.1.4. Let $X = \mathbb{A}^1_{\mathbb{C}}$, i.e. the affine scheme $\operatorname{Spec} \mathbb{C}[x]$. Let Y be another copy of of $\mathbb{A}^1_{\mathbb{C}}$ but instead use the variable y for book keeping purposes (i.e. $Y = \operatorname{Spec} \mathbb{C}[y]$). Now examine $U_x \subset X$ and $U_y \subset Y$, both of these are affine schemes isomorphic to $\operatorname{Spec} \mathbb{C}[x, 1/x]$ and $\operatorname{Spec} \mathbb{C}[y, 1/y]$ respectively. Note that $\mathbb{C}$ is algebraically closed, so the only prime ideals of $\mathbb{C}[x]$ are of the form $x - z$ for some $z \in \mathbb{C}$, and of course the zero ideal $\langle 0 \rangle$. It follows that U_x and U_y contain every ideal but the ideal $\langle x \rangle$. Furthermore, with this identification we can truly vies $\mathbb{A}^1_{\mathbb{C}}$ as $\mathbb{C}$ with an extra point $\langle 0 \rangle$ which is 'close' to every other point. Obviously the usual topology on $\mathbb{C}$ differs from the one on $\mathbb{A}^1_{\mathbb{C}}$; in particular $\mathbb{A}^1_{\mathbb{C}}$ is clearly non Hausdorff.

We wish to glue these two schemes together along U_x and U_y. Since U_x and U_y are affine schemes, it suffices to give a ring isomorphism $\mathbb{C}[x, 1/x] \to \mathbb{C}[y, 1/y]$. We give the obvious one induced by the map $\mathbb{C}[x] \to \mathbb{C}[y]$ given by $x \mapsto y$. Clearly this isomorphism descends to an isomorphism $\mathbb{C}[x, 1/x] \mapsto \mathbb{C}[y, 1/y]$ which takes $x \mapsto y$ and $1/x \mapsto 1/y$. Since there are only two schemes to glue, there is only one subset of X and Y respectively to glue, and only one isomorphism $\phi_{xy} : U_x \to U_y$ so the conditions of Theorem 2.1.1 are trivially satisfied. Denote the induced scheme by Z, and note that the topological space:

$$Z = \left(X \coprod Y \right) / \sim$$

looks like $\mathbb{A}^1_{\mathbb{C}}$ with two origins $\langle x \rangle$ and $\langle y \rangle$. Indeed, the embeddings $\psi_x : X \to Z$ and $\psi_y : Y \to Z$ satisfies:

$$\psi_x(\langle x - z \rangle) = \psi_y(\langle y - z \rangle)$$

for all $z \neq 0$, and also agree on the zero ideal. We wish to show that this scheme is not affine, and we do so by calculating the ring of global sections. Now note that:

$$Z = \mathscr{X} \cup \mathscr{Y}$$

and so:

$$\mathscr{O}_Z(Z) = \{(s_x, s_y) \in \mathscr{O}_{\mathscr{X}}(\mathscr{X}) \times \mathscr{O}_{\mathscr{Y}}(\mathscr{Y}) : s_x|_{\mathscr{X} \cap \mathscr{Y}} \cong \beta_{yx}(s_y|_{\mathscr{X} \cap \mathscr{Y}})\}$$

We have that $\mathscr{X} \cong \mathscr{Y} \cong \operatorname{Spec} \mathbb{C}[x]$, and that:

$$\mathscr{X} \cap \mathscr{Y} = \psi_x(X) \cap \psi_y(Y) = \psi_x(U_x) \cong \operatorname{Spec} \mathbb{C}[x, 1/x]$$

so under these identifications β_{yx} is equivalent to the map induced by $x \mapsto x$, hence:

$$\mathscr{O}_Z(Z) \cong \{(p, q) \in \mathbb{C}[x] \times \mathbb{C}[x] : \pi_x(p) = \pi_x(q)\}$$

where $\pi_x : \mathbb{C}[x] \to \mathbb{C}[x, 1/x]$ is the localization map. It follows that since the localization map of an integral domain is an injection that:

$$\mathscr{O}_Z(Z) \cong \mathbb{C}[x]$$

so if Z is affine then $Z \cong \operatorname{Spec}\mathbb{C}[x]$, but Z contains two copies of the zero ideal, hence cannot be affine by the same argument as in Example 2.1.3

This demonstrates an analogue of a failure of a scheme to be Hausdorff, in the sense that the $\mathbb{C}$ glued to itself everywhere except the origin is non Hausdorff. We will make this notion precise when we discuss separatedness. In our next example, we again glue two copies of an affine scheme together, just via a different isomorphism.

Example 2.1.5. Let X Y, $U_x \subset X$ and $U_y \subset Y$ be as previously defined in Example 2.1.4. Consider the map:

$$\mathbb{C}[x] \longrightarrow \mathbb{C}[y, 1/y]$$

induced by the assignment:

$$x \longmapsto 1/y$$

We note that $1/y$ is a unit in $\mathbb{C}[y, 1/y]$, hence this descend to a unique morphism:

$$\phi : \mathbb{C}[x, 1/x] \longrightarrow \mathbb{C}[y, 1/y]$$

We check that this is an isomorphism, $p \in \mathbb{C}[x, 1/x]$ be a polynomial such that $\phi(p) = 0$. We see that p can be written uniquely as:

$$p = \sum_{i=-n}^{m} z_i x^i$$

for some $n, m \in \mathbb{Z}^+$, and some $z_i \in \mathbb{C}$. It follows that:

$$\phi(p) = \sum_{i=-m}^{n} z_i y^i$$

and for this to be the zero polynomial, we must clearly have that $z_i = 0$ for all i, hence $p = 0$, so ϕ is injective. Clearly ϕ is surjective, as we can just invert any polynomial in $\mathbb{C}[y, 1/y]$ term by term and replace the variable y with x. It follows that ϕ is an isomorphism, with inverse induced by the assignment $y \mapsto 1/x$, so we obtain an isomorphism of schemes $U_x \mapsto U_y$ which trivially satisfy the criteria of Theorem 2.1.1.

As before, we first describe the topological space:

$$Z = \left(X \coprod Y\right) / \sim$$

and then calculate the ring of global sections. First note that the prime ideals $\langle x - z \rangle$ gets mapped to the prime ideal:

$$\eta(\langle x - z \rangle) = \left\{ \frac{p}{x^k} \in \mathbb{C}[x, 1/x] : p \in \langle x - z \rangle, k \geq 0 \right\}$$

so this is the ideal $\langle (x - z)/1 \rangle \subset \mathbb{C}[x, 1/x]$. Under the isomorphism $\phi : \mathbb{C}[y, 1/y] \to \mathbb{C}[x, 1/x]$[21] we see that ϕ induces a homeomorphism f given by:

$$f(\langle x - z \rangle) = \langle 1/y - z \rangle \in \operatorname{Spec}\mathbb{C}[y, 1/y]$$

We claim that $\langle 1/y - z \rangle = \langle y - 1/z \rangle$ in $\mathbb{C}[y, 1/y]$. Let $p \in \langle 1/y - z \rangle$, then:

$$p = q \cdot (1/y - z)$$

[21] Abuse of notation alert! This is the technically the inverse of ϕ, but for notational reasons we redefined ϕ as it's inverse.

for some $q \in \mathbb{C}[y, 1/y]$. Now note that element $(-y \cdot 1/z)/1$ is invertible in $\mathbb{C}[y, 1/y]$ hence we have that:

$$\begin{aligned} p =& q \cdot ((-y \cdot 1/z)/1) \cdot ((-y \cdot 1/z)/1)^{-1} \cdot (1/y - z) \\ =& q \cdot ((-y \cdot 1/z)/1)^{-1} \cdot (y - 1/z) \end{aligned}$$

so $p \in \langle y - 1/z \rangle$. The same argument in reverse demonstrates the other inclusion hence $\langle 1/y - z \rangle = \langle y - 1/z \rangle$. It follows that the ideal $\langle x - z \rangle$ is identified with ideal $\langle y - 1/z \rangle$ for all $z \neq 0$, and that $\langle 0 \rangle$ is identified with $\langle 0 \rangle$. As a set, we can make more this feel more familiar, identify X and Y with $\mathbb{C} \cup \{\langle 0 \rangle\}$ and define the map:

$$F : X \coprod Y \longrightarrow \mathbb{P}^2 \cup \{\langle 0 \rangle\}$$

where $\mathbb{P}^1 = \mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{C}^\times$, by:

$$F(z) = \begin{cases} \{0\} \text{ if } z \in X \text{ or } x \in Y \text{ and } z = \{0\} \\ [z, 1] \text{ if } z \in X \\ [1, z] \text{ if } z \in Y \end{cases}$$

We see that $z \neq 0 \in X$ and $1/z \in Y$ then:

$$F(z) = [z, 1] = [1, 1/z] = F(1/z)$$

and similarly for $1/z \in X$ and $z \in Y$, hence there is a unique set map:

$$F' : Z \longrightarrow \mathbb{P}^1 \cup \{\langle 0 \rangle\}$$

This is surjective, as if $[w, z] \in \mathbb{P}^1$, both of which are non zero, then $[w, z] = [1, z/w]$ so $[z/w] \in Z$ maps to $[1, z/w]$. If either w or z is zero then $[w, z] = [0, 1]$ or $[1, 0]$ respectively, and the elements $[0_x]$ and $[0_y]$[22] map to $[0, 1]$ and $[1, 0]$ respectively. Moreover, the ideal $\langle 0 \rangle$ gets mapped to $\langle 0 \rangle$. The same argument in reverse essentially proves that F' is injection, and is thus a set isomorphism. For this reason, we see that Z is an algebraic geometry analogue of projective space, and thus we denote Z by $\mathbb{P}^1_{\mathbb{C}}$.

To see that $\mathbb{P}^1_{\mathbb{C}}$ is not affine, we calculate the ring of global sections. We see that:

$$\mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(\mathbb{P}^1_{\mathbb{C}}) = \{(s_x, s_y) \in \mathscr{O}_{\mathscr{X}}(\mathscr{X}) \times \mathscr{O}_{\mathscr{Y}}(\mathscr{Y}) : s_x|_{\mathscr{X} \cap \mathscr{Y}} = \beta_{yx}(s_y|_{\mathscr{X} \cap \mathscr{Y}})\}$$

As before, we have that $\mathscr{O}_{\mathscr{X}}(\mathscr{X}) \cong \mathscr{O}_{\mathscr{Y}}(\mathscr{Y}) \cong \mathbb{C}[x]$, and that:

$$\mathscr{X} \cap \mathscr{Y} = \psi_x(X) \cap \psi_y(Y) = \psi_x(U_x) \cong \operatorname{Spec} \mathbb{C}[x, 1/x]$$

so under these identifications, β_{yx} is equivalent to the map given by $x \mapsto 1/x$. It follows that:

$$\mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(\mathbb{P}^1_{\mathbb{C}}) \cong \{(p, q) \in \mathbb{C}[x] \times \mathbb{C}[y] : \pi(p) = \beta_{yx}(\pi(q))\}$$

Let $p = \sum_i z_i x^i$, and $q = \sum_i w_i x^i$, then we see that if $\pi(p) = \beta_{yx}(\pi(q))$ we must have that:

$$\sum_i z_i x^i = \sum_i w_i x^{-i}$$

hence $z_i = w_i = 0$ for $i > 0$, and $z_0 = w_0$. It follows that:

$$\mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(\mathbb{P}^1_{\mathbb{C}}) \cong \mathbb{C}$$

so if $\mathbb{P}^1_{\mathbb{C}}$ was affine we would have that $\mathbb{P}^1_{\mathbb{C}} \cong \operatorname{Spec} k = \langle 0 \rangle$, which obviously cannot be the case.

This is our first example of what we will call a projective scheme. The entirety of the next section will be dedicated to the construction of the map (not a functor!)Proj : Ring $\to$ Scheme. In particular, we will have that $\mathbb{P}^n_k = \operatorname{Proj}(k[x_0, x_1, \ldots, x_n])$.

We continue with an extension of Example 2.1.3.

Proposition 2.1.1. *Let X and Y be schemes, then the topological space $X \coprod Y$ has the natural structure of a scheme, and is the coproduct in the category of schemes.*

[22]We use this notation to denote the image of $0 \in X$ and $0 \in Y$ under the open embeddings ψ_x and ψ_y respectively.

Proof. Note that $\emptyset \subset X$ and $\emptyset \subset Y$, and since $\emptyset$ is an open subset of X and Y, it follows that $\emptyset$ is an open subscheme of X and Y, and there is an obvious isomorphism between the two. Since there are only two schemes to glue, it follows that this satisfies the criteria of Theorem 2.1.1, hence:

$$Z = \left(X \coprod Y\right) / \sim$$

has the natural structure of a scheme. However, this equivalence relation is the trivial one, hence:

$$Z = X \coprod Y$$

It follows that $X \coprod Y$ has the natural structure of a scheme, and we have have that the canonical injections $\psi_X : X \to Z$ and $\psi_Y : Y \to Z$ are scheme isomorphisms onto their images.

Let $U \subset Z$ be open, then we have that since the gluing is trivial:

$$\begin{aligned}
\mathscr{O}_Z(U) =& \mathscr{O}_{\mathscr{X}}(U \cap \mathscr{X}) \times \mathscr{O}_{\mathscr{Y}}(U \cap \mathscr{Y}) \\
=& \mathscr{O}_{\mathscr{X}}(U \cap \psi_X(X)) \times \mathscr{O}_{\mathscr{Y}}(U \cap \psi_Y(Y)) \\
=& \mathscr{O}_X(\psi_X^{-1}(U) \cap X) \times \mathscr{O}_Y(\psi_Y^{-1}(U) \cap Y) \\
=& \mathscr{O}_X(\psi_X^{-1}(U)) \times \mathscr{O}_Y(\psi_Y^{-1}(U))
\end{aligned}$$

which is exactly the structure sheaf we put on $\operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2$. Note that the Z already satisfies the universal property of the coproduct in the category of topological, spaces, i.e. for any topological space W and morphisms $\phi_X : X \to W$ and $\phi_Y : Y \to W$ there exists a unique morphism $\phi : Z \to W$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, "\psi_X"] \arrow[ddr, bend right, "\phi_X"'] & & Y \arrow[dl, "\psi_Y"'] \arrow[ddl, bend left, "\phi_Y"] \\
& Z \arrow[d, "\phi"] & \\
& W &
\end{tikzcd}$$

We thus need to show that the sheaf morphisms commute ‘in the opposite direction’. Suppose that W is actually a scheme, and the ϕ_X and ϕ_Y are morphisms of schemes, then note that we have:

$$\psi_X^\sharp : \mathscr{O}_Z \to \psi_{X*}\mathscr{O}_X \qquad \text{and} \qquad \phi_X^\sharp : \mathscr{O}_W \to \phi_{X*}\mathscr{O}_X^\sharp$$

and similarly for the Y morphisms. We thus need to construct a unique morphism $\phi^\sharp : \mathscr{O}_W \to \phi_*\mathscr{O}_Z$ such that the following diagram commutes:

$$\begin{tikzcd}
& \mathscr{O}_W \arrow[d, "\phi^\sharp"] \arrow[ddl, bend right, "\phi_X^\sharp"'] \arrow[ddr, bend left, "\phi_Y^\sharp"] & \\
& \phi_*\mathscr{O}_Z \arrow[dl, "\phi_*\psi_X^\sharp"] \arrow[dr, "\phi_*\psi_Y^\sharp"'] & \\
\phi_{X*}\mathscr{O}_X & & \phi_{Y*}\mathscr{O}_Y
\end{tikzcd}$$

First, as a sanity check, lets make sure this diagram makes sense. Note that:

$$\phi_*\psi_X^\sharp : \phi_*\mathscr{O}_Z \longrightarrow \phi_*(\psi_{X*}\mathscr{O}_X)$$

however, we have that $\phi \circ \psi_X = \phi_X$, so:

$$\phi_*(\psi_{X*}\mathscr{O}_X) = (\phi \circ \psi_X)_*\mathscr{O}_X = \phi_{X*}\mathscr{O}_X$$

so the diagram does indeed make sense. Now let U be an open subset of W, we define a map:

$$\phi^\sharp_U : \mathscr{O}_W(U) \longrightarrow \phi_*\mathscr{O}_Z(U)$$

by first noting that:

$$\begin{aligned}
\phi_*\mathscr{O}_Z(U) =& \mathscr{O}_Z(\phi^{-1}(U)) \\
=& \mathscr{O}_X(\psi_X^{-1}(\phi^{-1}(U))) \times \mathscr{O}_Y(\psi_Y^{-1}(\phi^{-1}(U))) \\
=& \mathscr{O}_X((\phi \circ \psi_X)^{-1}(U)) \times \mathscr{O}_Y((\phi \circ \psi_Y)^{-1}(U)) \\
=& \mathscr{O}_X(\phi_X^{-1}(U)) \times \mathscr{O}_Y(\phi_Y^{-1}(U)) \\
=& \phi_{X*}\mathscr{O}_X(U) \times \phi_{Y*}\mathscr{O}_Y(U)
\end{aligned}$$

so the only reasonable definition of U is:

$$\phi^\sharp_U(s) = \Big((\phi^\sharp_X)_U(s), (\phi^\sharp_Y)_U(s)\Big)$$

We check that this commutes with restriction maps. Let $V \subset U$, and $s \in \mathscr{O}_W(U)$ then:

$$\begin{aligned}
\phi^\sharp_V \circ \theta^U_V(s) &= \Big((\phi^\sharp_X)_V, (\phi^\sharp_Y)_V\Big) \circ \theta^U_V(s) \\
&= \Big((\phi^\sharp_X)_V \circ \theta^U_V(s), (\phi^\sharp_Y)_V \circ \theta^U_V(s)\Big)
\end{aligned}$$

since $\phi^\sharp_X$ and $\phi^\sharp_Y$ are natural transformations, we have that:

$$\phi^\sharp_V \circ \theta^U_V(s) = \Big(\theta^U_V \circ (\phi^\sharp_X)_U(s), \theta^U_V \circ (\phi^\sharp_Y)_U(s)\Big)$$

However, note that restriction maps on $\phi_{X*}\mathscr{O}_X$ are given by $\theta^U_V = \theta^{\phi_X^{-1}(U)}_{\phi_X^{-1}(V)}$, so we have that:

$$\phi^\sharp_V \circ \theta^U_V = \left(\theta^{\phi_X^{-1}(U)}_{\phi_X^{-1}(V)} \times \theta^{\phi_Y^{-1}(U)}_{\phi_Y^{-1}(V)}\right) \circ \phi^\sharp_U$$

Now note that the restriction maps on $\phi_*\mathscr{O}_Z$ are given by:

$$\begin{aligned}
\theta^U_V =& \theta^{\phi^{-1}(U)}_{\phi^{-1}(V)} \\
=& \left(\theta^{\psi_X^{-1}(\phi^{-1}(U))}_{\psi_X^{-1}(\phi^{-1}(V))} \times \theta^{\psi_Y^{-1}(\phi^{-1}(U))}_{\psi_Y^{-1}(\phi^{-1}(V))}\right) \\
=& \left(\theta^{\phi_X^{-1}(U)}_{\phi_X^{-1}(V)} \times \theta^{\phi_Y^{-1}(U)}_{\phi_Y^{-1}(V)}\right)
\end{aligned}$$

hence:

$$\phi^\sharp_V \circ \theta^U_V = \theta^U_V \circ \phi^\sharp_U$$

so it follows that $\phi^\sharp : \mathscr{O}_W \to \phi_*\mathscr{O}_Z$ is indeed a natural transformation. We now need to check that $\phi_*\psi^\sharp_X \circ \phi^\sharp = \phi^\sharp_X$, and it suffices to check that they agree on all open sets of W. Recall that $\psi^\sharp_X$ is defined to be the identity on open sets when $U \subset Z$ is entirely contained in $\psi_X(X)$; since $\mathscr{O}_Z(U) = \mathscr{O}_X(\psi_X^{-1}(U)) \times \mathscr{O}_Y(\psi_Y^{-1}(U))$, it follows that $(\psi^\sharp_X)_U$ is the projection:

$$\mathscr{O}_X(\psi_X^{-1}(U)) \times \mathscr{O}_Y(\psi_Y^{-1}(U)) \longrightarrow \mathscr{O}_X(\psi_X^{-1}(U))$$

Now let $U \subset W$ be open, then:

$$\begin{aligned}
(\phi_*\psi^\sharp_X \circ \phi^\sharp)_U =& (\phi_*\psi^\sharp_X)_U \circ \phi^\sharp_U \\
=& (\psi^\sharp_X)_{\phi^{-1}(U)} \circ \Big((\phi^\sharp_X)_U \times (\phi^\sharp_Y)_U\Big)
\end{aligned}$$

Now note that $(\phi_X^\sharp)_U$ has image in $\mathscr{O}_X(\phi_X^{-1}(U)) = \mathscr{O}_X((\phi\circ\psi_X)^{-1}(U))$, and that $(\psi_X^\sharp)_{\phi^{-1}(U)}$ is the projection:

$$\mathscr{O}_X((\phi\circ\psi_X)^{-1}(U)) \times \mathscr{O}_Y((\phi\circ\psi_X)^{-1}(U)) \longrightarrow \mathscr{O}_X((\phi\circ\psi_X)^{-1}(U))$$

hence:

$$(\phi_*\psi_X^\sharp \circ \phi^\sharp)_U = (\phi_X^\sharp)_U$$

for all U. It follows that $\phi_*\psi_X^\sharp \circ \phi^\sharp = \phi_X^\sharp$, and similarly for Y, hence we have that $Z = X \coprod Y$ satisfies the universal property of the coproduct in the category of schemes as desired. $\square$

Before moving on to discuss closed subschemes, we prove the following cute result, which is an analogue of Corollary 1.4.3.

Proposition 2.1.2. *Let X be a scheme and $Y = \operatorname{Spec} A$ be an affine scheme. Then the set of morphisms $\operatorname{Hom}(X,Y)$ is in natural bijection with the set of ring morphisms $\operatorname{Hom}(\mathscr{O}_Y(Y), \mathscr{O}_X(X)) \cong \operatorname{Hom}(A, \mathscr{O}_X(X))$.*

Proof. Let $(f, f^\sharp)$ be a morphism $X \to Y$, then $f_Y^\sharp : \mathscr{O}_Y(Y) \to \mathscr{O}_X(f^{-1}(Y)) = \mathscr{O}_X(X)$ is a ring morphism. Define a set map:

$$\Phi : \operatorname{Hom}(X,Y) \longrightarrow \operatorname{Hom}(\mathscr{O}_Y(Y), \mathscr{O}_X(X))$$

by:

$$(f, f^\sharp) \longmapsto f_Y^\sharp$$

We want to define a map in the other direction, and show that these are inverses of another. Let $\psi : \mathscr{O}_Y(Y) \to \mathscr{O}_X(X)$ be a ring homomorphism, we want to define a map:

$$(f_\psi, f_\psi^\sharp) : X \longrightarrow Y$$

We first determine the topological map $f_\psi : X \to Y$; note that every point in $Y = \operatorname{Spec} A$ can be identified with prime ideal of $\mathscr{O}_Y(Y)$ via the isomorphism $\mathscr{O}_Y(Y) \cong A$. It thus suffices to assign a prime ideal of $\mathscr{O}_Y(Y)$ to each $x \in X$. Let $x \in X$, then we have that the ring $(\mathscr{O}_X)_x$ has a unique maximal (and thus prime) ideal $\mathfrak{m}_x$, and there is a unique stalk map map $\pi_x : \mathscr{O}_X(X) \to (\mathscr{O}_X)_x$. It follows that $\pi_x^{-1}(\mathfrak{m}_x)$ is a prime ideal of $\mathscr{O}_X(X)$, and $\psi^{-1}(\pi_x^{-1}(\mathfrak{m}_x))$ is a prime ideal of $\mathscr{O}_Y(Y)$. Let φ_A be the natural isomorphism $A \to \mathscr{O}_Y(Y)$, then we see that:

$$\varphi_A^{-1}(\psi^{-1}(\pi_x^{-1}(\mathfrak{m}_x))) \subset A$$

is a prime ideal of A, hence we define $f_\psi : X \to Y$ by:

$$f_\psi(x) = \varphi_A^{-1}(\psi^{-1}(\pi_x^{-1}(\mathfrak{m}_x)))$$

We check that this is continuous, and it suffices to check this on distinguished opens $U_a \subset \operatorname{Spec} A$. We see that:

$$\begin{aligned} f^{-1}(U_a) =& \{x \in X : f(x) \in U_a\} \\ =& \{x \in X : a \notin f(x)\} \\ =& \{x \in X : a \notin \varphi_A^{-1}(\psi^{-1}(\pi_x^{-1}(\mathfrak{m}_x)))\} \\ =& \{x \in X : \varphi_A(a) \notin \psi^{-1}(\pi_x^{-1}(\mathfrak{m}_x))\} \\ =& \{x \in X : \pi_x(\psi(\varphi_A(a))) \notin \mathfrak{m}_x\} \end{aligned}$$

Let $\psi(\varphi_A(a)) = g \in \mathscr{O}_X(X)$, then we have that:

$$f_\psi^{-1}(U_a) = \{x \in X : g_x \notin \mathfrak{m}_x\}$$

It thus suffices to show that for every $g \in \mathscr{O}_X)(X)$ the set:

$$X_g = \{x \in X : g_x \notin \mathfrak{m}_x\}$$

is an open set. Cover X with affine open sets $U_i = \operatorname{Spec} B_i$, then we see that:

$$X_g = \bigcup X_g \cap U_i$$

It thus suffices to check that $X_g \cap U_i$ is an open set for each i. Let $\beta_i : U_i \to \operatorname{Spec} B_i$ be an isomorphism of affine schemes, then we have an isomorphism $\mathscr{O}_X(U_i) \cong B_i$, which we denote by $\gamma_i : \mathscr{O}_X(U_i) \to B_i$. We claim that:

$$\beta_i(U_i \cap X_g) = U_{\gamma_i(g|_{U_i})}$$

which would imply that $U_i \cap X_g$ is an open subset of U_i, and thus an open subset of X. First note that since $(g|_{U_i})_x = g_x$:

$$U_i \cap X_g = \{x \in U_i : (g|_{U_i})_x \notin \mathfrak{m}_x\}$$

The homeomorphism β_i associates to each x a unique prime ideal $\mathfrak{p}_x \subset B_i$. Moreover, the unique maximal ideal $\mathfrak{m}_x$ is then isomorphic to $(B_i)_{\mathfrak{p}_x}$, hence:

$$\beta_i(U_i \cap X_g) = \{\mathfrak{p}_x \in \operatorname{Spec} B_i : \gamma(g|_{U_i})_{\mathfrak{p}_x} \notin (B_i)_{\mathfrak{p}_x}\}$$

Now $\gamma(g|_{U_i}) \in B_i$, so $\gamma(g|_{U_i})_{\mathfrak{p}_x}$ is given by:

$$\gamma(g|_{U_i})_{\mathfrak{p}_x} = \frac{\gamma(g|_{U_i})}{1} \notin (B_i)_{\mathfrak{p}_x}$$

We wish to show that if $b/1 \notin (B_i)_{\mathfrak{p}_x}$ then $b \notin \mathfrak{p}_x$, however, this clear by the contrapositive, i.e. if $b \in \mathfrak{p}_x$ then $b/1 \in (B_i)_{\mathfrak{p}_x}$ as:

$$(B_i)_{\mathfrak{p}_x} = \left\{\frac{p}{s} \in (B_i)_{\mathfrak{p}_x} : p \in \mathfrak{p}_x\right\}$$

hence if $b \in \mathfrak{p}_x$ then clearly $b/1 \in (B_i)_{\mathfrak{p}_x}$. It follows that:

$$\begin{aligned}\beta_i(U_i \cap X_g) =&\{\mathfrak{p}_x \in \operatorname{Spec} B_i : \gamma(g|_{U_i}) \notin \mathfrak{p}_x\}\\ =&U_{\gamma(g|_{U_i})}\end{aligned}$$

so $X_g \cap U_i$ is open for all i implying that X_g is open. It follows that $f_\psi^{-1}(U_a)$ is open, and thus f_ψ is continuous.

We now define the the map $f_\psi^\sharp : \mathscr{O}_Y \to f_{\psi *}\mathscr{O}_X$, and by Theorem 1.4.1 it suffices to define $f_\psi^\sharp$ on distinguished open set U_a. We thus need to define morphisms:

$$A_a \cong \mathscr{O}_Y(U_a) \longrightarrow f_{\psi *}\mathscr{O}_X(U_a) = \mathscr{O}_X(X_g)$$

where $g = \psi(\varphi_A(a))$. First consider the restriction map $\theta^X_{X_g} : \mathscr{O}_X(X) \to \mathscr{O}_X(X_g)$, then we want to show that image of g is a unit in $\mathscr{O}_X(X_g)$. Recall that for a local ring any element not in $\mathfrak{m}_x$ is a unit, hence $g_x \in (\mathscr{O}_X)_x$ is a unit for all $x \in X_g$. It follows that $(g_x) \in \prod_{x \in X_g} (\mathscr{O}_X)_x$ is a unit, in particular there is a sequence $(h_x) \in \prod_{x \in X_g} (\mathscr{O}_X)_x$ such that:

$$(h_x) \cdot (g_x) = (1_x)$$

Now for each $x \in X_g$, we write $h_x = [V_x, s^x]$ for some $V_x \subset X_g$, and some $s^x \in \mathscr{O}_X(V_x)$, it follows that:

$$h_x \cdot g_x = [V_x, s^x] \cdot [X, g] = [V_x \cap X, s^x|_{V_x \cap X} \cdot g|_{V_x \cap X}] = [V_x, s^x \cdot g|_{V_x}] = 1_x$$

There is an open subset of $W_x \subset V_x$ such that $(s^x \cdot g|_{V_x})|_{W_x} = 1$, implying that $s^x|_{W_x} = (g|_{W_x})^{-1}$. Repeating this for all $x \in X_g$ gives us an open cover of X_g by W_x, along with sections $s^x|_{W_x} := t^x \in \mathscr{O}_X(W_x)$. Now that since $t^x = (g|_{W_x})^{-1}$, we have that for all $y \in W_x$:

$$t^x_y = g_y^{-1} = h_y$$

hence $(h_x)/ \in \mathscr{O}^\sharp_X(X_g)$. Since $\mathscr{O}^\sharp_X \cong \mathscr{O}_X$ it follows that there is unique element $h \in \mathscr{O}_X(X_g)$ such that $h = (g|_{X_g})^{-1}$, hence $\theta^X_{X_g}(g)$ is a unit in $\mathscr{O}_X(X_g)$ so there exists a unique morphism:

$$\begin{aligned}\mathscr{O}_X(X)_g &\longrightarrow \mathscr{O}_X(X_g)\\ s/g^k &\longmapsto s|_{X_g} \cdot h^k\end{aligned}$$

We now show that there is a map $A_a \to \mathscr{O}_X(X)_g$, however this again follows from the universal property of localization, as we have that $\psi(\varphi_A(a)) = g$, so the image of a is a unit in $\mathscr{O}_X(X)_g$. We thus have have map:

$$\begin{aligned} A_a &\longrightarrow \mathscr{O}_X(X)_g \\ b/a^k &\longmapsto \psi(\varphi_A(b))/g^k \end{aligned}$$

and hence a morphism:

$$\begin{aligned} A_a &\longrightarrow \mathscr{O}_X(X_g) \\ b/a^k &\longmapsto \psi(\varphi_A(b))|_{X_g} \cdot h^k \end{aligned}$$

This clearly commutes with restrictions maps on the base, hence we get a sheaf morphism:

$$f_\psi^\sharp : \mathscr{O}_Y \to f_{\psi_*}\mathscr{O}_X$$

The assignment $\psi \mapsto (f_\psi, f_\psi^\sharp)$ then defines a set map $\Psi : \mathrm{Hom}(\mathscr{O}_Y(Y), \mathscr{O}_X(X)) \to \mathrm{Hom}(X, Y)$.

We check that Φ and Ψ are inverses of one another. Let $\psi \in \mathrm{Hom}(\mathscr{O}_Y(Y), \mathscr{O}_X(X))$, then we see that:

$$\Phi \circ \Psi(\psi) = (f_\psi^\sharp)_Y$$

It suffices to check that:

$$\psi \circ \varphi_A = (f_\psi^\sharp)_Y \circ \varphi_A$$

Well, note that by construction $(f_\psi^\sharp)_Y \circ \varphi_A$ is equivalent to the composition:

$$A \longrightarrow \mathscr{O}_X(X)_1 \longrightarrow \mathscr{O}_X(X)$$

which since there is nothing to invert is the map $b \mapsto \psi(\varphi_A(b))$, hence $(f_\psi^\sharp)_Y = \psi$, and $\Phi \circ \Psi = \mathrm{Id}$.

Now let $(f, f^\sharp) \in \mathrm{Hom}(X, Y)$, and set $\phi = f_Y^\sharp$, then we want to show that:

$$(f, f^\sharp) = (f_\phi, f_\phi^\sharp)$$

We first check that the topological maps are equal, in particular, we want to show that:

$$f(x) = \varphi_A^{-1}(\phi^{-1}(\pi_x^{-1}(\mathfrak{m}_x)))$$

Let $a \in f(x)$, then since $f(x)$ is a prime ideal, we see that $\varphi_A(a)$ is a section which vanishes at $f(x)$, i.e $\varphi_A(a)_{f(x)}$ lies in the unique maximal ideal $\mathfrak{m}_{f(x)} \cong A_{f(x)}$. It follows that:

$$\pi_x(\phi(\varphi_A(a))) = (f_Y^\sharp(\varphi_A(a)))_x = [X, f_Y^\sharp(\varphi_a)] = [f^{-1}(Y), f_Y^\sharp(\varphi_a)] = f_x(\varphi_A(a)_{f(x)})$$

since f_x is a morphism of local rings, we must have that $\pi_x(\phi(\varphi_A(a))) \in \mathfrak{m}_x$, hence $a \in \varphi_A^{-1}(\phi^{-1}(\pi_x^{-1}(\mathfrak{m}_x)))$. Now suppose that $a \in \varphi_A^{-1}(\phi^{-1}(\pi_x^{-1}(\mathfrak{m}_x)))$, then it follows that $f_x(\varphi_A(a)_{f(x)}) \in \mathfrak{m}_x$, and if $\varphi_A(a)_{f(x)} \notin \mathfrak{m}_{f(x)}$, then $\varphi_A(a)_{f(x)}$ is a unit in $(\mathscr{O}_Y)_{f(x)}$, implying that $\mathfrak{m}_x = (\mathscr{O}_X)_x$ contradicting the fact that $\mathfrak{m}_x$ is maximal, hence $a \in f(x)$ as well, so $f(x) = f_\phi(x)$ as desired.

To check that $f_\phi^\sharp = f^\sharp$, it suffices to check they agree on distinguished opens $U_a \subset \operatorname{Spec} A$ by Corollary 1.4.2. In particular, it suffices to check that the induced maps:

$$A_a \longrightarrow \mathscr{O}_X(X_g)$$

where $g = \phi(\varphi_A(a))$ agree. Let $b/a^k \in A_a$, then:

$$\begin{aligned} (f_\phi^\sharp)_{U_a}(\varphi_A(b)/\varphi_A(a^k)) =& \phi(\varphi_A(b))|X_g \cdot h^k \\ =& f_{U_a}^\sharp(\varphi_A(b)|_{U_a}) \cdot h^k \\ =& f_{U_A}^\sharp(\varphi_A(b)|_{U_a}) \cdot (g^k|_{X_g})^{-1} \end{aligned}$$

Now note that:

$$g|_{X_g} = f_Y^\sharp(\varphi_A(a))|_{X_g} = f_{U_a}^\sharp(\varphi_A(a)|_{U_a})$$

hence:

$$(g^k|_{X_g})^{-1} = f_{U_a}^\sharp(\varphi_A(a)|_{U_a})^{-k}$$

however $\varphi_A(a)|_{U_a}$ invertible in $\mathscr{O}_Y(U_a)$ hence:

$$(g^k|_{X_g})^{-1} = f_{U_a}^\sharp(\varphi_A(a)|_{U_a}^{-k})$$

so:

$$\begin{aligned}(f_\phi^\sharp)_{U_a}(\varphi_A(b)/\varphi_A(a^k)) =& f_{U_A}^\sharp(\varphi_A(b)|_{U_a}) \cdot f_{U_a}^\sharp(\varphi_A(a)|_{U_a}^{-k}) \\ =& f_{U_a}^\sharp(\varphi_A(b)|_{U_a} \cdot \varphi_A(a)|_{U_a}^{-k}) \\ =& f_{U_a}^\sharp\left(\frac{\varphi_A(b)}{1} \cdot \frac{1}{\varphi_A(a)^k}\right) \\ =& f_{U_a}^\sharp(\varphi_A(b)/\varphi_A(a^k))\end{aligned}$$

implying that $f_\phi^\sharp = f^\sharp$. We thus have that:

$$\Psi \circ \Phi(f, f^\sharp) = (f_\phi, f_\phi^\sharp) = (f, f^\sharp)$$

hence $\Psi \circ \Phi = \mathrm{Id}$ implying the claim. □

Note that since $\mathbb{Z}$ is the initial object in the category of rings, there exists a unique morphism $X \to \operatorname{Spec} \mathbb{Z}$ for every scheme X. As promised earlier, we now discuss how to put an induced subscheme structure on Zariski closed subsets of a scheme.

Definition 2.1.3. Let $(X, \mathscr{O}_X)$ be a scheme, and Y a Zariski closed subset of X, then the **sheaf of ideals** is given by the assignment $U \mapsto I(U)$, where:

$$I(U) = \{s \in \mathscr{O}_X(U) : \forall x \in Y \cap U, s_x \in \mathfrak{m}_x\}$$

That is $I(U)$ is the subgroup of sections on U which vanish on $Y \cap U$.

We quickly check that this is a sheaf:

Lemma 2.1.4. *The assignment $U \mapsto I(U)$ defines a sheaf on X.*

Proof. Clearly if $s \in I(U)$, and $V \cap U \neq \emptyset$, then $s|_V \in I(U)$ as $(s|_V)_x = s_x$ for all $x \in V$, so the restriction maps are precisely the same as the ones on X. Now let U_i be an open cover for U, and $s \in I(U)$ such that $s|_{U_i} = 0$ for all i. Then since $0 \in I(U)$, and $\mathscr{O}_X$ is a sheaf it follows that $s \in I(U)$ is equal to zero, implying sheaf axiom one. To prove sheaf axiom two, take U_i as before, and let $s_i \in I(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists an $s \in \mathscr{O}_X(U)$ such that $s|_{U_i} = s_i$ for all U_i. For all $x \in U$ we have that $x \in U_i$ for some i, hence $s_x = (s_i)_x$, so if $x \in Y$ then $s_x \in \mathfrak{m}_x$ implying that $s \in I(U)$ so U is a sheaf. □

Given that we have just constructed a sheaf of ideals on X, it should be obvious that we are about construct a new 'quotient sheaf' of rings on X. Our plan of action is as follows: to construct this sheaf $\mathscr{O}_X/I$, then define the structure of sheaf on Y to be $\mathscr{O}_Y = \iota^{-1}(\mathscr{O}_X/I)$, and finally to show that this gives Y the structure of a scheme, when equipped with the subspace topology.

Let us first examine the affine case. Let $X = \operatorname{Spec} A$, and $Y = \mathbb{V}(I)$ for some radical ideal I of A. Then for a distinguished open $U_g \subset \operatorname{Spec} A$, we have that:

$$\mathscr{O}_X(U_g) \cong A_g$$

Now note that $\mathbb{V}(I) \cap U_g$ is a closed subset of U_g when equipped with the subspace topology, so $\mathbb{V}(I) \cap U_g$ corresponds to the vanishing set of an ideal $I_g \subset A_g$. We claim that:

$$I_g = \left\{a/g^k \in A_g : a \in I\right\}$$

As an abuse of notation, and confidence, denote the above set by I_g, then we need to show that:

$$\mathbb{V}(I_g) = \eta(\mathbb{V}(I) \cap U_g)$$

where η is the homeomorphism $\operatorname{Spec} A \to \operatorname{Spec} A_g$. We have that:

$$\mathbb{V}(I) \cap U_g = \{\mathfrak{p} \in \operatorname{Spec} A : I \subset \mathfrak{p} \text{ and } g \notin \mathfrak{p}\}$$

so:

$$\eta(\mathbb{V}(I) \cap U_g) = \{\eta(\mathfrak{p}) \in \operatorname{Spec} A_g : I \subset \mathfrak{p} \text{ and } g \notin \mathfrak{p}\}$$

However,

$$\eta(\mathfrak{p}) = \left\{ \frac{p}{g^k} : p \in \mathfrak{p}, k \geq 0 \right\}$$

Clearly if $g \notin \mathfrak{p}$, then $\eta(\mathfrak{p}) \in \operatorname{Spec} A_g$, as otherwise η is not defined. If $I \subset \mathfrak{p}$, then we also clearly have that $I_g \subset \mathfrak{p}$, as for $a/g^k \in I_g$ we have that $a \in I \subset \mathfrak{p}$. It follows that $\eta(\mathbb{V}(I) \cap U_g) \subset \mathbb{V}(I_g)$. Now let $\mathfrak{q} \in \mathbb{V}(I_g)$, then $I_g \subset \mathfrak{q}$, and $\mathfrak{q}$ is of the form $\eta(\mathfrak{p})$ for some $\mathfrak{p} \in U_g$. Moreover, we have that $\pi^{-1}(I_g) = I \subset \mathfrak{p}$, so $\mathfrak{p} \in \mathbb{V}(I) \cap U_g$, and thus $q = \eta(\mathfrak{p}) \in \eta(\mathbb{V}(I) \cap U_g)$. It follows that $\mathbb{V}(I_g) \subset U_g \cap \mathbb{V}(I_g)$, so we obtain the desired equality. We can now calculate $I(U_g)$ to be

$$\begin{aligned}
I(U_g) =& \{s \in \mathscr{O}_X(U_g) : \forall \mathfrak{q} \in \mathbb{V}(I) \cap U_g, s \in \mathfrak{q}\} \\
\cong& \{a/g^k \in A_g : \forall \mathfrak{q} \in \mathbb{V}(I_g), a/g^k \in \mathfrak{q}\} \\
\cong& \bigcap_{\mathfrak{q} \in \mathbb{V}(I_\mathfrak{q})} \mathfrak{q} \\
\cong& \sqrt{I_q}
\end{aligned}$$

Note that $I_q \subset \sqrt{I_q}$ automatically, and that if $a/g^k \in I_q$ we have that there is some r such that $a^r/g^{kr} \in I_q$, implying that $a^r \in I$, so $a \in \sqrt{I} = I$ as I is radical. It follows that:

$$I(U_g) \cong I_g$$

hence:

$$\mathscr{O}_X(U_g)/I(U_g) \cong A_g/I_g$$

We now have the following lemma:

Lemma 2.1.5. *Let $X = \operatorname{Spec} A$, and $Y = \mathbb{V}(I)$ for some radical ideal I, then the assignment $U_g \mapsto A_g/I_g$ defines a sheaf on the base of distinguished opens.*

Proof. Let $U_g \subset U_f$, then recall $\sqrt{\langle g \rangle} \subset \sqrt{\langle f \rangle}$, so there exists an $m > 0$, and $b \in A$ such that $g^m = f \cdot b$. It follows that the image of f is a unit in A_g, so we get a restriction map given by:

$$\frac{a}{f^k} \longmapsto \frac{a \cdot b^k}{g^{mk}}$$

If $a/f^k \in I_f$, then $a \in I$, and certainly $a \cdot b^k \in I$ hence $a/f^k|_{U_g} \in I_g$. It follows that we get well defined restriction maps given by $A_f/I_f \to A_g/I_g$:

$$[a/f^k] \longmapsto [a \cdot b^k/g^{mk}]$$

so we have a presheaf on the distinguished opens.

By Lemma 1.4.1 and Lemma 1.4.2 it suffices to take all covers to be finite. Now let U_{g_i} be an open cover U_f, and $[a/f^k] \in A_f/I_f$ such that $[a/f^k]|_{U_{g_i}} = 0$ for all i. Note, that $[a/f^k]$ induced a unique element in $\mathscr{O}_X(U_f)/I(U_f)$, and similarly for it's restrictions. Denote this element by s, if the restrictions are all 0, then $s_\mathfrak{p} \in \mathfrak{m}_\mathfrak{p}$ for all $\mathfrak{p} \in Y \cap U_f$, as $s_\mathfrak{p} = (s|_{U_{g_i}})_\mathfrak{p}$ and U_{g_i} cover U_f. It follows that $s \in I(U_f)$, hence $[a/f^k] \in I_g$, so $[a/f^k] = 0$.

Now let U_{g_i} be an open cover of U_f and $[a/g_i^{k_i}] \in A_{g_i}/I_{g_i}$ such that:

$$\left[\frac{a \cdot g_j^{k_i}}{(g_i g_j)^{k_i k_j}}\right] = \left[\frac{a \cdot g_i^{k_j}}{(g_i g_j)^{k_i k_j}}\right] \tag{2.1.4}$$

for all $U_{g_i g_j} = U_{g_i} \cap U_{g_j}$. We first show that $A_{g_i}/I_{g_i} \cong (A/I)_{[g_i]}$. Define the map:

$$\begin{aligned} A &\longrightarrow (A/I)_{[g_i]} \\ a &\longmapsto [a]/1 \end{aligned}$$

and note that g_i is clearly a unit in this map, with inverse given by $1/[g_i]$ so we have a unique homomorphism

$$\begin{aligned} A_{g_i} &\longrightarrow (A/I)_{[g_i]} \\ a/g_i^k &\longmapsto [a][g_i]^k \end{aligned}$$

This map is clearly surjective; now let $a/g_i^k \in I_{g_i}$, then $a \in I$, so $[a] = 0$, hence $a/g_i^k \mapsto 0/[g_i]^k = 0$. Suppose that $a/g_i^k \mapsto 0$, then we have that:

$$\frac{[a]}{[g_i]^k} = 0 \Rightarrow [g_i]^M \cdot [a] = 0$$

We see that this implies that $g_i^M \cdot a \in I$, so $g_i^M \cdot a/1 \in I_{g_i}$, implying that $a/1 \in I_{g_i}$, hence $a/g_i^k \in I_{g_i}$. It follows that the kernel of the map is equal to I_{g_i} hence the induced unique homomorphism:

$$\begin{aligned} A_{g_i}/I_{g_i} &\longrightarrow (A/I)_{[g_i]} \\ [a/g_i^k] &\longmapsto [a]/[g_i]^k \end{aligned}$$

is an isomorphism. The expression (2.3) is then equivalent to:

$$\frac{[a \cdot g_j^{k_i}]}{[(g_i g_j)^{k_i k_j}]} = \frac{[a \cdot g_i^{k_j}]}{[(g_i g_j)^{k_i k_j}]}$$

The same argument in Proposition 1.4.3 then proves the claim, as we are now just dealing with localizations of some ring A/I.

□

Take any $g \in I$, then note that $\mathbb{V}(I) \cap U_g = \emptyset$, indeed if $\mathfrak{p} \in \mathbb{V}(I)$, then $I \subset \mathfrak{p}$, hence $g \in \mathfrak{p}$, so $\mathfrak{p} \notin U_g$. Moreover, if $\mathfrak{p} \in U_g$, then $g \notin \mathfrak{p}$, so $\mathfrak{p} \notin \mathbb{V}(I)$. It follows that $\mathbb{V}(I_g) = \emptyset$, implying that $I_g = A_g$, hence:

$$\mathscr{O}_X(U_g)/I(U_g) = \{0\}$$

Lemma 2.1.6. *The assignment $U \mapsto \mathscr{O}_X(U)/I(U)$, where U is open and affine defines a sheaf on the basis of affine opens for X.*

Proof. Let U and be V be open affines in X such that $V \subset U$. Then, we define restriction maps by:

$$[s] \in \mathscr{O}_X(U)/I(U) \longmapsto [\theta_V^U(s)] \in \mathscr{O}_X(V)/I(U)$$

i.e. we choose a class representative $s \in [s]$, restrict to $\mathscr{O}_X(V)$ and the project again. Since I is a sheaf of ideals, it follows that this is independent of the class representative chosen, and thus well defined.

Now let U be open affine, U_i an open cover of U by open affines, and $[s] \in \mathscr{O}_X(U)/I(U)$ such that $[s]|_{U_i} = 0$ for all i. This implies that $s \in [s]$ restricts to an element in $I(U_i)$ for all i, and since $s_x = (s|_{U_i})_x$ for all $x \in U_i$, it follows that for all $x \in Y \cap U$ we have $s_x \in \mathfrak{m}_x$, implying that $s \in I(U)$, hence $[s] = 0$, so sheaf on a base axiom one is satisfied.

Now let U be open and affine, U_i be an open cover of U by open affines and $[s_i] \in \mathscr{O}_X(U_i)/I(U_i)$ be sections such that for all open affines $U_{ij} \subset U_i \cap U_j$ we have:

$$[s_i]|_{U_{ij}} = [s_j]|_{U_{ij}}$$

Now note that U is isomorphic as a scheme to $\operatorname{Spec} A$ for some ring A, so we can take $\{U_i\}$ to be a finite open cover by Lemma 1.4.1 and Lemma 1.4.2. We also have that $U \cap Y \cong \mathbb{V}(J)$ for some radical ideal $J \subset A$. Under this identification, each U_i can be written as a finite union of distinguished opens of $\operatorname{Spec} A$:

$$U_i = \bigcup_{a_i \in A} U_{a_i}$$

and see that:

$$\begin{aligned} U_i \cap U_j &= \left(\bigcup_{a_i} U_{a_i} \right) \cap \left(\bigcup_{a_j} U_{a_j} \right) \\ &= \bigcup_{a_i, a_j} U_{a_i} \cap U_{a_j} \\ &= \bigcup_{a_i, a_j} U_{a_i \cdot a_j} \end{aligned}$$

Now note that $U_{a_i \cdot a_j}$ is then an affine open subset of $U_i \cap U_j$, hence:

$$[s_i]|_{U_{a_i \cdot a_j}} = [s_j]|_{U_{a_i \cdot a_j}}$$

Moreover, we have that:

$$[s_i]|_{U_{a_i}}|_{U_{a_i \cdot a_j}} = [s_i]|_{U_{a_i \cdot a_j}}$$

and that for a_i and b_i we clearly have that $[s_i]|_{U_{a_i \cdot b_i}} = [s_i]|_{U_{a_i \cdot b_i}}$, so by reindexing to include all a_i, we obtain a finite open cover of $\operatorname{Spec} A$ by distinguished opens $\{U_{a_i}\}_{i \in I}$, and sections $[t_i] := [s_i]|_{U_{a_i}} \in \mathscr{O}_{\operatorname{Spec} A}(U_{a_i})/I(U_{a_i}) \cong (A/J)_{[a_i]}$ such that:

$$[t_i]|_{U_{a_i} \cap U_{a_j}} = [t_j]|_{U_{a_i} \cap U_{a_j}}$$

for all $U_{a_i} \cap U_{a_j}$. Lemma 2.1.5 then gives us an element $[s] \in \mathscr{O}_{\operatorname{Spec} A}(U)/I(U) \cong A/J$ such that $[s]|_{U_{a_i}} = [t_i]$ for all i. We show that $[s]|_{U_i} = [s_i]$. Recall that U_i is covered by distinguished opens U_{a_i}, and that for each a_i:

$$\left([s]|_{U_i} - [s_i] \right) |_{U_{a_i}} = [t_i] - [t_i] = 0 \tag{2.1.5}$$

it follows by sheaf on a base axiom one that $[s]|_{U_i} = [s_i]$, implying the claim. ☐

Proposition 2.1.3. *Let $X = \operatorname{Spec} A$, $Y = \mathbb{V}(J)$ for some radical ideal J, I be the sheaf of ideals induced by Y, $\mathscr{O}_X/I$ the sheaf induced by Lemma 2.1.5, and $\iota : Y \to X$ the inclusion map. Then Y, equipped with subspace topology, and the structure sheaf $\mathscr{O}_Y = \iota^{-1}\mathscr{O}_X/I$ is an affine scheme isomorphic to* $\operatorname{Spec} A/J$.

Proof. We first define a homeomorphism $f : \mathbb{V}(J) \to \operatorname{Spec} A/J$. Let $\pi : A \to A/J$ be the projection map, and $\mathfrak{p} \subset \mathbb{V}(J)$, then we claim that $\pi(\mathfrak{p}) \subset A/J$ is a prime ideal in A/J. It is clear that $\pi(\mathfrak{p})$ is a group, we check that $\pi(\mathfrak{p})$ is an ideal. Suppose that $[a] \in \pi(\mathfrak{p})$ and $[b] \in A/J$, then there we see there is some $i \in J$ such that $a + i \in \mathfrak{p}$, and it follows that $(a + i) \cdot b \in \mathfrak{p}$. We thus must have that $[(a + i) \cdot b] \in \pi(\mathfrak{p})$, however:

$$[(a + i) \cdot b] = [ab + ib] = [ab] + i[b] = [ab] = [a] \cdot [b]$$

so $\pi(\mathfrak{p})$ swallows multiplication and is thus an ideal. We now show that $\pi(\mathfrak{p})$ is prime, let $[a]$ and $[b] \in A/J$, such that $[a] \cdot [b] \in \pi(\mathfrak{p})$. It follows that $[a \cdot b] \in \pi(\mathfrak{p})$, hence there is some $j_{ab} \in J$ such that $a \cdot b + j_{ab} \in \mathfrak{p}$. Since $\mathfrak{p}$ is closed under addition, and $-j_{ab} \in J \subset \mathfrak{p}$, it follows that $a \cdot b \in \mathfrak{p}$, hence either a or $b \in \mathfrak{p}$, implying that either $[a]$ or $[b]$ lies in $\pi(\mathfrak{p})$.

We thus define:

$$f : \mathbb{V}(J) \longrightarrow \operatorname{Spec} A/J$$

by $\mathfrak{p} \mapsto \pi(\mathfrak{p})$. This map is surjective, as if $\mathfrak{q} \in \operatorname{Spec} A/J$, we have that $\pi(\pi^{-1}(\mathfrak{q})) = \mathfrak{q}$, since π is surjective. Now suppose that $\pi(\mathfrak{p}) = \pi(\mathfrak{q})$, then we need to show that $\mathfrak{p} = \mathfrak{q}$. Let $a \in \mathfrak{p}$, then $[a] \in \pi(\mathfrak{p})$, and $[a] \in \pi(\mathfrak{q})$. Since $[a] \in \pi(\mathfrak{q})$, there is a $j \in J$ such that $a + j \in \mathfrak{q}$. However $J \subset \mathfrak{q}$, and again $\mathfrak{q}$ is closed under subtraction so $a + j - j = a \in \mathfrak{q}$, and $\mathfrak{p} \subset \mathfrak{q}$. The same argument shows that $\mathfrak{q} \subset \mathfrak{p}$, implying injectivity.

We claim that this map is continuous, and it suffices to check this on basic opens. Let $U_{[g]}$ be a distinguished open, then:

$$\begin{aligned} f^{-1}(U_{[g]}) =& \{\mathfrak{p} \in \mathbb{V}(J) : [g] \notin \pi(\mathfrak{p})\} \\ =& \{\mathfrak{p} \in \mathbb{V}(J) : \langle [g] \rangle \not\subset \pi(\mathfrak{p})\} \\ =& \{\mathfrak{p} \in \mathbb{V}(J) : \pi^{-1}(\langle [g] \rangle) \not\subset \mathfrak{p}\} \end{aligned}$$

We claim that:

$$\{\mathfrak{p} \in \mathbb{V}(J) : \pi^{-1}(\langle [g] \rangle) \not\subset \mathfrak{p}\} = \{\mathfrak{p} \in \mathbb{V}(J) : \langle g \rangle \not\subset \mathfrak{p}\}$$

Let $\mathfrak{p} \in \mathbb{V}(J)$ such that $\pi^{-1}(\langle [g] \rangle) \not\subset \mathfrak{p}$, then we want to show that $\langle g \rangle \not\subset \mathfrak{p}$. Well, we have that there exists an $a \in \pi^{-1}(\langle [g] \rangle) \not\subset \mathfrak{p}$, and $a = b \cdot g^k + j$ for some $j \in J$. Clearly, $b \cdot g^k + j \notin \mathfrak{p}$, but $j \in J \subset \mathfrak{p}$, so the only way this holds is if $b \cdot g^k \notin \mathfrak{p}$. We have that $b \cdot g^k \in \langle g \rangle$, so $\langle g \rangle \not\subset \mathfrak{p}$. Now suppose that $\langle g \rangle \not\subset \mathfrak{p}$, then there exists some $a \in \langle g \rangle$ such that $a \notin \mathfrak{p}$. However, $a = b \cdot g^k$, so $[a] = [b] \cdot [g]^k \in \langle [g] \rangle$, hence $a \in \pi^{-1}(\langle [g] \rangle)$ implying that $\langle [g] \rangle \not\subset \mathfrak{p}$. It follows that:

$$\begin{aligned} f^{-1}(U_{[g]}) =& \{\mathfrak{p} \in \mathbb{V}(J) : \langle g \rangle \not\subset \mathfrak{p}\} \\ =& \mathbb{V}(J) \cap U_g \end{aligned}$$

which is open the subspace topology.

We claim that this map is open and thus a homeomorphism. Note that $\{\mathbb{V}(J) \cap U_g\}_{g \in A}$ is a basis for $\mathbb{V}(J)$, and since f is a bijection, we have that:

$$f(\mathbb{V}(J) \cap U_g) = f(f^{-1}(U_{[g]}) = U_{[g]}$$

so f is open.

Now note that if $\iota : \mathbb{V}(J) \to \operatorname{Spec} A$ is the inclusion map, and $f^{-1} : \operatorname{Spec} A/J \to \mathbb{V}(J)$ is the homeomorphism, we have that $\iota \circ f^{-1} : \operatorname{Spec} A/J \to \operatorname{Spec} A$ comes from the ring homomorphism $\pi : A \to A/J$. We want to construct a sheaf isomorphism:

$$(f^{-1})^\sharp : \iota^{-1}\mathscr{O}_X/I \longrightarrow f_*^{-1}\mathscr{O}_{\operatorname{Spec} A/J}$$

and by Theorem 1.3.1 and Corollary 2.1.1 it suffices to define a sheaf isomorphism:

$$F : \mathscr{O}_X/I \longrightarrow \iota_* f_*^{-1}\mathscr{O}_{\operatorname{Spec} A/J} = (\iota \circ f^{-1})_*\mathscr{O}_{\operatorname{Spec} A/J}$$

We do so on a basis of distinguished opens U_g. Since $f^{-1} \circ \iota$ is topological map coming from the the projection $\pi : A \to A/J$, we have that if $g \in J$, then $(\iota \circ f^{-1})^{-1}(U_g) = U_{[g]} = U_0 = \emptyset$. By our earlier discussion we have that $\mathscr{O}_X/I(U_g) = \{0\}$ so our isomorphism on these open sets is trivial.

In the case where $g \notin J$, we have that $\mathscr{O}_X/I(U_g) = A_g/J_g \cong (A/J)_{[g]}$, while $\mathscr{O}_{\operatorname{Spec} A/J}(U_{[g]}) = (A/J)_{[g]}$. These isomorphisms clearly commute with restrictions on a distinguished base, so F is an isomorphism. We define $(f^{-1})^\sharp$ to be the sheaf isomorphism induced by the isomorphism in Theorem 1.3.1, hence $(f^{-1}, (f^{-1})^\sharp) : \operatorname{Spec} A/J \to \mathbb{V}(J)$ is an isomorphism as desired. $\square$

We can now prove the desired claim:

Theorem 2.1.2. *Let X be a scheme, Y a Zariski closed subset of X, and I the sheaf of ideals on X induced by Y. Then there exists the natural structure of a scheme on Y, such that for all affine opens $U \subset X$, $\mathscr{O}_Y(U \cap Y) \cong \mathscr{O}_X(U)/I(U)$.*

Proof. Equip Y with the subspace topology, and the sheaf $\iota^{-1}\mathscr{O}_X/I$, where $\mathscr{O}_X/I$ is the sheaf induced by Lemma 2.1.6, and $\iota : Y \to X$ is the inclusion map. We need to show that every point in Y has an open neighborhood isomorphic to an affine scheme. Let $y \in Y$, then since $Y \subset X$, there is an open neighborhood U of y in X, such that $U \cong \operatorname{Spec} A$, and let $f : U \to \operatorname{Spec} A$ be the isomorphism. Now note that $U \cap Y$ is open in subspace topology on Y, and closed in the subspace topology on U. It follows that there is radical ideal $J \subset A$, such that $f(U \cap Y) = \mathbb{V}(J) \subset \operatorname{Spec} A$. Now $f^\sharp$ gives an isomorphism:

$$f^\sharp : \mathscr{O}_{\operatorname{Spec} A} \longrightarrow f_* \mathscr{O}_U = f_*(\mathscr{O}_X|_U)$$

We claim that this induces an isomorphism:

$$f^\sharp : I_{\mathbb{V}(J)} \longrightarrow f_*(I|_U)$$

where $I_{\mathbb{V}(J)}$ is the sheaf of ideals on Spec A induced by $\mathbb{V}(J)$. Indeed, let $V \subset \operatorname{Spec} A$; if $s \in I_{\mathbb{V}(J)}(V)$, then $s_\mathfrak{p} \in \mathfrak{m}_\mathfrak{p}$ for all $\mathfrak{p} \in V \cap \mathbb{V}(J)$. For all all $\mathfrak{p} \in V \cap \mathbb{V}(J)$ let $\mathfrak{p} = f(x)$ for some unique $x \in f^{-1}(V) \cap U \cap Y$. Then since f is a morphism of locally ringed spaces we have that $f_x(s_\mathfrak{p}) \in \mathfrak{m}_x$ for all $x \in f^{-1}(V)$, hence:

$$f_x(s_\mathfrak{p}) = f_x([V, s]_\mathfrak{p}) = [f^{-1}(V), f_V^\sharp(s)]_x = (f_V^\sharp(s))_x \in \mathfrak{m}_x$$

so $f_V^\sharp(s) \in I(f^{-1}(V))$ for all $s \in I_{\mathbb{V}(J)}(V)$. Now let $t \in I(f^{-1}(V))$, then since $f_V^\sharp$ is an isomorphism there exists an $s \in \mathscr{O}_{\operatorname{Spec} A}(V)$ such that $f_V^\sharp(s) = t$, so $f_x(s_\mathfrak{p}) = t_x \in \mathfrak{m}_x$ for all $x \in f^{-1}(V) \cap Y$, where $\mathfrak{p} = f(x)$. However, f_x is an isomorphism, so since $\mathfrak{m}_x$ is the unique maximal ideal, and isomorphisms map maximal ideals to maximal ideals, we must have that $f_x(\mathfrak{m}_\mathfrak{p}) = \mathfrak{m}_x$, hence $s_\mathfrak{p} \in \mathfrak{m}_\mathfrak{p}$ for all $\mathfrak{p} \in \mathbb{V}(J) \cap V$. It follows that $f^\sharp$ induces an isomorphism of ideal sheafs, as desired.

We now claim that this induces an isomorphism:

$$\tilde{f}^\sharp : \mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)} \longrightarrow f_*(\mathscr{O}_X/I|_U)$$

Indeed, note that for any distinguished open set U_g, we clearly have that $f^{-1}(U_g) \subset U \subset X$ is then clearly an affine open, and we have that:

$$(\mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)})(U_g) \cong \mathscr{O}_{\operatorname{Spec} A}(U_g)/I_{\mathbb{V}(J)}(U_g)$$

while:

$$f_*(\mathscr{O}_X/I|_U)(U_g) = \mathscr{O}_X/I(f^{-1}(U_g)) \cong \mathscr{O}_X(f^{-1}(U_g))/I(f^{-1}(U_g))$$

By Corollary 1.4.2 it thus suffices to define morphisms:

$$\psi_{U_g} : \mathscr{O}_{\operatorname{Spec} A}(U_g)/I_{\mathbb{V}(J)}(U_g) \longrightarrow \mathscr{O}_X(f^{-1}(U_g))/I(f^{-1}(U_g))$$

which commute with restriction maps, but we clearly already have one induced by $f^\sharp$. Indeed, set:

$$\psi_{U_g}([s]) = [f_V^\sharp(s)]$$

which is clearly well defined, and obviously commute with said restrictions. The maps then induce the desired isomorphism of sheaves $\tilde{f}^\sharp : \mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)} \longrightarrow f_*(\mathscr{O}_X/I|_U)$.

We now switch to the topological picture and equip $U \cap Y$ with the subspace topology induced by Y, and $\mathbb{V}(J)$ equipped with the subspace topology on Spec A, we want $f|_{U \cap Y} : U \cap Y \to \mathbb{V}(J)$ to be a homeomorphism. We first see that it is continuous, as if $W \subset \mathbb{V}(J)$ is open, then $W = V \cap \mathbb{V}(J)$ for some open subset $V \subset \operatorname{Spec} A$. It follows that:

$$f|_{U \cap Y}^{-1}(W) = f^{-1}(V \cap \mathbb{V}(J)) = f^{-1}(V) \cap f^{-1}(\mathbb{V}(J)) = f^{-1}(V) \cap (U \cap Y)$$

We see that $f^{-1}(V)$ is open in U, and thus open in X, so it follows that $f^{-1}(V) \cap Y$ is open in Y. Since:

$$f^{-1}(V) \cap (U \cap Y) = (f^{-1}(V) \cap Y) \cap (U \cap Y)$$

it follows that $f|_{U \cap Y}^{-1}(W)$ is open in $U \cap Y$, so f is continuous. Now let $W \subset U \cap Y$ be open, then $W = V \cap (U \cap Y)$ for some open subset $V \subset Y$, but for V to be open in Y we must have that $V = Z \cap Y$ for some Z open in X. We see that:

$$f|_{U \cap Y}(W) = f(Z \cap Y \cap (U \cap Y)) = f(Z \cap (U \cap Y)) = f((Z \cap U) \cap (U \cap Y))$$

and since f is a bijection:

$$f|_{U \cap Y}(W) = f(Z \cap U) \cap f(U \cap Y) = f(Z \cap U) \cap \mathbb{V}(J)$$

since $f : U \to \operatorname{Spec} A$ is a homeomorphism, it follows that $f(Z \cap U)$ is open in $\operatorname{Spec} A$, hence $f(Z \cap U) \cap \mathbb{V}(J)$ is open in $\mathbb{V}(J)$. We thus that have that $f|_{U\cap Y}$ is a homeomorphism $U \cap Y \to \mathbb{V}(J)$.

So now we have a homeomorphism $g = f|_{U\cap Y} : U \cap Y \longrightarrow \mathbb{V}(J)$, we claim that there then exists an isomorphism of sheaves:

$$\iota_{\mathbb{V}(J)}^{-1}(\mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)}) \cong g_*(\iota^{-1}(\mathscr{O}_X/I)|_{U\cap Y})$$

We shall prove this by use of Theorem 1.3.1 and Corollary 2.1.1, and by noting that we have the following commutative square of topological maps:

$$\begin{tikzcd} U \arrow[r, "f"] & \operatorname{Spec} A \\ U \cap Y \arrow[u, "\iota_{U\cap Y}"] \arrow[r, "g"] & \mathbb{V}(J) \arrow[u, "\iota_{\mathbb{V}(J)}"'] \end{tikzcd}$$

Now note that by Corollary 2.1.1 suffices to show that:

$$g^{-1}\left(\iota_{\mathbb{V}(J)}^{-1}(\mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)})\right) \cong \iota^{-1}(\mathscr{O}_X/I)|_{U\cap Y}$$

Now note that since $(\iota_{\mathbb{V}(J)} \circ g)_* = g_* \circ \iota_{\mathbb{V}(J)*}$, so by Theorem 1.3.1, we have that $(\iota_{\mathbb{V}(J)} \circ g)^{-1} = g^{-1} \circ \iota_{\mathbb{V}(J)}^{-1}$, and by the diagram above we have that $\iota_{\mathbb{V}(J)} \circ g = f \circ \iota_{U\cap Y}$, so it suffices to show that:

$$(f \circ \iota_{U\cap Y})^{-1}(\mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)}) \cong \iota^{-1}(\mathscr{O}_X/I)|_{U\cap Y}$$

Now we have that:

$$(f \circ \iota_{U\cap Y})^{-1}(\mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)}) = \iota_{U\cap Y}^{-1}\left(f^{-1}\left(\mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)}\right)\right)$$

Now by our earlier, we work we have that the image of $\tilde{f}^\sharp$ under the isomorphism in Theorem 1.3.1 gives an isomorphism $f^{-1}(\mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)}) \cong \mathscr{O}_X/I|_U$, hence we have that:

$$(f \circ \iota_{U\cap Y})^{-1}(\mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)}) \cong \iota_{U\cap Y}^{-1}(\mathscr{O}_X/I|_U)$$

so it suffices to show that:

$$\iota_{U\cap Y}^{-1}(\mathscr{O}_X/I|_U) \cong \iota^{-1}(\mathscr{O}_X/I)|_{U\cap Y}$$

Recall that $\mathscr{O}_X/I|_U \cong \iota_U^{-1}\mathscr{O}_X/I$ by Corollary 1.3.2, so we have that the left hand side satisfies:

$$\iota_{U\cap Y}^{-1}(\mathscr{O}_X/I|_U) \cong (\iota_U \circ \iota_{U\cap Y})^{-1}\mathscr{O}_X/I$$

while the right hand side satisfies:

$$\iota^{-1}(\mathscr{O}_X/I)|_{U\cap Y} \cong (\iota \circ \iota_{U\cap Y})^{-1}\mathscr{O}_X/I$$

Now the issue is that technically have two different inclusion maps $\iota_{U\cap Y}$. The first is $\iota_{U\cap Y} : U \cap Y \to U$, and the the second is $\iota_{U\cap Y} : U \cap Y \to Y$, however, clearly when composed with $\iota_U : U \to X$, and $\iota : Y \to X$, we find that $\iota \circ \iota_{U\cap Y} = \iota_U \circ \iota_{U\cap Y}$, as topological maps. It follows that:

$$(\iota_U \circ \iota_{U\cap Y})^{-1}\mathscr{O}_X/I = (\iota \circ \iota_{U\cap Y})^{-1}\mathscr{O}_X/I$$

So reversing this chain of isomorphisms gives the desired result:

$$g^\sharp : \iota_{\mathbb{V}(J)}^{-1}(\mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)}) \longrightarrow g_*(\iota^{-1}(\mathscr{O}_X/I)|_{U\cap Y})$$

It follows that $(U \cap Y, \mathscr{O}_Y|_{U\cap Y}) \cong (\mathbb{V}(J), \mathscr{O}_{\mathbb{V}(J)})$, and hence by Proposition 2.1.3 that $(U \cap Y, \mathscr{O}_Y|_{U\cap Y}) \cong (\operatorname{Spec} A/J, \mathscr{O}_{\operatorname{Spec} A/J})$, so Y is indeed a scheme.

Moreover, we see that:

$$\mathscr{O}_Y(U \cap Y) = \mathscr{O}_Y|_{U\cap Y}(U \cap Y) \cong \mathscr{O}_{\operatorname{Spec} A/J}(\operatorname{Spec} A/J) \cong A/J$$

while:

$$\mathscr{O}_X(U) \cong A$$

and:

$$I(U) \cong J \subset A$$

hence:

$$\mathscr{O}_X(U)/I(U) \cong A/J$$

implying the claim. □

2.2 The Proj Construction

Our very first examples of schemes were affine ones, and now pretty much all the examples we have encountered are either open/closed subschemes of affine schemes, or a gluing of two affine schemes. In fact Example 2.1.5 is the motivating example for this section, it being the simplest example of what we will call a projective scheme. Indeed, our goal in this section is to discuss the analogue of projective space in differential geometry. We begin with the following example; reader be warned this is a mildly messy computation, and the checking of certain details are most likely best done on your own.

Example 2.2.1. Consider the variables $x_0, \dots, x_n$, and $n+1$ rings:

$$A_i = \mathbb{C}\left[\frac{x_0}{x_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i}\right] \cong \mathbb{C}\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] / \langle x_i/x_i - 1\rangle$$

which gives us $n+1$ schemes $X_i = \operatorname{Spec} A_i$. We note that for each i, the x_j/x_i is just a dummy variable to remind us of how this object is related to the coordinate charts on $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus 0/\mathbb{C}^*$.

For all i, j and we set $U_{ij} \subset X_i$ to be U_{x_j/x_i}, i.e the distinguished open set corresponding to the localization of A_i at x_j/x_i. We need to write down isomorphisms $\phi_{ij} : U_{ij} \to U_{ji}$, and since all schemes are affine, it suffices to provide ring homomorphisms:

$$\phi_{ij}^\sharp : (A_j)_{x_i/x_j} \longrightarrow (A_i)_{x_j/x_i}$$

We suggestively denote $1/(x_j/x_i)$ by x_i/x_j, and consider the morphism:

$$\xi_{ij}^\sharp : \mathbb{C}\left[\frac{x_0}{x_j}, \dots, \frac{\hat{x}_j}{x_j}, \dots, \frac{x_n}{x_j}\right] \longrightarrow \mathbb{C}\left[\frac{x_0}{x_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right]$$

induced by the following assignment on generators:

$$x_k/x_j \longmapsto \begin{cases} (x_k/x_i) \cdot (x_i/x_j) \cdot & \text{if } k \neq i \\ x_i/x_j & \text{if } k = i \end{cases}$$

Per our suggestive notation, we see that x_i/x_j is then a unit under the image of $\xi_{ij}^\sharp$ as:

$$\xi_{ij}^\sharp(x_i/x_j) \cdot x_j/x_i = (x_i/x_j) \cdot (x_j/x_i) = (1/(x_j/x_i))(x_j/x_i) = 1$$

We thus set $\phi_{ij}^\sharp$ to be the unique morphism induced by the universal property of localization, which is given on generators by:

$$\begin{aligned} \phi_{ij}^\sharp : \mathbb{C}\left[\frac{x_0}{x_j}, \dots, \frac{\hat{x}_j}{x_j}, \dots, \frac{x_n}{x_j}, \frac{x_j}{x_i}\right] &\longrightarrow \mathbb{C}\left[\frac{x_0}{x_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right] \\ x_l/x_m &\longmapsto \begin{cases} (x_l/x_i) \cdot (x_i/x_j) & \text{if } l \neq i, m = j \\ x_i/x_j & \text{if } l = i, m = j \\ x_j/x_i & \text{if } l = j, m = i \end{cases} \end{aligned}$$

Note that this is an isomorphism, as the map in the other direction

$$\phi_{ji}^\sharp : \mathbb{C}\left[\frac{x_0}{x_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_j}{x_i}\right] \longrightarrow \mathbb{C}\left[\frac{x_0}{x_j}, \dots, \frac{\hat{x}_j}{x_j}, \dots, \frac{x_n}{x_j}, \frac{x_j}{x_i}\right]$$
$$x_l/x_m \longmapsto \begin{cases} (x_l/x_j)\cdot(x_j/x_i) & \text{if } l \neq j, m = i \\ x_j/x_i & \text{if } l = j, m = i \\ x_i/x_j & \text{if } l = i, m = j \end{cases}$$

satisfies $\phi_{ji}^\sharp = (\phi_{ij}^\sharp)^{-1}$, hence, we alsow have that the induced scheme morphisms must satisfy $\phi_{ij} = \phi_{ji}^{-1}$. Now we note that:

$$U_{ij} \cap U_{ik} = U_{x_j/x_i} \cap U_{x_k/x_i} = U_{(x_j/x_i)(x_k/x_i)} \cong \operatorname{Spec}\mathbb{C}\left[\frac{x_0}{x_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}, \frac{x_i}{x_k}\right]$$

while:

$$U_{ji} \cap U_{jk} = U_{x_i/x_j} \cap U_{x_k/x_j} = U_{(x_i/x_j)(x_k/x_j)} \cong \operatorname{Spec}\mathbb{C}\left[\frac{x_0}{x_j}, \dots, \frac{\hat{x}_j}{x_j}, \dots, \frac{x_n}{x_j}, \frac{x_j}{x_i}, \frac{x_j}{x_k}\right]$$

where again we have that $(x_i/x_k) := (x_k/x_i)^{-1}$ and $(x_j/x_k) := (x_k/x_j)^{-1}$. We thus want $\phi_{ij}(U_{(x_j/x_i)(x_k/x_i)}) = U_{(x_i/x_j)(x_k/x_j)}$, and consider $U_{(x_j/x_i)(x_k/x_i)}$ and $U_{(x_i/x_j)(x_k/x_j)}$ as distinguished open sets of the affine schemes $U_{x_j/x_i} \cong \operatorname{Spec}\mathbb{C}[\{x_k/x_i\}_{k\neq i}, x_i/x_j]$ and $U_{x_i/x_j} \cong \operatorname{Spec}\mathbb{C}[\{x_k, x_j\}_{k\neq j}, x_j/x_i]$. Note that if $k = i$, then the statement is trivial.

Now suppose that $\mathfrak{p} \in U_{(x_i/x_j)(x_k/x_j)}$, then we have that $\mathfrak{p} \in \operatorname{Spec}\mathbb{C}[\{x_k, x_j\}_{k\neq j}, x_j/x_i]$, and $x_k/x_j \notin \mathfrak{p}$, it follows that since $\phi_{ij}^\sharp : (A_j)_{x_i/x_j} \to (A_i)_{x_j/x_i}$ is an isomorphism, that $\phi_{ij}^\sharp(\mathfrak{p})$ is a prime ideal of $(A_i)_{x_j/x_i}$, which satisfies $(\phi_{ij}^\sharp)^{-1}(\phi_{ij}^\sharp(\mathfrak{p})) = \phi_{ij}(\phi_{ij}^\sharp(\mathfrak{p})) = \mathfrak{p}$. Moreover, since $x_k/x_j \notin \mathfrak{p}$, we have that $\phi_{ij}^\sharp(x_k/x_j) = x_k/x_i \cdot x_i/x_j \notin \phi_{ij}^\sharp(\mathfrak{p})$, hence $x_k/x_i \notin \phi_{ij}^\sharp(\mathfrak{p})$ as x_i/x_j is a unit in $(A_i)_{x_j/x_i}$. It follows that $\phi_{ij}^\sharp(\mathfrak{p}) \in U_{(x_j/x_i)(x_k/x_i)}$, so $\mathfrak{p} \in \phi_{ij}(U_{(x_j/x_i)(x_k/x_i)})$.

We now let $\mathfrak{p} \in \phi_{ij}(U_{(x_j/x_i)(x_k/x_i)})$, then $\mathfrak{p} = (\phi_{ij}^\sharp)^{-1}(\mathfrak{q})$, for some $\mathfrak{q} \in U_{(x_j/x_i)(x_k/x_i)}$, implying that $x_k/x_i \notin \mathfrak{q}$. Since $x_k/x_i \notin \mathfrak{q}$, we have that $(\phi_{ij}^\sharp)^{-1}(x_k/x_i) = x_k/x_j \cdot (x_j/x_i) \notin (\phi_{ij}^\sharp)^{-1}(\mathfrak{q}) = \mathfrak{p}$, hence $x_k/x_j \notin \mathfrak{p}$. It follows by the same argument that $\mathfrak{p} \in U_{(x_i/x_j)(x_k/x_j)}$, so $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ as desired.

We now need to check that on $U_{ij} \cap U_{ik}$ we have:

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij}$$

Now note that $\phi_{ik}(U_{ij} \cap U_{ik}) = U_{kj} \cap U_{ki} = U_{(x_j/x_k)\cdot(x_k/x_i)} \cong \operatorname{Spec}\mathbb{C}\{\{x_l/x_k\}_{l\neq k}, x_k/x_j, x_k/x_i\}$, so the ring map which induces the morphism of schemes $\phi_{ik}|_{U_{ij}\cap U_{ik}} : U_{ij} \cap U_{ik} \longrightarrow U_{kj} \cap U_{kj}$ is given on generators by:

$$(\phi_{ik}^\sharp)_{U_{(x_j/x_k)\cdot(x_k/x_i)}} : \mathbb{C}[\{x_l/x_k\}_{l\neq k}, x_k/x_j, x_k/x_i] \longrightarrow \mathbb{C}[\{x_l/x_i\}_{l\neq i}, (x_i/x_j), (x_i/x_k)]$$
$$x_l/x_m \longmapsto \begin{cases} (x_l/x_i)\cdot(x_i/x_k) & \text{if } l \neq i, m = k \\ x_i/x_k & \text{if } l = i, m = k \\ x_k/x_i & \text{if } l = k, m = i \\ (x_i/x_j)\cdot(x_k/x_i) & \text{if } l = k, m = j \end{cases}$$

Now we essentially want to show that:

$$(\phi_{ik}^\sharp)_{U_{(x_j/x_k)\cdot(x_k/x_i)}} = (\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}} \circ (\phi_{jk}^\sharp)_{(x_j/x_k)\cdot(x_k/x_i)}$$

well similarly we have that $(\phi_{jk}^\sharp)_{(x_j/x_k)\cdot(x_k/x_i)}$ is given on generators by:

$$(\phi_{jk}^\sharp)_{(x_j/x_k)\cdot(x_k/x_i)} : \mathbb{C}[\{x_l/x_k\}_{l\neq k}, x_k/x_j, x_k/x_i] \longrightarrow \mathbb{C}[\{x_l/x_j\}_{l\neq j}, (x_j/x_i), (x_j/x_k)]$$
$$x_l/x_m \longmapsto \begin{cases} (x_l/x_j)\cdot(x_j/x_k) & \text{if } l \neq j, m = k \\ x_j/x_k & \text{if } l = j, m = k \\ x_k/x_j & \text{if } l = k, m = j \\ (x_j/x_i)\cdot(x_k/x_j) & \text{if } l = k, m = i \end{cases}$$

while:

$$(\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}} : \mathbb{C}[\{x_l/x_j\}_{l\neq j},(x_j/x_i),(x_j/x_k)] \longrightarrow \mathbb{C}[\{x_l/x_i\}_{l\neq i},(x_i/x_j),(x_i/x_k)]$$

$$x_l/x_m \longmapsto \begin{cases} (x_l/x_i)\cdot(x_i/x_j) & \text{if } l\neq i, m=j \\ x_i/x_j & \text{if } l=i, m=j \\ x_j/x_i & \text{if } l=j, m=i \\ (x_i/x_k)\cdot(x_j/x_i) & \text{if } l=j, m=k \end{cases}$$

We now check that these agree on generators. Let $x_l/x_k \in \mathbb{C}[\{x_l,x_k\}_{l\neq k}, x_k/x_i, x_k/x_j]$, such that $l \neq j$, then:

$$(\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}} \circ (\phi_{jk}^\sharp)_{(x_j/x_k)\cdot(x_k/x_i)}(x_l/x_k) = (\phi_{ij}^\sharp)_{U_{(x_i/x_j)}}((x_l/x_j)\cdot(x_j/x_k))$$

If $l \neq i$, we have that:

$$(\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}}((x_l/x_j)\cdot(x_j/x_k)) = (x_l/x_i)\cdot(x_i/x_j)\cdot(x_i/x_k)\cdot(x_j/x_i) = (x_l/x_i)\cdot(x_i/x_k)$$

however:

$$(\phi_{ik}^\sharp)_{U_{(x_j/x_i)\cdot(x_k/x_i)}}(x_l/x_k) = (x_l/x_i)\cdot(x_i/x_k)$$

If $l = i$, then we have that:

$$(\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}}((x_i/x_j)\cdot(x_j/x_k)) = (x_i/x_j)\cdot(x_i/x_k)\cdot(x_j/x_i) = x_i/x_k$$

but:

$$(\phi_{ik}^\sharp)_{U_{(x_j/x_i)\cdot(x_k/x_i)}}(x_i/x_k) = x_i/x_k$$

Now suppose that $l = j$, then:

$$\begin{aligned}(\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}} \circ (\phi_{jk}^\sharp)_{(x_j/x_k)\cdot(x_k/x_i)}(x_j/x_k) =&(\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}}(x_j/x_k) \\ =&(x_i/x_k)\cdot(x_j/x_i)\end{aligned}$$

while:

$$(\phi_{ik}^\sharp)_{U_{(x_j/x_i)\cdot(x_k/x_i)}}(x_j/x_k) = (x_j/x_i)\cdot(x_i/x_k)$$

Now for x_k/x_j, we have that:

$$\begin{aligned}(\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}} \circ (\phi_{jk}^\sharp)_{(x_j/x_k)\cdot(x_k/x_i)}(x_k/x_j) =&(\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}}(x_k/x_j) \\ =&(x_k/x_i)\cdot(x_i/x_j)\end{aligned}$$

while:

$$(\phi_{ik}^\sharp)_{U_{(x_j/x_i)\cdot(x_k/x_i)}}(x_k/x_j) = (x_i/x_j)\cdot(x_k/x_i)$$

And finally for x_k/x_i:

$$\begin{aligned}(\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}} \circ (\phi_{jk}^\sharp)_{(x_j/x_k)\cdot(x_k/x_i)}(x_k/x_i) =&(\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}}((x_j/x_i)\cdot(x_k/x_j)) \\ =&(x_j/x_i)\cdot(x_k/x_i)\cdot(x_i/x_j) \\ =&x_k/x_j\end{aligned}$$

while:

$$(\phi_{ik}^\sharp)_{U_{(x_j/x_i)\cdot(x_k/x_i)}}(x_k/x_i) = x_k/x_i$$

so indeed we have that:

$$(\phi_{ik}^\sharp)_{U_{(x_j/x_k)\cdot(x_k/x_i)}} = (\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}} \circ (\phi_{jk}^\sharp)_{(x_j/x_k)\cdot(x_k/x_i)}$$

implying that:

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij}$$

as desired. It follows by Theorem 2.1.1 that the affine schemes $\operatorname{Spec} A_i$ glue together to form a scheme which we denote by $\mathbb{P}^n_{\mathbb{C}}$. We denote the open embeddings $\operatorname{Spec} A_i \to \mathbb{P}^n_{\mathbb{C}}$ by ψ_i, their topological images in $\mathbb{P}^n_{\mathbb{C}}$ by $\mathcal{A}_i$, and the sheaf isomorphisms $\mathscr{O}_{\mathcal{A}_i}|_{\mathcal{A}_i \cap \mathcal{A}_j} \to \mathscr{O}_{\mathcal{A}_j}|_{\mathcal{A}_i \cap \mathcal{A}_j}$ by β_{ij}.

We see that $\mathbb{P}^n_{\mathbb{C}}$ is not affine by calculating it's global ring of sections. We have that:

$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}(\mathbb{P}^n_{\mathbb{C}}) = \left\{ (s_i) \in \prod_{i=0}^{n} \mathscr{O}_{\mathcal{A}_i}(\mathcal{A}_i) : \forall i, j, \beta_{ij}(s_i|_{\mathcal{A}_i \cap \mathcal{A}_j}) = s_j|_{\mathcal{A}_i \cap \mathcal{A}_j} \right\}$$

We first note that:

$$\mathcal{A}_i \cap \mathcal{A}_j = \psi_i(U_{ij}) = \psi_i((U_{x_j/x_i}))$$

and that:

$$\mathscr{O}_{\mathcal{A}_i}(\mathcal{A}_i) \cong A_i$$

Denote by π_{ij} the localization map $A_i \to (A_i)_{x_j/x_i}$, then it follows that:

$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}(\mathbb{P}^n_{\mathbb{C}}) \cong \left\{ (s_i) \in \prod_{i=0}^{n} A_i : \forall i, j, \phi^\sharp_{ji}(\pi_{ij}(s_i)) = \pi_{ji}(s_j) \right\}$$

We know that any element in A_i and A_j can only be written as polynomials in the variables x_l/x_i and x_m/x_j, where $l \neq i$ and $m \neq j$, and the localization maps are the inclusions into the polynomial rings discussed above. We see that for $l \neq j$:

$$\phi^\sharp_{ji}(x_l/x_i) = (x_l/x_j) \cdot (x_j/x_i) \notin \operatorname{im} \pi_{ji}$$

and that if $l = j$ then:

$$\phi^\sharp_{ji}(x_j/x_i) = x_j/x_i \notin \operatorname{im} \pi_{ji}$$

hence the only polynomials $s_i \in A_i$ which can possibly satisfy $\phi^\sharp_{ji}(\pi_{ij}(s_i)) = \pi_{ji}(s_j)$ are the constant ones. However, $\phi^\sharp_{ji} \circ \pi_{ij}$ is the identity on constant polynomials, hence we must have that:

$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}(\mathbb{P}^n_{\mathbb{C}}) \cong \left\{ (s_i) \in \prod_{i=0}^{n} \mathbb{C} : \forall i, j, s_i = s_j \right\} \cong \mathbb{C}$$

so $\mathbb{P}^n_{\mathbb{C}}$ is certainly not affine.

We now discuss why we denote this by $\mathbb{P}^n_{\mathbb{C}}$, by showing that is an analogue of complex projective space $\mathbb{P}^n$ from differential geometry. First note that $\mathbb{C}$ is algebraically closed, so by the weak Nullstellensatz[23], the maximal ideals of A_i are of the form:

$$(z_0, \ldots, \hat{z}_i, \ldots, z_n) := \left\langle \frac{x_0}{x_i} - z_0, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i} - z_n \right\rangle$$

where $z_j \in \mathbb{C}$ for all j. It is easy that any maximal ideal corresponds to a closed point of $\operatorname{Spec} A_i$ and any closed point of $\operatorname{Spec} A_i$ must in turn be a maximal ideal. Since the embeddings ψ_i determine the topology on $\mathbb{P}^n_{\mathbb{C}}$, so a point $[x] \in \mathbb{P}^n_{\mathbb{C}}$ is closed, if only if $\psi_i^{-1}([x])$ is a maximal ideal of A_i for all i[24]. Let $[x] \in \mathcal{A}_i$ be closed, if $[x] \notin \mathcal{A}_j$ for any other $j \neq i$ then we claim that is the origin in $\operatorname{Spec} A_i$:

$$\psi^{-1}([x])_i = (0, \ldots, 0) := \left\langle \frac{x_0}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i} \right\rangle$$

[23]One could also potentially argue this fact using Zariski's lemma (Theorem 6.1.3), and the fact that $\mathbb{C}$ is algebraically closed.

[24]Note that if $x \notin \mathcal{A}_i$, then we have that $\psi_i^{-1}([x])$ is empty and thus closed

Indeed, if $[x] \notin \mathcal{A}_j$, then we must have that $\psi_i^{-1}([x]) \notin U_{ij}$ for all j, implying that $x_j/x_i \in \psi_i^{-1}([x])$ for all j, so clearly $\psi_i^{-1}([x])$ is the origin. Now if $[x] \in \mathcal{A}_i \cap \mathcal{A}_j$, we have that $[x] \in \psi_i(U_{ij}) = \psi_j(U_{ji})$, hence $\psi_i^{-1}([x])$, is equivalent to $\psi_j^{-1}([x])$. Indeed, we have that $\psi_i^{-1}([x]) \in U_{ij}$, and $\psi_j^{-1}([x]) \in U_{ji}$. We need to show that:

$$\phi_{ij}(\psi_i^{-1}([x])) = \psi_j^{-1}([x])$$

Well, apply ψ_j to the left hand side:

$$\psi_j(\phi_{ij}(\psi_i^{-1}([x]))) = \psi_i(\psi_i^{-1}([x])) = [x]$$

while clearly $\psi_j(\psi_j^{-1}([x])) = [x]$, so since ψ_j is injective we have the desired equality, implying $\psi_i^{-1}([x]) \sim \psi_j^{-1}([x])$. Now let:

$$\psi_i^{-1}([x]) = x_i = (z_0, \ldots, \hat{z}_i, \ldots, z_n) = \left\langle \frac{x_0}{x_i} - z_0, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i} - z_n \right\rangle$$

Now we see that under the isomorphism $U_{ij} \cong (\operatorname{Spec} A_i)_{x_j/x_i}$, x_i gets mapped to the ideal:

$$\left\langle \frac{x_0}{x_i} - z_0, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i} - z_n \right\rangle \subset \mathbb{C}[\{x_k/x_i\}_{k\neq i}, x_i/x_j]$$

and that under ϕ_{ij} we have that:

$$\phi_{ij}(x_i) = \left\langle \frac{x_0}{x_j} \cdot \frac{x_j}{x_i} - z_0, \ldots, \frac{\hat{x}_j}{x_j}, \ldots, \frac{x_j}{x_i} - z_j, \ldots, \frac{x_n}{x_i}\frac{x_j}{x_i} - z_n \right\rangle$$

which by the same argument as in Example 2.1.5 can be rewritten as:

$$\phi_{ij}(x_i) = \left\langle \frac{x_0}{x_j} \cdot \frac{x_j}{x_i} - z_0, \ldots, \frac{\hat{x}_j}{x_j}, \ldots, \frac{x_i}{x_j} - \frac{1}{z_j}, \ldots, \frac{x_n}{x_j}\frac{x_j}{x_i} - z_n \right\rangle$$

We claim that this ideal is equal to:

$$J = \left\langle \frac{x_0}{x_j} - \frac{z_0}{z_j}, \ldots, \frac{\hat{x}_j}{x_j}, \ldots, \frac{x_i}{x_j} - \frac{1}{z_j}, \ldots, \frac{x_n}{x_j} - \frac{z_n}{z_j} \right\rangle$$

Clearly, we need only show that any generator of J lies in $\phi_{ij}(x_i)$ and vice versa. Note that:

$$\frac{x_i}{x_j} \cdot \left(\frac{x_k}{x_j}\frac{x_j}{x_i} - z_k \right) + z_k \cdot \left(\frac{x_i}{x_j} - \frac{1}{z_j} \right) = \frac{x_k}{x_j} - z_k\frac{x_i}{x_j} + z_k\frac{x_i}{x_j} - \frac{z_k}{z_j} = \frac{x_k}{x_j} - \frac{z_k}{z_j}$$

so $x_k/x_j - z_k/z_j \in \phi_{ij}(x_i)$. Meanwhile, we see that:

$$\frac{x_k}{x_j}\frac{x_j}{x_i} - z_k = \frac{x_j}{x_i} \cdot \left(\frac{x_k}{x_j} - \frac{z_k}{z_i} \right) - z_k \cdot \frac{x_j}{x_i}\left(\frac{x_i}{x_j} - \frac{1}{z_j} \right)$$

so $(x_k/x_J) \cdot (x_j/x_i) - z_k \in J$ as well. It follows that:

$$x_j = \left(\frac{z_0}{z_j}, \ldots, \frac{1}{z_j}, \ldots, \frac{z_n}{z_j} \right)$$

We denote the set of closed points of $X_i = \operatorname{Spec} A_i$ by $|X_i|$, and construct a set map:

$$F : \coprod_{i=0}^{n} |X_i| \longrightarrow \mathbb{P}^n$$
$$x_i = (z_0, \ldots, \hat{z}_i, \ldots, z_n) \in X_i \longmapsto [z_0, \ldots, z_{i-1}, 1, z_i, \ldots, z_n]$$

and note that such a map is clearly surjective, as every equivalence class $[w_0, \ldots, w_n]$ must have at least one non zero entry w_k, so we can always rescale to obtain something of the above form. Moreover, we see that if $x_i \sim x_j$, then we have that by our previous work:

$$x_j = \left(\frac{z_0}{z_j}, \ldots, \frac{z_{i-1}}{z_j}, \frac{1}{z_j}, \frac{z_{i+1}}{z_j}, \ldots, \hat{z}_j, \ldots, \frac{z_n}{z_j} \right)$$

so:

$$\begin{aligned}F(x_j) = &\left[\frac{z_0}{z_j}, \dots, \frac{z_{i-1}}{z_j}, \frac{1}{z_j}, \frac{z_{i+1}}{z_j}, \dots, \frac{z_{j-1}}{z_j}, 1, \frac{z_{j+1}}{z_j} \dots, \frac{z_n}{z_j}\right] \\ =&[z_0, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n] \\ =&F(x_i)\end{aligned}$$

so F factors through the quotient, and we thus obtain a map:

$$\begin{aligned}\tilde{F} : |\mathbb{P}^n_{\mathbb{C}}| &\longrightarrow \mathbb{P}^n \\ [x] &\longmapsto F(\psi_i^{-1}(x))\end{aligned}$$

for any i such that $[x] \in \mathcal{A}_i$. It is already surjective as F is surjective, so it suffices to check that if $F(x_i) = F(y_j)$ then $x_i \sim y_j$. First note that if $j = i$, then we clearly have that $x_i = y_j$, as the only way for:

$$(z_1, \dots, z_{i-1}, 1, z_{i+1}, z_n) = \lambda(w_1, \dots, z_{i-1}, 1, z_{i+1}, w_n))$$

is if $1 = \lambda$. Now suppose that:

$$[z_0, \dots, z_{i-1}, 1, z_{i+1}, \cdots z_n] = [w_0, \dots, w_{j-1}, 1, w_{j+1}, \dots, w_n]$$

then we must clearly have that $w_i, z_j \neq 0$, hence $x_i \in U_{ij}$ and $y_j \in U_{ji}$. It follows that we can rewrite the right hand side as:

$$\left[\frac{w_0}{w_i}, \dots, \frac{w_{i-1}}{w_i}, 1, \frac{w_{i+1}}{w_i}, \dots, \frac{w_{j-1}}{w_i}, 1, \frac{w_{j+1}}{w_i}, \dots, \frac{w_n}{w_i}\right]$$

Since the right hand side now has 1 in the ith spot, it follows that:

$$z_k = \frac{w_k}{w_i}$$

hence:

$$x_i = \phi_{ji}(y_j)$$

so the $x_i \sim y_j$ and $\tilde{F}$ injective. It follows that $\tilde{F}$ is a set isomorphism, so we can identify the closed points of $\mathbb{P}^n_{\mathbb{C}}$ with the classical projective space $\mathbb{P}^n$. We thus call $\mathbb{P}^n_{\mathbb{C}}$ the n-dimensional projective scheme over $\mathbb{C}$.

Note, that in the gluing process, we never used the fact that $\mathbb{C}$ was a field, or algebraically closed, so we could easily repeat this process with any ring or field A and obtain a projective scheme $\mathbb{P}^n_A$. We will however, lose the identification of closed points with classical projective space. Indeed, if we were to look at $\mathbb{P}^1_{\mathbb{R}}$, then $\left\langle (x/y)^2 + 1 \right\rangle \in \operatorname{Spec} \mathbb{R}[x/y] \subset \mathbb{P}^1_{\mathbb{R}}$ is a closed point, but has no corresponding element in $\mathbb{RP}^1$.

Now before we move onto to discussing projective schemes in generality, we quickly show that $\mathbb{P}^n_{\mathbb{C}}$ satisfies another property which make it's remarkably similar to $\mathbb{P}^n$. Indeed, there exits a canonical map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ given by:

$$(z_0, \dots, z_n) \longmapsto [z_0, \dots, z_n]$$

We show now show a similar statement for the scheme $\mathbb{P}^n_{\mathbb{C}}$.

Lemma 2.2.1. *Let $\mathbb{P}^n_{\mathbb{C}}$ be the scheme constructed in Example 2.1.5, and $\mathbb{A}^{n+1} \setminus \{0\}$ the affine scheme* $\operatorname{Spec} \mathbb{C}[x_0, \dots, x_n]$ *minus the closed point $\langle x_0, \dots, x_n \rangle$. Then there exists a morphism:*

$$\mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n_{\mathbb{C}}$$

which one closed points satisfies:

$$(z_0, \dots, z_n) \longrightarrow [z_0, \dots, z_n]$$

under the identification of $|\mathbb{P}^n_{\mathbb{C}}|$ with $\mathbb{P}^n$.

Proof. Note that $\mathbb{A}^{n+1} \setminus \{0\}$ is indeed an open subscheme of $\mathbb{A}^{n+1}$, and admits an open cover of distinguished opens by:

$$\mathbb{A}^{n+1} \setminus 0 = \bigcup_{i=0}^{n} U_{x_i}$$

We also note by Corollary 1.2, that the structure sheaf on $\mathbb{A}^{n+1} \setminus \{0\}$, which we denote by Y going forward, is isomorphic to the one obtained by gluing the sheafs $\mathscr{O}_{U_{x_i}} = \mathscr{O}_{\mathbb{A}^{n+1}}|_{U_{x_i}}$ together, where the transition functions $\beta_{ij} : \mathscr{O}_{U_{x_i}}|_{U_{x_i} \cap U_{x_j}} \to \mathscr{O}_{U_{x_j}}|_{U_{x_i} \cap U_{x_j}}$ are the identity maps. It is then clear that the sheaf of global sections is satisfies $\mathscr{O}_Y(Y) \cong \mathbb{C}[x_0, \ldots, x_n]$ by the same argument in Example 2.1.2.

We will now make use of of Proposition 1.3.3 to obtain a morphism of locally ringed spaces. We note that $X_i = \operatorname{Spec} \mathbb{C}[\{x_l/x_i\}_{l \neq i}]$ is an embedded open subscheme of $\mathbb{P}^n_{\mathbb{C}}$ for all $0 \leq i \leq n$, such that the closed points X_i correspond precisely to the closed points of $\mathcal{A}_i = \psi_i(X_i)$ which can be written in the form:

$$[z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n]$$

It thus makes sense to define morphisms $\xi_i : U_{x_i} \longrightarrow X_i$, and then compose with compose with the embedding $\psi_i : X_i \to \mathbb{P}^n_{\mathbb{C}}$. Since U_{x_i} and X_i are both affine schemes, we need only define a ring map:

$$\xi_i^\sharp : \mathbb{C}[\{x_l/x_i\}_{l \neq i}] \longrightarrow \mathbb{C}[x_0, \ldots, x_n, x_i^{-1}]$$

Given our suggestive choice of notation, it should be no surprise that we define this map on generators by:

$$\frac{x_j}{x_i} \longrightarrow x_j \cdot x_i^{-1} \tag{2.2.1}$$

We see that any closed point of U_{x_i} is of the form $(z_0, \ldots, z_i, \ldots, z_n)$, where $z_i \neq 0$, and that this is the ideal:

$$I = \langle x_0 - z_0, \ldots, x_i - z_i, \ldots, x_n - z_n \rangle \subset \mathbb{C}[x_0, \ldots, x_n, x_i^{-1}]$$

under the identification $U_{x_i} \cong \operatorname{Spec} \mathbb{C}[x_0, \ldots, x_n, x_i^{-1}]$. We claim that:

$$(\xi_i^\sharp)^{-1}(I) = \left\langle \frac{x_0}{x_i} - \frac{z_0}{z_i}, \ldots, \frac{x_n}{x_i} - \frac{z_n}{z_i} \right\rangle$$

Since the right hand side is clearly a maximal ideal, we clearly need only show that each generator lies in $(\xi_i^\sharp)^{-1}(I)$. Now, note that:

$$\xi_i^\sharp(x_l/x_i - z_l/z_i) = x_l \cdot x_i^{-1} - z_l/z_i$$

however:

$$\begin{aligned} x_i^{-1}(x_l - z_l) + z_l \cdot (-x_i^{-1} \cdot z_i^{-1})(x_i - z_i) =& x_l x_i^{-1} - x_i^{-1} - z_l x_i^{-1} - z_l z_i^{-1} + z_l x_i^{-1} \\ =& x_l x_i^{-1} - z_l z_i^{-1} \end{aligned}$$

implying the claim. It follows that under the embedding ψ_i, we have that:

$$f_i((z_0, \ldots, z_n)) = \psi_i \circ (\xi_i^\sharp)^{-1}((z_0, \ldots, z_n)) = \psi_i((z_0/z_i, \ldots, \hat{z}_i, \ldots, z_n/z_i))$$

which is identified with $[z_0/z_i, \ldots, z_{i-1}/z_i, 1, z_{i+1}/z_i, \ldots, z_n] = [z_0, \ldots, z_n] \in \mathbb{P}^n$. So, on closed points f_i provides the correct map.

We now check that $f_i|_{U_{x_i} \cap U_{x_j}} = f_j|_{U_{x_i} \cap U_{x_j}}$. Note, that $f_i = \psi_i \circ \xi_i$, where ξ_i is the scheme morphism $U_{x_i} \to X_i$ induced by the ring map defined by (2.5). It thus suffices to check that a), $f_i|_{U_{x_i} \cap U_{x_j}}$ has image in U_{ij}, and b) that $\phi_{ij} \circ \xi_i|_{U_{x_i} \cap U_{x_j}} = \xi_j|_{U_{x_i} \cap U_{x_j}}$. Indeed, if b) holds, then we have that:

$$\psi_j \circ \phi_{ij} \circ \xi_i|_{U_{x_i} \cap U_{x_j}} = \psi_j \circ \xi_j|_{U_{x_i} \cap U_{x_j}} \Longrightarrow \psi_i \circ \xi_i|_{U_{x_i} \cap U_{x_j}} = \psi_j \circ \xi_j|_{U_{x_i} \cap U_{x_j}}$$

We first check that:

$$\xi_i(U_{x_i} \cap U_{x_j})45 = U_{x_j/x_i} \subset X_i = \operatorname{Spec} \mathbb{C}[\{x_l/x_i\}_{l \neq i}]$$

Note that $U_{x_i} \cap U_{x_j} = U_{x_i x_j} \subset U_{x_i}$, is the distinguished open of U_{x_i} consisting of prime ideals $\mathfrak{p} \subset \mathbb{C}[x_0, \ldots, x_n, x_i^{-1}]$ such that $x_j \notin \mathfrak{p}$. We need to show then that $x_j/x_i \notin (\xi_i^\sharp)^{-1}(\mathfrak{p})$. Well, if $x_j/x_i \in (\xi_i^\sharp)^{-1}(\mathfrak{p})$, then $\xi^\sharp(x_j/x_i) = x_j \cdot x_i^{-1} \in \mathfrak{p}$, implying that $x_j \in \mathfrak{p}$, as x_i^{-1} is a unit. It follows that ξ_i has image contained in $U_{x_i} \cap U_{x_j}$.

Now note that the ring map inducing the scheme morphism $U_{x_i x_j} \to U_{ij}$ is given by:

$$(\xi_i^\sharp)_{U_{x_i x_j}} : \mathbb{C}[\{x_l/x_i\}_{l \neq i}, x_i/x_j] \longrightarrow \mathbb{C}[x_0, \ldots, x_n, x_i^{-1}, x_j^{-1}]$$
$$x_l/x_m \longmapsto \begin{cases} x_l \cdot x_i^{-1} & \text{if } l \neq i, m = i \\ x_i \cdot x_j^{-1} & \text{if } l = i, m = j \end{cases}$$

while for $U_{x_i x_j} \to U_{ji}$ it is given by:

$$(\xi_j^\sharp)_{U_{x_i x_j}} : \mathbb{C}[\{x_l/x_j\}_{l \neq j}, x_j/x_i] \longrightarrow \mathbb{C}[x_0, \ldots, x_n, x_i^{-1}, x_j^{-1}]$$
$$x_l/x_m \longmapsto \begin{cases} x_l \cdot x_j^{-1} & \text{if } l \neq j, m = j \\ x_j \cdot x_i^{-1} & \text{if } l = j, m = i \end{cases}$$

and it now suffices to check that:

$$\xi_i^\sharp|_{U_{x_i x_j}} \circ \phi_{ij}^\sharp = \xi_j^\sharp|_{U_{x_i x_j}}$$

and we do so on generators. Let $x_l/x_j \in \mathbb{C}[\{x_l/x_j\}_{l \neq j}, x_j/x_i]$ such that $l \neq i$. Then:

$$\phi_{ij}^\sharp(x_l/x_j) = (x_l/x_i) \cdot (x_i/x_j)$$

and:

$$(\xi_i^\sharp)_{U_{x_i x_j}}((x_l/x_i) \cdot (x_i/x_j)) = x_l \cdot x_i^{-1} \cdot x_i \cdot x_j^{-1} = x_l \cdot x_j^{-1} = \xi_j^\sharp|_{U_{x_i x_j}}(x_l/x_j)$$

Now examine x_i/x_j, then:

$$\xi_i^\sharp|_{U_{x_i x_j}} \circ \phi_{ij}^\sharp(x_i/x_j) = \xi_i^\sharp|_{U_{x_i x_j}}(x_i/x_j) = x_i \cdot x_j^{-1}$$

while:

$$\xi_j^\sharp(x_i/x_j) = x_i \cdot x_j^{-1}$$

Finally, for x_j/x_i, we have that:

$$(\xi_i^\sharp)_{U_{x_i x_j}} \circ \phi_{ij}^\sharp(x_j/x_i) = (\xi_i^\sharp)_{U_{x_i x_j}}(x_j/x_i) = x_j \cdot x_i^{-1}$$

while:

$$(\xi_j^\sharp)_{U_{x_i x_j}}(x_j/x_i) = x_j \cdot x_i^{-1}$$

It thus follows that:

$$\xi_i^\sharp|_{U_{x_i x_j}} \circ \phi_{ij}^\sharp = \xi_j^\sharp|_{U_{x_i x_j}}$$

hence:

$$\phi_{ij} \circ \xi_i|_{U_{x_i x_j}} = \xi_j|_{U_{x_i x_j}}$$

and it follows that the scheme morphisms $f_i : \psi_i \circ \xi_i$ glue together to form a map:

$$f : \mathbb{A}^{n+1} \smallsetminus \{0\} \longrightarrow \mathbb{P}^n_{\mathbb{C}}$$

which clearly sends closed points:

$$(z_0, \ldots, z_n) \longmapsto [z_0, \ldots, z_n]$$

$\square$

Now, as promised we move forward with the Proj construction. Much like Spec, we will see that Proj takes a commutative ring to a scheme, however, in this case, we will have that *a*) the ring must have a *grading*, *b*) the scheme will not in general be affine, and *c*) Proj is not a functor. We will in fact find that:

$$\mathbb{P}^n_{\mathbb{C}} \cong \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_n]$$

We need the following definition:

Definition 2.2.1. A **$\mathbb{Z}$-graded ring** is a direct sum of abelian groups :

$$A = \bigoplus_{n \in \mathbb{Z}} A_n$$

equipped with a ring structure such that $A_i \cdot A_j \subset A_{i+j}$ for all $i, j \in \mathbb{Z}$. We call elements of A_i **homogenous elements of degree** i. A **homogenous ideal** is an ideal generated by homogeneous elements, and a **graded ideal** is an ideal such that

$$I = \bigoplus_{n \in \mathbb{Z}} (I \cap A_n)$$

Clearly, we have that $A_0 \subset A$ is a subring, A_i is an A_0 module for all i, and A itself is and A_0 algebra. We often make the mild sin of referring to a $\mathbb{Z}$-graded ring as a graded ring, and so the reader should always assume we mean a ring with with $\mathbb{Z}$-graded structure unless state otherwise. Indeed there are other notions of a graded ring over other abelian groups, and so we will clarify should the need arise. We prove the following facts from commutative algebra:

Lemma 2.2.2. *Let A be a graded ring, and $I, J \subset A$ ideals of A. Then the following hold:*

a) *I is homogeneous if and only if it is graded*

b) *If I and J are homogenous, then IJ, $I + J$, $I \cap J$, and $\sqrt{I}$ are homogenous.*

c) *If I is homogenous, then I is prime if $I \neq A$ and for any homogenous elements $a, b \in A$, $a \cdot b \in I$ implies that $a \in I$ or $b \in I$.*

Proof. We start with *a*). Suppose that I is graded, then any $i \in I$ can be written as the finite sum:

$$i = \sum_i a_i$$

where each $a_i \in I \cap A_n$. Each a_i is homogenous, so it follows that I is generated by homogenous elements.

Suppose that I is generated by homogenous elements, then any $i \in I$ can be written as the finite sum:

$$i = \sum_i a_i \cdot b_i$$

where $a_i \in A$, and each $b_i \in I \cap A_i$. Since A is graded, for each a_i we can write:

$$a_i = \sum_j a_{ij}$$

where each $a_{ij} \in A_j$. It follows that:

$$i = \sum_{i,j} a_{ij} b_i$$

It follows that $a_{ij} b_i \in A_{i+j}$ for all i and j, so we can rewrite i as the finite sum:

$$i = \sum_n \sum_{i+j=n} a_{ij} b_i$$

then for each n set:

$$d_n = \sum_{i+j=n} a_{ij} b_i$$

It is then clear that $d_n \in I \cap A_n$ hence:

$$i = \sum_n d_n$$

Since $(I \cap A_n) \cap (I \cap A_m) = I \cap (A_n \cap A_m) = I \cap \{0\} = \{0\}$ it thus follows that:

$$I = \bigoplus_{n \in \mathbb{Z}} (I \cap A_n)$$

Now let I and J be homogenous ideals of A. We see that:

$$IJ = \langle ij : i \in I, j \in J \rangle$$

if S_I is the generating set of I, and S_J is the generating set of J, then we see that:

$$IJ = \langle S_I \cdot S_J \rangle$$

where:

$$S_I \cdot S_J = \{s \cdot t : s \in S_I, t \in S_J\}$$

Since all s and t are homogenous, it follows that $s \cdot t$ is homogenous, hence IJ is generated by homogenous elements, implying that IJ is homogenous.

The sum $I + J$ is the ideal:

$$I + J = \langle S_I \cup S_J \rangle$$

so $I + J$ is indeed generated by homogenous elements, implying that $I + J$ is homogenous.

Now consider $I \cap J$; we have that by a):

$$I = \bigoplus_{n \in \mathbb{Z}} (I \cap A_n) \qquad \text{and} \qquad J = \bigoplus_{n \in \mathbb{Z}} (J \cap A_n)$$

We claim that:

$$\bigoplus_{n \in \mathbb{Z}} (I \cap A_n) \cap \bigoplus_{n \in \mathbb{Z}} (J \cap A_n) = \bigoplus_{n \in \mathbb{Z}} (I \cap J \cap A_n)$$

Let $i \in \bigoplus_{n \in \mathbb{Z}} (I \cap J \cap A_n)$, then:

$$i = \sum_n a_n$$

where $a_n \in I \cap J \cap A_n$. It follows that $a_n \in I \cap A_n$, and $a_n \in \cap J \cap A_n$ for all n, hence $i \in \bigoplus_{n \in \mathbb{Z}} (I \cap A_n) \cap \bigoplus_{n \in \mathbb{Z}} (J \cap A_n)$. Now suppose that $i \in \bigoplus_{n \in \mathbb{Z}} (I \cap A_n) \cap \bigoplus_{n \in \mathbb{Z}} (J \cap A_n)$, then:

$$i = \sum_n a_n$$

where each $a_n \in I \cap A_n$ and:

$$i = \sum_n b_n$$

where $b_n \in J \cap A_n$. It follows that:

$$\sum_n a_n - b_n = 0$$

and since the intersection $A_n \cap A_m = \{0\}$ we must have that $a_n = b_n$ for all n. It follows that $a_n \in I \cap J \cap A_n$ for all n, implying the claim.

Now consider the radical of I:

$$\sqrt{I} = \{a \in A : a^n \in I\}$$

Let $a \in \sqrt{I}$, then since $a^n \in I$ we can write:

$$a^n = \sum_j b_j$$

where each $b_j \in I \cap A_n$. We can write a as:

$$a = \sum_i a_i$$

Then there exists a top degree element a_m, and it follows that $a_m^n = b_{nm} \in I \cap A_{nm}$, hence $a_m \in \sqrt{I} \cap A_m$. Now $a - a_m \in \sqrt{I}$, so we can apply the same argument to the next highest graded piece. It follows that $a_i \in \sqrt{I} \cap A_i$ for all i, so:

$$\sqrt{I} = \bigoplus_{n \in \mathbb{Z}} \sqrt{I} \cap A_n$$

and is thus homogenous by a).

To prove c) suppose I homogenous, not equal I, and suppose that for any homogenous elements $a, b \in A$, $a \cdot b \in I$ implies that $a \in I$ or $b \in I$. Now let $a, b \in A$ be arbitrary, and write:

$$a = \sum_i a_i \qquad \text{and} \qquad b = \sum_i b_i$$

where $a_i, b_i \in A_i$. Suppose that $a \cdot b \in I$, but neither a nor $b \in I$; we will prove the contrapositive. Since $a, b \notin I$, and I is graded, there is lowest degree m and l such that $a_m, b_l \notin I$, and $a_i, b_j \in I$ for all $i < m$ and $j < l$[25]. Now note that the product is given by:

$$a \cdot b = \sum_{i,j} a_i b_j$$

and we have the $m + l$th component of $a \cdot b$ is given by:

$$(a \cdot b)_{m+l} = \sum_{i+j=m+l} a_i b_j \in I \cap A_{m+l}$$

For each such i and j not equal to m and l, we must either have that $i > m$ or $j > l$, but if $i > m$ or $j > l$ then $a_i \in I$ or $b_j \in I$, hence all such $a_i b_j \in I$. It follows that $a_m b_l \in I$, but neither $a_m \in I$ nor $b_l \in I$, so the claim follows by the contrapositive. □

Definition 2.2.2. Let A and B be rings, and $\phi : A \to B$ a ring homomorphism. Then, ϕ is a **graded ring homomorphism** if for all $n \in \mathbb{Z}$, $\phi(A_n) \subset B_n$. A graded ring homomorphism is a **graded ring isomorphism** if it is graded, and an isomorphism as a ring homomorphism.

We note that homogenous ideals are precisely those that lead to graded quotients.

Lemma 2.2.3. *Let A be a graded ring, and I a homogenous ideal, then:*

$$A/I = \bigoplus_{n \in \mathbb{Z}} \pi(A_n) \cong \bigoplus_{n \in \mathbb{Z}} A_n/I_n$$

where π is the quotient map, and $I_n = A_n \cap I_n$.

Proof. We note that since $\pi : A \to A/I$ is a surjective homomorphism that:

$$[a] = \left[\sum_n a_n\right] = \sum_n [a_n]$$

[25] If this is lowest degree is zero, then the elements are homogenous and the contrapositive is immediate.

clearly each $[a_n] \in \pi(A_n)$, so any $[a]$ can be written as a finite sum, where each element lies in $\pi(A_n)$. To see that this admits a grading, we check that $\pi(A_n) \cap \pi(A_m) = \{0\}$. Let $[a] \in \pi(A_n) \cap \pi(A_m)$, then we have that there is $a_m \in A_m$ and $a_n \in A_n$ such that:

$$[a_m] = [a_n]$$

which implies that there exists an $i \in I$ such that:

$$a_m + i = a_n$$

Since i is graded we can write this as:

$$a_m + \sum_j b_j = a_n$$

where $b_j \in A_j \in I$. It follows that:

$$a_n - a_m = \sum_j b_J$$

but a_n and a_m are homogenous, so $b_i = 0$ for $i \neq m, n$, and $b_m = -a_m$ and $b_n = a_n$. This then implies that both a_m and a_n lie in I, hence $[a] = 0$. It follows that:

$$A/I = \bigoplus_{n \in \mathbb{Z}} \pi(A_n)$$

Moreover, we see that $[a_m] \cdot [a_n] = [a_m \cdot a_n] \in \pi(A_{m+n})$, so A/I is a graded ring.

We now define the following homomorphism of abelian groups:

$$\begin{aligned} \phi_n : A_n &\longrightarrow A/I \\ a_n &\longmapsto [a_n] \end{aligned}$$

clearly this is a surjection onto $\pi(A_n)$, and clearly $\ker \phi_n = I_n$, hence ϕ_n descends to an isomorphism ψ_n:

$$\begin{aligned} \psi_n : A_n/I_n &\longrightarrow \pi(A_n) \\ [a_n]_n &\longmapsto [a_n] \end{aligned}$$

where the n subscript denotes taking the equivalence class in A_n/I_n. We take the direct sum of abelian modules:

$$\bigoplus_{n \in \mathbb{Z}} A_n/I_n$$

and equip with it the ring structure defined on homogenous elements by:

$$[a_n]_n \cdot [a_m]_m = [a_m \cdot a_n]_{m+n}$$

and extend to linearly. This is clearly well defined as I is a graded ideal, hence we define the isomorphism:

$$\begin{aligned} \Psi : \bigoplus_{n \in \mathbb{Z}} A_n/I_n &\longrightarrow \bigoplus_{n \in \mathbb{Z}} \pi(A_n) \\ \sum_n [a_n]_n &\longmapsto \sum_n \psi_n([a_n]_n) = \sum_n [a_n] \end{aligned}$$

Which is clearly a ring homomorphism as:

$$\psi_{m+n}([a_m]_m \cdot [a_n]_n) = [a_m \cdot a_n]$$

while:

$$\psi_m([a_m]_m) \cdot \psi_n([a_n]_n) = [a_m \cdot a_n]$$

so it is clearly a graded isomorphism of rings. □

We have a similar result for localization:

Lemma 2.2.4. *Let A be a graded ring, and S be a multiplicatively closed subset of A containing only homogenous elements. Then $S^{-1}A$ has the natural structure of a graded ring.*

Proof. We define a grading on $S^{-1}A$ by first defining the homogenous elements of $S^{-1}A$ to be those of the form:

$$H = \left\{ \frac{a}{s} : s \in S, a \text{ is homogenous} \right\}$$

Note that this indeed makes sense as S contains only homogenous elements. We then define the degree of any $a/s \in H$ as $\deg a - \deg s$. We check that this well defined, suppose that:

$$\frac{a}{s} = \frac{b}{t}$$

then there is $u \in U$ such that:

$$u(at - bs) = 0$$

We note that since $a \cdot t$ and $b \cdot s$ are homogenous, we must have that $\deg(a \cdot t) = \deg a + \deg t = \deg b + \deg s = \deg(d \cdot s)$, hence:

$$\deg a - \deg s = \deg b - \deg t$$

so the degree of an element is well defined. We define the set:

$$(S^{-1}A)_m = \left\{ \frac{a}{s} \in H : \deg(a/s) = m \text{ or } \exists u \in S, u \cdot a = 0 \right\}$$

We claim that this is a subgroup of $S^{-1}A$; indeed $0 \in (S^{-1}A)_m$ as $0/s$ satisfies $u \cdot 0 = 0$ for all $u \in S$. Now, suppose suppose that $a/s \in (S^{-1}A)_m$, then $\deg(-a/s) = m$, and $(a/s) + (-a/s) = 0$, so $(S^{-1}A)_m$ contains inverses. Now, let a/s and b/t in $(S^{-1}A)_m$, then:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$

so:

$$\deg \left(\frac{at + bs}{st} \right) = \deg(at + bs) - \deg(st)$$

Note that:

$$\deg(st) = \deg(s) + \deg(t)$$

while:

$$\deg(at) = \deg(a) + \deg(t) \qquad \text{and} \qquad \deg(bs) = \deg(b) + \deg(s)$$

Since:

$$\deg a - \deg s = \deg b - \deg t$$

it follows that $\deg(at) = \deg(bs)$, hence:

$$\deg \left(\frac{at + bs}{st} \right) = \deg a + \deg t - \deg s - \deg t = \deg a - \deg s = m$$

so $(S^{-1}A)_m$ is closed under addition. Since the degree of an element is well defined, it follows by the construction of $(S^{-1}A)_m$ that for $m \neq n$, we have that $(S^{-1}A)_m \cap (S^{-1}A)_n = \{0\}$. Now finally, let $a/s \in S^{-1}A$, then a can be written as the sum:

$$a = \sum_i a_i$$

where $A_i \in a_i$ for each i. It follows that:

$$\frac{a}{s} = \sum_i \frac{a_i}{s}$$

each element is then homogenous of degree $i - \deg s$. It follows that any element can be written as sum of homogenous elements, hence:

$$S^{-1}A = \bigoplus_{n\in\mathbb{Z}} (S^{-1}A)_n$$

We now need only check that $(S^{-1}A)_n \cdot (S^{-1}A)_m \subset (S^{-1}A)_{m+n}$. Let $a/s \in (S^{-1}A)_n$, and $b/t \in (S^{-1}A)_m$, then we see that:

$$\deg\left(\frac{ab}{ts}\right) = \deg(ab) - \deg(st) = \deg a + \deg b - \deg s - \deg t = \deg(a/s) + \deg(b/t) = m + n$$

implying the claim. □

We say that a $\mathbb{Z}$-graded ring A is $\mathbb{Z}^{\geq 0}$ graded if for all $n < 0$ we have $A_n = \{0\}$. Going forward, we assume that all rings are $\mathbb{Z}^{\geq 0}$ graded.

Definition 2.2.3. We fix a **base ring** B, and say that a graded ring A is **graded over B** if $A_0 = B$. Moreover, the subset:

$$A_+ = \bigoplus_{i>0} A_i$$

is a prime ideal called the **irrelevant ideal**. If the irrelevant ideal is finitely generated, then we say that A is a **finitely graded ring over B**. Finally, if A is generated by A_1 as a B-algebra, we say that A is **generated in degree 1**.

We now begin the Proj construction:

Definition 2.2.4. Let A be a graded ring, then as a set **Proj A** is defined by:

$$\operatorname{Proj} A = \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \text{ is homogenous and } A_+ \not\subset \mathfrak{p}\}$$

i.e. $\operatorname{Proj} A$ are the set of all homogenous prime ideals which do not contain the irrelevant ideal.

If $f \in A$ is homogenous, we denote by A_f the localization of A by the multiplicatively closed subset generated by f, equipped with the natural $\mathbb{Z}$ grading given by Lemma 2.2.4. We define $(A_f)_0$ to be the degree zero elements of A_f.

Proposition 2.2.1. *Let $f \in A_+$ be homogenous, then there is a bijection between the prime ideals of $(A_f)_0$, the homogenous prime ideals of A_f, and the homogenous prime ideals of A which do not contain f.*

Proof. First note that there is a bijection between the prime ideals of A_f, and the prime ideals of A which does not contain f. Clearly, if $\mathfrak{p} \subset A$ is homogenous, then the corresponding ideal $\mathfrak{p}_f$ is homogenous in A_f, when equipped with the natural $\mathbb{Z}$ grading from Lemma 2.2.4. Now note that if $\mathfrak{p} \subset A_f$ is homogenous, then $\mathfrak{p}$ is generated by homogenous elements, and if $\pi : A \to A_f$, then $\pi^{-1}(\mathfrak{p})$ is the prime ideal of A corresponding to $\mathfrak{p}$. Now suppose that $\pi^{-1}(\mathfrak{p})$ is not generated by homogenous elements, then if $\{a_i\}_i$ are the generators of $\pi^{-1}(\mathfrak{p})$, we have that $\{a_i/1\}_i$ are the generators of $\mathfrak{p}$, implying $\mathfrak{p}$ is not generated generated by homogenous elements, so by the contrapositive, we have that $\pi^{-1}(\mathfrak{p})$ is homogeneous. It follows that the bijection between primes not containing f, and primes of A_f preserves homogenous primes, implying the claim.

Now we have a natural inclusion homomorphism of rings $\iota : (A_f)_0 \hookrightarrow A_f$, so any homogenous prime of A_f pulls back to a prime ideal of $(A_f)_0$. Given a prime $\mathfrak{p}_0 \in (A_f)_0$, then we set $\phi(\mathfrak{p}_0) = \sqrt{\mathfrak{p}_0 A_f}$, where $p_0 A_f$ is the the in A_f generated by $\mathfrak{p}_0$ as a subset of A_f. Since $\mathfrak{p}_0 A_f$ is generated by degree zero elements, it is homogenous, so by Lemma 2.2.2 $\sqrt{\mathfrak{p}_0 A_f}$ is homogenous. By Lemma 2.2.2 part *c)*, we need only prove this for homogenous elements of $\sqrt{\mathfrak{p}_0 A_f}$. Let a and b be homogenous elements of degree k and l, such that $a \cdot b \in \sqrt{\mathfrak{p}_0 A_f}$, so there must exist some $r \geq 0$ such that $(a \cdot b)^r \in \mathfrak{p}_0 A_f$ by the definition of the radical. Now, $(a \cdot b)^r$ has degree $(k + l)r$, so let $j = \deg f$, then:

$$\frac{(a \cdot b)^{jr}}{f^{(k+l)r}} \in (\mathfrak{p}_0 A_f)_0 = \mathfrak{p}_0$$

It follows that since $\mathfrak{p}_0$ is prime, either $a^{jr}/f^{kr} \in \mathfrak{p}_0$, or $b^{jr}/f^{kr} \in \mathfrak{p}_0$, hence either $a^{jr} \in \mathfrak{p}_0 A_f$, or $b^{jr} \in \mathfrak{p}_0 A_f$. Again by the definition of the radical we have that either $a \in \sqrt{\mathfrak{p}_0 A_f}$, or $b \in \sqrt{\mathfrak{p}_0 A_f}$, so $\sqrt{\mathfrak{p}_0 A_f}$ is indeed prime.

Now if we have $\mathfrak{p}_0 \in (A_f)_0$, then $\iota^{-1}(\sqrt{\mathfrak{p}_0 A_f}) = (\mathfrak{p}_0 A_f)_0 = \mathfrak{p}_0$, so one direction of the bijection is immediate. Now let $\mathfrak{p} \subset A_f$ be a homogenous prime ideal, we want to show that:

$$\mathfrak{p} = \sqrt{\iota^{-1}(\mathfrak{p}) A_f}$$

Note that $\iota^{-1}(\mathfrak{p}) = (\mathfrak{p})_0$, i.e. the degree zero elements of $\mathfrak{p}$. Since both primes are homogenous, it suffices to check equality on homogenous elements. Let $a \in \mathfrak{p}$ have degree k, then $a^j/f^k \in (\mathfrak{p})_0$, so $a^j \in (\mathfrak{p})_0 A_f$, hence $a \in \sqrt{\iota^{-1}(\mathfrak{p}) A_f}$. Now suppose that $a \in \sqrt{\iota^{-1}(\mathfrak{p}) A_f}$, then there exists some r such that $a^r \in (\mathfrak{p})_0 A_f$, but this implies that $a^r \in \mathfrak{p}$, as $(\mathfrak{p})_0 \mathbb{A}_f \subset \mathfrak{p}$, Since $\mathfrak{p}$ is prime it follows that $a \in \mathfrak{p}$, implying the second direction of the bijection. □

So to sum up the result of the last proposition, which is an analogue of Proposition 1.1.3 minus the topological information, we have that a homogenous prime ideal which does not contain f induces a unique homogenous prime ideal of A_f, which then induces induces a unique prime ideal of the subring $(A_f)_0$. Our next step is to put a topology on $\operatorname{Proj} A$.

Definition 2.2.5. Let T be a subset of homogenous elements then the **projective vanishing set of T**, denoted $\mathbb{V}(T)$ is defined by:

$$\mathbb{V}(T) = \{\mathfrak{p} \in \operatorname{Proj} A : T \subset \mathfrak{p}\}$$

Similarly, if f is a homogenous element of positive degree, and $I \subset A$ is a homogenous ideal, we set:

$$\mathbb{V}(f) := \mathbb{V}(\langle f \rangle) = \{\mathfrak{p} \in \operatorname{Proj} A : f \in \mathfrak{p}\} \qquad \text{and} \qquad \mathbb{V}(I) = \{\mathfrak{p} \in \operatorname{Proj} A : I \subset \mathfrak{p}\}$$

This leads us to our next lemma, which follows a very similar argument to Proposition 1.1.1:

Lemma 2.2.5. *Let A be a graded ring, then defining the closed sets of* $\operatorname{Proj} A$ *to be* $\mathbb{V}(I)$ *for all homogenous ideals defines a topology on* $\operatorname{Proj} A$

Proof. We first see that zero element is contained in $(A)_d$ for every d, so 0 has any degree we wish. It follows that since $0 \subset \mathfrak{p}$ for all homogenous primes, that:

$$\mathbb{V}(0) = \operatorname{Proj} A$$

so $\operatorname{Proj} A$ is closed. We also have that that:

$$\mathbb{V}(A_+) = \emptyset$$

as no $\mathfrak{p} \in \operatorname{Proj} A$ contains A_+, so the empty set is closed.

Now let I and J be homogenous prime ideals, then we want to show that:

$$\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cap J)$$

Let $\mathfrak{p} \in \mathbb{V}(I) \cup \mathbb{V}(J)$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$, if $I \subset \mathfrak{p}$, then $I \cap J \subset I \subset \mathfrak{p}$, and similarly for J, hence $\mathfrak{p} \in \mathbb{V}(I \cap J)$. If $\mathfrak{p} \in \mathbb{V}(I \cap J)$, then $I \cap J \subset \mathfrak{p}$. Let $r \in I \cdot J$, then $r = i \cdot j$ for some $i \in I$ and some $j \in J$. It follows that $r \in I \cap J$, so $I \cdot J \subset I \cap J$. , hence $I \cdot J \subset \mathfrak{p}$. Now suppose that $I \not\subset \mathfrak{p}$, then there exists an $i \in I$ such that $i \notin \mathfrak{p}$, however since $I \cdot J \subset \mathfrak{p}$, we have that for all $j \in J$, $i \cdot j \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime it follows that $J \subset \mathfrak{p}$, and if $J \not\subset \mathfrak{p}$, the same argument demonstrates $I \subset \mathfrak{p}$. Note that if neither $I \subset \mathfrak{p}$, nor $J \subset \mathfrak{p}$, then $\mathfrak{p}$ can't be prime, as there exists $i \in I$ and $j \in J$ such that $i, j \notin \mathfrak{p}$, but $i \cdot j \in \mathfrak{p}$. It follows that $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$, hence $\mathfrak{p} \in \mathbb{V}(I) \cup \mathbb{V}(J)$, implying the second direction.

Now let $\{I_\alpha\}$ be an arbitrary family of homogenous ideals. We claim that:

$$\bigcap_\alpha \mathbb{V}(I_\alpha) = \mathbb{V}\left(\sum_\alpha I_\alpha\right)$$

where $\sum_\alpha I_\alpha$ is the smallest ideal containing all I_α. Suppose that $\mathfrak{p} \in \bigcap_\alpha \mathbb{V}(I_\alpha)$, then we have that $I_\alpha \subset \mathfrak{p}$ for all α. Now since any $i \in \sum_\alpha I_\alpha$ can be written as the finite sum $\sum_{j=1}^n r_j$ where each $r_j \in I_\alpha \subset \mathfrak{p}$, we have that $i \in \mathfrak{p}$, hence $\sum_\alpha I_\alpha \subset \mathfrak{p}$, so $\mathfrak{p} \in \mathbb{V}(\sum_\alpha I_\alpha)$. Now suppose that $\mathfrak{p} \in \mathbb{V}(\sum_\alpha I_\alpha)$, then since $I_\alpha \subset \sum_\alpha I_\alpha$, we have that $I_\alpha \subset \mathfrak{p}$ for all α, hence $\mathfrak{p} \in \bigcap_\alpha \mathbb{V}(I_\alpha)$. □

As before call this topology on $\operatorname{Proj} A$ the Zariski topology. Note that Lemma 1.1.1 holds in the sense that for any homogenous ideals I and J, the following hold:

a) $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$

b) $J \subset I \Longrightarrow \mathbb{V}(J) \supset \mathbb{V}(I)$

c) $\mathbb{V}(I) \subset \mathbb{V}(J) \Longleftrightarrow \sqrt{I} \supset \sqrt{J}$

We define a basis of open sets similarly, though impose more restrictions on what our basic opens can be:

Definition 2.2.6. Let A be a graded ring, and f a homogenous element of positive degree, then we define the **(projective) distinguished open** to be:

$$U_f = \mathbb{V}(f)^c$$

Lemma 2.2.6. *The set of (projective) distinguished opens form a basis for the Zariski topology on* $\operatorname{Proj} A$*.*

Proof. Let $U \subset \operatorname{Proj} A$ be open, then we have that for some homogenous ideal $I \subset A$:

$$U = \mathbb{V}(I)^c$$

Note that:

$$I = \sum_{i \in I} \langle i \rangle$$

hence:

$$\begin{aligned} U =& \mathbb{V}\left(\sum_{i \in I} \langle i \rangle\right)^c \\ =& \left(\bigcap_{i \in I} \mathbb{V}(i)\right)^c \\ =& \bigcup_{i \in I} \mathbb{V}(i)^c \end{aligned}$$

Now we can split this into the following union:

$$U = \bigcup_{i \in I_+} U_i \cup \bigcup_{j \in I_0} \mathbb{V}(j)^c$$

where I_+ denotes the elements of I with positive degree, and I_0 are the degree zero elements of positive degree. Let $\{f_k\}$ be the generators of the irrelevant ideal A_+, then:

$$\emptyset = \mathbb{V}(A_+) = \bigcap_k \mathbb{V}(f_k) \Rightarrow \operatorname{Proj} A = \bigcup_k U_{f_k}$$

We claim that if $j \in I_0$, then:

$$U_j = \bigcup_k U_{jf_k}$$

Let $\mathfrak{p} \in U_j$, then $j \notin \mathfrak{p}$; since $A_+ \not\subset \mathfrak{p}$, we must have that there exists some k such that $f_k \notin \mathfrak{p}$. It follows that $jf_k \notin \mathfrak{p}$, hence $\mathfrak{p} \in \bigcup_k U_{jf_k}$. Now suppose that $\mathfrak{p} \in \bigcup U_{jf_k}$, then for some k we have that $jf_k \notin \mathfrak{p}$, hence $j \notin \mathfrak{p}$, and $f_k \notin \mathfrak{p}$, so $\mathfrak{p} \in U_j$. It follows that:

$$U = \bigcup_{i \in I_+} U_i \cup \bigcup_{j \in I_0} \bigcup_k U_{jf_k}$$

so the distinguished opens generate the Zariski topology on $\operatorname{Proj} A$. □

It should be no surprise that we are about to prove a similar result to Proposition 1.1.3, and from there we will use the projective distinguished opens to put the structure of a scheme on $\operatorname{Proj} A$ for any graded ring A.

Proposition 2.2.2. *Let A be a graded ring, and f a homogenous element of positive degree. Then $U_f \subset \operatorname{Proj} A$ is homeomorphic to* $\operatorname{Spec}(A_f)_0$.

Proof. Recall that $U_f \subset \operatorname{Proj} A$ is defined by $\mathbb{V}(f)^c$, hence:

$$U_f = \{\mathfrak{p} \in \operatorname{Proj} A : f \notin \mathfrak{p}\}$$

From Proposition 2.2.1 we have a bijection $U_f \leftrightarrow$ homogenous primes of $A_f \leftrightarrow \operatorname{Spec}(A_f)_0$, given by $F : \mathfrak{p} \mapsto \mathfrak{p}_f \mapsto \iota^{-1}(\mathfrak{p}_f)$, where $\iota : (A_f)_0 \to A_f$ is the inclusion map, and $\mathfrak{p}_f$ is the prime ideal generated by the image of $\mathfrak{p}$ under the localization map. In other words, $\mathfrak{p}_f = \eta(\mathfrak{p})$, where η is as defined in Let $\mathbb{V}(I) \subset \operatorname{Spec}(A_f)_0$ be a closed subset, for some radical ideal $I \subset (A_f)_0$, then we have that:

$$\begin{aligned} F^{-1}(\mathbb{V}(I)) =& \{\mathfrak{p} \in U_f : \iota^{-1}(\mathfrak{p}_f) \in \mathbb{V}(I)\} \\ =& \{\mathfrak{p} \in U_f : I \subset \iota^{-1}(\mathfrak{p}_f)\} \end{aligned}$$

We first claim that $I \subset \iota^{-1}(\mathfrak{p}_f)$ if and only if $\iota(I) \subset \mathfrak{p}_f$. Suppose that $I \subset \iota^{-1}(\mathfrak{p}_f)$, then $i \in I$ implies that $i \in \iota^{-1}(\mathfrak{p}_f)$. By definition, it follows that $\iota(i) \in \mathfrak{p}_f$, hence $\iota(I) \subset \mathfrak{p}_f$. Now suppose that $\iota(I) \subset \mathfrak{p}_f$, since ι is injective, we thus have that $\iota^{-1}(I) = I$, hence $I \subset \iota^{-1}(\mathfrak{p}_f)$. It follows that:

$$F^{-1}(\mathbb{V}(I)) = \{\mathfrak{p} \in U_f : \iota(I) \subset \mathfrak{p}_f\}$$

Note that since $I \subset (A_f)_0$, we have that $\iota(I)$ consists of degree zero elements of A_f, and is thus homogenous. We see that if $\pi : A \to A_f$ is the localization map, then $\pi^{-1}(\mathfrak{p}_f) = \mathfrak{p}$, hence:

$$F^{-1}(\mathbb{V}(I)) = \{\mathfrak{p} \in U_f : \pi^{-1}(\iota(I)) \subset \mathfrak{p}\} = U_f \cap \mathbb{V}(\pi^{-1}(\iota(I)))$$

which is a closed in the subspace topology on U_f hence F is continuous. We note that $f \notin \pi^{-1}(\iota(I))$, as this would imply that $f/1 \in I$ which can't be true as $I \subset (A_f)_0$. It follows that:

$$F^{-1}(\mathbb{V}(I)) = \mathbb{V}(\pi^{-1}(\iota(I))) \subset U_f$$

Now take $V \subset U_f$ be a closed subset. We must have that $V = \mathbb{V}(I) \cap U_f$ for some homogenous ideal I. Moreover, if $f \in I$, then $\mathbb{V}(I) \cap U_f = \emptyset$, hence we actually have that $V = \mathbb{V}(I) \subset U_f$, and $f \notin I$. Now note that by Proposition 2.2.1:

$$\begin{aligned} F(\mathbb{V}(I)) =& \{\mathfrak{q} \in \operatorname{Spec}(A_f)_0 : I \subset \pi_f^{-1}(\sqrt{\mathfrak{q}A_f})\} \\ =& \{\mathfrak{q} \in \operatorname{Spec}(A_f)_0 : \pi_f(I) \subset \sqrt{\mathfrak{q}A_f}\} \\ =& \{\mathfrak{q} \in \operatorname{Spec}(A_f)_0 : \iota^{-1}(\pi_f(I)) \subset \mathfrak{q}\} \\ =& \mathbb{V}(\iota^{-1}(\pi_f(I))) \subset \operatorname{Spec}(A_f)_0 \end{aligned}$$

which is closed. It follows that F is a continuous closed bijection, and hence a homeomorphism as desired. $\square$

Our goal is to now equip $\operatorname{Proj} A$ with the structure of a scheme via Proposition 1.2.11. Note that we could also glue the affine schemes $\operatorname{Spec}(A_f)_0$ together via Theorem 2.1.1 and get the same result, but this would 'overkill', given that we have already in a sense glued the topological spaces $\operatorname{Spec}(A_f)_0$ together by our construction of the topology on $\operatorname{Proj} A$. We could also define the structure sheaf to be the sheaf on a base given by $U_f \mapsto \mathscr{O}_{\operatorname{Spec}(A_f)_0}$, and show that this truly defines a sheaf on the base of distinguished opens, but as we are about to see this equivalent description would be much more involved. We need the following lemma:

Lemma 2.2.7. *Let A be a graded ring, and $f, g \in A$ homogenous elements of positive degree. Then:*

$$(A_{fg})_0 \cong ((A_f)_0)_h$$

where $h = g^{\deg f}/f^{\deg g}$. In particular, with $h^{-1} = f^{\deg g}/g^{\deg f}$ we have that:

$$((A_f)_0)_h \cong ((A_g)_0)_{h^{-1}}$$

Proof. We first examine the map:

$$\begin{aligned}\psi : A_f &\longrightarrow A_{fg}\\ \frac{a}{f^k} &\longmapsto \frac{a\cdot g^k}{(fg)^k}\end{aligned}$$

and note that if $a/f^k \in (A_f)_0$, then clearly $\psi(a/f^k) \in (A_{fg})_0$, so this descends to a morphism $\psi_0 : (A_f)_0 \to (A_{fg})_0$. We now see that

$$\psi_0 : h \longmapsto \frac{g^{\deg f}\cdot g^{\deg g}}{(fg)^{\deg g}}$$

which has an inverse in $(A_{fg})_0$ given by:

$$\frac{f^{\deg g+\deg f}}{(fg)^{\deg f}}$$

so there exists a unique map:

$$\begin{aligned}\theta_0 : ((A_f)_0)_h &\longrightarrow (A_{fg})_0\\ \frac{a}{f^l}\cdot h^{-k} &\longmapsto \frac{a\cdot g^l}{(fg)^l}\cdot\left(\frac{f^{\deg g+\deg f}}{(fg)^{\deg f}}\right)^k\end{aligned}$$

We first claim this map injective. Suppose that $(a/f^l)\cdot h^{-k}\mapsto 0$, then we have that:

$$\frac{a\cdot g^l\cdot f^{k\deg g+k\deg f}}{(fg)^{l+k\deg f}}=0$$

implying there exists a K such that:

$$(fg)^K(a\cdot g^l\cdot f^{k\deg g+k\deg f})=0$$

We want to then show that there exists an L such that:

$$\frac{a\cdot g^{L\deg f}}{f^{l\cdot L\deg f}}=0$$

meaning that we really want to show there exists an L' such that:

$$f^{L'}\cdot(ag^{L\deg f})=0$$

Well set $L=K+l$, and $L'=K+k\deg g+k\deg f$, then:

$$f^{K+k\deg g+k\deg f}(a\cdot g^{K+l})=(fg)^K(a\cdot g^l\cdot f^{k\deg g+k\deg f})=0$$

so $a/f^l\cdot h^{-k}=0$ as desired, and the map is injective. Now let $a/(fg)^k\in(A_{fg})_0$, then we see that:

$$\begin{aligned}\theta_0\left(\frac{g^{k\deg f-k}a}{f^{k\deg g+k}}\cdot h^{-k}\right)&=\frac{a\cdot g^{k\deg f-k+k+k\deg g}}{(fg)^{k\deg g+k}}\cdot\frac{f^{k\deg f+k\deg g}}{(fg)^{k\deg f}}\\&=\frac{ag^{k\deg f+k\deg g}\cdot f^{k\deg f+k\deg g}}{(fg)^{k\deg g+k\deg f+k}}\\&=\frac{a}{(fg)^k}\end{aligned}$$

so the map is also surjective, and thus an isomorphism. Clearly the second claim follows from the first. □

Theorem 2.2.1. *There exists a unique (up to unique isomorphism) sheaf of rings* $\mathscr{O}_{\operatorname{Proj} A}$ *on* $\operatorname{Proj} A$ *which makes* $\operatorname{Proj} A$ *into a scheme, such that* $(U_f,\mathscr{O}_{\operatorname{Proj} A}|_{U_f})\cong(\operatorname{Spec} A_f,\mathscr{O}_{\operatorname{Spec}(A_f)_0})$ *for any homogenous element* f *of positive degree.*

Proof. Let A_+^{hom} be the set of homogenous elements of A of positive degree, and consider the cover of $\operatorname{Proj} A$ by $\{U_f\}_{f\in A_+^{\text{hom}}}$. For each U_f, let $\psi_f : \operatorname{Spec}(A_f)_0 \to U_f$ be the aforementioned homeomorphism, and set $\mathscr{F}_f := \psi_{f*}\mathscr{O}_{\operatorname{Spec}(A_f)_0}$. We need to define sheaf isomorphisms $\phi_{fg} : \mathscr{F}_f|_{U_f\cap U_g} \to \mathscr{F}_g|_{U_f\cap U_g}$ which satisfy $\phi_{fg} = \phi_{lg}\circ\phi_{fl}$ on triple overlaps $U_f \cap U_l \cap U_g$. First note that that:

$$U_f \cap U_g = \{\mathfrak{p} \in \operatorname{Proj} A : f, g \notin \mathfrak{p}\}$$

Since $\mathfrak{p}$ is prime, we have that $f, g \notin \mathfrak{p} \Leftrightarrow f \cdot g \notin \mathfrak{p}$, hence:

$$U_f \cap U_g = U_{fg} \cong \operatorname{Spec}(A_{fg})_0$$

Now by Lemma 2.2.7 and Corollary 1.4.3, we have that as affine schemes:

$$U_h = \operatorname{Spec}((A_f)_0)_h \cong \operatorname{Spec}(A_{fg})_0 \cong \operatorname{Spec}((A_g)_0)_{h^{-1}} = U_{h^{-1}}$$

where $h = g^{\deg f}/f^{\deg g}$, $h^{-1} = f^{\deg g}/g^{\deg f}$, and $U_h \subset \operatorname{Spec}(A_f)_0$, $U_{h^{-1}} \subset \operatorname{Spec}(A_g)_0$ are the distinguished open sets.

Moreover, we have $U_{fg} \subset U_f$, so we can examine the open set $\psi_f^{-1}(U_{fg}) \subset \operatorname{Spec}(A_f)_0$. We claim that this is equal to $U_h \subset \operatorname{Spec}(A_f)_0$; indeed, we have that if $\mathfrak{q} \in \psi_f^{-1}(U_{fg})$, then $\mathfrak{q} = \iota^{-1}(\mathfrak{p}_f)$ for some $\mathfrak{p} \in U_{fg}$. Since $f \cdot g \notin \mathfrak{p}$, we have that $g \notin \mathfrak{p}_f$, hence $h \notin \mathfrak{p}_f$, but $h \in (A_f)_0$, hence $h \notin \iota^{-1}(\mathfrak{p}_f)$ so $\mathfrak{q} \in U_h$. Now suppose that $\mathfrak{q} \in U_h$, we want to show that $\psi_f(\mathfrak{q}) \in U_{fg}$; well clearly $\psi_f(\mathfrak{q}) \in U_f$, and $g^{\deg f}/1 \notin \sqrt{\mathfrak{q}_0 A_f}$, hence $g^{\deg f} \notin \pi^{-1}(\sqrt{(}\mathfrak{q}_0)A_f) = \psi_f(\mathfrak{q})$, so $g \notin \psi_f(\mathfrak{q})$, implying that $\psi_f(\mathfrak{q}) \in U_{fg}$. Respectively, we have that $\psi_g^{-1}(U_{fg}) = U_{h^{-1}} \subset \operatorname{Spec}(A_g)_0$.

Now note since the isomorphism

$$(U_h, \mathscr{O}_{\operatorname{Spec}(A_f)_0}|_{U_h}) \cong (U_{h^{-1}}, \mathscr{O}_{\operatorname{Spec}(A_g)_0}|_{U_{h^{-1}}})$$

is induced by the by the unique ring isomorphisms from Lemma 2.2.7, that the homeomorphism $U_h \to U_{h^{-1}}$ must be given by the restriction $\psi_g^{-1}\circ\psi_f|_{U_h}$. In particular, we have the following sheaf isomorphism:

$$\mathscr{O}_{\operatorname{Spec}(A_g)_0}|_{U_{h^{-1}}} \longrightarrow (\psi_g^{-1}\circ\psi_f|_{U_h})_*\mathscr{O}_{\operatorname{Spec}(A_f)_0}|_{U_h}$$

Since ψ_g is a homeomorphism, we thus obtain the following isomorphism of sheaves:

$$(\psi_g|_{U_{fg}})_*(\mathscr{O}_{\operatorname{Spec}(A_g)_0}|_{U_{h^{-1}}}) \longrightarrow (\psi_f|_{U_{fg}})_*(\mathscr{O}_{\operatorname{Spec}(A_f)_0}|_{U_h})$$

By noting that

$$(\psi_g|_{U_{fg}})_*(\mathscr{O}_{\operatorname{Spec}(A_g)_0}|_{U_{h^{-1}}}) = (\psi_{g*}\mathscr{O}_{\operatorname{Spec}(A_g)_0})|_{U_{fg}} = \mathscr{F}_g|_{U_{fg}}$$

and similarly for the ψ_f, we have the desired isomorphisms ϕ_{gf}.

Now let $f, g, l \in A_+^{\text{hom}}$, then we have the following unique ring isomorphisms:

$$(A_{fgl})_0 \cong ((A_f)_0)_{(gl)^{\deg f}/f^{\deg gl}} \cong ((A_l)_0)_{(fg)^{\deg l}/l^{\deg fg}} \cong ((A_g)_0)_{(fl)^{\deg g}/f^{\deg fl}}$$

If we denote the ring isomorphisms by β_{fl}, and β_{lg}, then by uniqueness we have that $\beta_{lg}\circ\beta_{fl} = \beta_{fg}$. Since these ring isomorphisms are what induce the sheaf isomorphisms ϕ_{lg}, ϕ_{fg}, ϕ_{fl}, it is then clear that on $U_g \cap U_l \cap U_f$ we have that $\phi_{fg} = \phi_{lg}\circ\phi_{fl}$, this gluing defines a unique (up to unique isomorphism) sheaf of rings on $\operatorname{Proj} A$.

All that remains to show is that $\operatorname{Proj} A$ is a scheme however this is now clear, as for any homogenous element of postive degree f, we have a homeomorphism $\psi_f : \operatorname{Spec}(A_f)_0 \to U_f$, and a sheaf morphism:

$$\mathscr{O}_{\operatorname{Proj} A}|_{U_f} = \psi_{f*}\mathscr{O}_{\operatorname{Spec}(A_f)_0} \longrightarrow \psi_{f^*}\mathscr{O}_{\operatorname{Spec}(A_f)_0}$$

given by the identity map, so every point in x has an open neighborhood isomorphic to an affine scheme. □

We now recall that the construction in Example 2.2.1 is valid for any commutative ring A, hence we have the following proposition:

Proposition 2.2.3. *Let A be a commutative ring, and consider the polynomial ring $A[x_0, \dots, x_n]$ with the standard grading induced by $\deg x_i = 1$, then:*

$$\mathbb{P}^n_A \cong \operatorname{Proj} A[x_0, \dots, x_n]$$

where $\mathbb{P}^n_A$ is the scheme constructed as in Example 2.2.1.

Proof. We first claim the distinguished opens $U_{x_i} \subset \operatorname{Proj} A[x_0, \dots, x_n]$ cover $\operatorname{Proj} A[x_0, \dots, x_n]$. Let $\mathfrak{p} \in \operatorname{Proj} A$, then we have that $\mathfrak{p}$ is a homogenous prime ideal which does not contain the trivial idea, then $\mathfrak{p}$ can not be of the form:

$$\mathfrak{p} = \langle x_0, \dots, x_n \rangle$$

or contain such an ideal. It follows that at least one $x_i \in A[x_0, \dots, x_n]$ does not lie in $\mathfrak{p}$, hence we have that:

$$\operatorname{Proj} A[x_0, \dots, x_n] = \bigcup_{i=0}^{n} U_{x_i}$$

We now note that for each i we have that as schemes:

$$U_{x_i} \cong \operatorname{Spec}(A[x_0, \dots, x_n]_{x_i})_0$$

and that the ring homomorphism:

$$\begin{gathered} \phi_i : (A[x_0, \cdots, x_n]_{x_i})_0 \longrightarrow A[\{x_k/x_i\}_{k \neq i}] \\ x_m/x_i \mapsto x_m/x_i \end{gathered}$$

is an isomorphism. We thus have scheme isomorphisms:

$$\operatorname{Spec} A[\{x_k/x_i\}_{k\neq i}] \longrightarrow U_{x_i} \subset \operatorname{Proj} A[x_0, \dots, x_n]$$

Now by noting we have that $\mathbb{P}^n_A$ is given by gluing the schemes $X_i = \operatorname{Spec} A[\{x_k/x_i\}_{k\neq i}]$ together as in Example 2.2.1, and via the open embeddings $\psi_i : X_i \to \mathbb{P}^n_A$, we have scheme isomorphisms:

$$f_i : \psi(X)_i \subset \mathbb{P}^n_A \longrightarrow U_{x_i} \subset \operatorname{Proj} A[x_0, \dots, x_n]$$

which trivially agree on overlaps, so we have a scheme isomorphism:

$$f : \mathbb{P}^n_A \longrightarrow \operatorname{Proj} A[x_0, \dots, x_n]$$

as desired. ☐

With the above proposition, we now define projective schemes precisely:

Definition 2.2.7. A scheme X is a **projective scheme over B** if it is of the form $\operatorname{Proj} A$ for some graded ring A with $A_0 = B$, and A finitely generated as a B-algebra. In particular, if A is any commutative ring, then the projective scheme $\mathbb{P}^n_A$ is defined by:

$$\mathbb{P}^n_A = \operatorname{Proj}(A[x_0, \dots, x_n])$$

Now let k be any algebraically closed field, and recall the argument that the closed points of $\mathbb{P}^n_k$ are in bijection with standard projective space over k. We wish to identify the closed points $[z_0, \dots, z_n]$ with homogenous prime ideals of $k[x_0, \dots, x_n]$. Note that at least one of these z_i must not be zero, so we can rewrite this point as:

$$[z_0/z_i, \dots, 1, \dots, z_n/z_i] \in \psi(X_i)$$

which then corresponds to the maximal ideal:

$$\mathfrak{p}_0 = \left\langle \frac{x_0}{x_i} - \frac{z_0}{z_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i} - \frac{z_n}{z_i} \right\rangle \in \operatorname{Spec} k[\{x_l/x_i\}_{l\neq i}]$$

which is the the same ideal in $\mathrm{Spec}(k[x_0, \ldots, x_n]_{x_i})_0$. Under the bijection between prime ideals of $(k[x_0, \ldots, x_n]_{x_i})_0$ and homogenous prime ideals missing the trivial ideal of $k[x_0, \ldots, x_n]$, we then have that this corresponds to $\pi^{-1}(\sqrt{\mathfrak{p}_0 A_{x_i}})$ where $A = k[x_0, \ldots, x_n]$. Since x_i is invertible in A_{x_i}, we see that:

$$\mathfrak{p}_0 A_{x_i} = \langle x_0 z_i - x_i z_0, \ldots, \hat{x}_i, \ldots, x_n z_i - x_i z_n \rangle \subset A_{x_i}$$

which is prime and thus radical. It follows that:

$$\pi^{-1}(\sqrt{\mathfrak{p}_0 A_{x_i}}) = \langle x_0 z_i - x_i z_0, \ldots, \hat{x}_i, \ldots, x_n z_i - x_i z_n \rangle \subset k[x_0, \ldots, x_n]$$

Now note that for any k and l we have that:

$$x_k z_l - x_l z_k = (x_k z_i - x_i z_k) \cdot (z_l / z_i) - (x_l z_i - x_i z_l) \cdot (z_k / z_i)$$

so we have that:

$$\pi^{-1}(\sqrt{\mathfrak{p}_0 A_{x_i}}) = \langle x_i z_j - x_j z_i | 0 \leq i, j \leq n \rangle$$

Therefore the correspondence between closed points and homogenous prime ideals of $k[x_0, \ldots, x_n]$ is given by:

$$[z_0, \ldots, z_n] \longleftrightarrow \langle x_i z_j - x_j z_i | 0 \leq i, j \leq n \rangle$$

Example 2.2.2. Let A be any commutative ring, we will examine $\mathrm{Proj}\, A[x]$ with two different gradings on $A[x]$. First, let $A[x]$ have the standard grading, and then note that $U_x = \mathrm{Proj}\, A[x]$. However, $U_x \cong \mathrm{Spec}(A[x]_x)_0$, and we see that:

$$(A[x]_x)_0 \cong (A[x, x^{-1}])_0 \cong A$$

so $\mathrm{Proj}\, A[x] \cong \mathrm{Spec}\, A$. Note that when $A = \mathbb{C}$, then we have that this implies that $\mathbb{P}^0_{\mathbb{C}} \cong \{\langle 0 \rangle\}$, i.e. the singleton set. This matches with the fact that $\mathbb{C} \setminus \{0\}/\mathbb{C}^*$ is just a point.

Now let $A[x]$ have the trivial grading so that every element is homogenous and of degree 0, we wish to describe $\mathrm{Proj}\, A[x]$, however this is easy. We see that

$$\mathrm{Proj}\, A[x] = \{\mathfrak{p} \in \mathrm{Spec}\, A[x] : \mathfrak{p} \text{ is homogenous and } (A[x])_+ \notin \mathfrak{p}\}$$

is empty as $(A[x])_+ = \langle 0 \rangle$ and every ideal contains 0. It follows that $\mathrm{Proj}\, A[x]$ is the empty scheme.

Example 2.2.3. Let $X = \mathrm{Proj}\, \mathbb{C}[x, y, z]$, where $\mathbb{C}[x, y, z]$ is equipped with the grading $\deg x = 0$, $\deg y = \deg z = 1$, and all elements of $\mathbb{C}$ are degree zero. We know that in the standard grading case the closed points $\mathbb{P}^2_{\mathbb{C}}$ are precisely the points of $\mathbb{CP}^2$, we now wish to see how this changes with this new grading. We claim that $X = U_y \cup U_z$, where U_y and U_z are the projective distinguished open sets. Let $\mathfrak{p} \in X$, then $\mathfrak{p}$ is a homogenous prime ideal which does not contain the trivial ideal. In particular, either y or x can't lie in $\mathfrak{p}$, so $\mathfrak{p} \in U_x \cup U_z$. We have that:

$$U_y \cong \mathrm{Spec}(\mathbb{C}[x, y, z]_y)_0 \cong \mathrm{Spec}\, \mathbb{C}[x, z/y] \qquad \text{and} \qquad U_z = \mathrm{Spec}(\mathbb{C}[x, y, z]_z)_0 \cong \mathrm{Spec}\, \mathbb{C}[x, y/z]$$

The closed points of each are of the form $\langle x - w_1, z/y - w_2 \rangle$ and $\langle x - w_1, y/z - w_2 \rangle$, and for a prime $\mathfrak{p}$ to be closed we necessitate that $\iota^{-1}(\mathfrak{p})$ in both U_x and U_y. We have that the gluing isomorphism along $U_z \cap U_y = U_{xy} \cong \mathrm{Spec}[x, y/z/, z/y]$ takes the closed point $\langle x - w_1, z/y - w_2 \rangle$ to $\langle x - w_1, y/z - 1/w_2 \rangle$. We define a set map

$$F : |U_x| \coprod |U_y| \longrightarrow \mathbb{C} \times \mathbb{P}^1$$

via the disjoint union set map induced by the maps:

$$\langle x - w_1, z/y - w_2 \rangle \longmapsto (w_1, [w_2, 1]) \qquad \text{and} \qquad \langle x - w_1, y/z - w_1 \rangle \longmapsto (w_1, [1, w_2])$$

Now this map is clearly surjective, and factors through the quotient condition, as if $\langle x - w_1, z/y - w_2 \rangle \sim \langle x - w_1, y/z - v \rangle$, then we have that $v = 1/w_2$, so $\langle x - w - 1, y/z - v \rangle$ maps to $(w_1, [1, 1/w_2]) = (w_1, [w_2, 1])$. It follows there is induced map $\tilde{F} : |X| \to \mathbb{C} \times \mathbb{P}^1$, which is then also clearly injective. Via the identification of $\mathbb{P}^1$ with $\mathbb{C} \cup \{\infty\}$ we see that the closed points of X are in bijection with $\mathbb{C} \times (\mathbb{C} \cup \{\infty\})$

Example 2.2.4. Let V be a vector space over a field k[26], then we define:

$$\mathbb{P}(V) := \mathrm{Proj}(\mathrm{Sym}\, V^*)$$

where $\mathrm{Sym}\, V^*$ is the symmetric algebra of the dual space to V. In particular:

$$\mathrm{Sym}\, V^* = T(V^*)/I = (k \oplus V^* \oplus V^* \otimes_k V^* \oplus \cdots)/I$$

where I is the homogenous ideal:

$$I = \langle \omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1 : \omega_i \in V^* \rangle$$

Note that with V finite dimensional, after fixing a basis $\{e_i\}_{i=1}^n$, and a corresponding dual basis $\{e^i\}_{i=1}^n$, we obtain an isomorphism

$$\mathrm{Sym}\, V^* \cong k[e^1, \ldots, e^n] \cong k[x_1, \ldots, x_n]$$

hence:

$$\mathbb{P}(V) \cong \mathbb{P}_k^{n-1}$$

We now suppose $k = \bar{k}$, and claim that any closed point of $\mathbb{P}(V)$ corresponds to a one dimensional linear subspace of $l \subset V$. Indeed, let $l \subset V$ be a one dimensional linear subspace, and define:

$$\mathfrak{p}_l = \langle \omega \in V^* : \omega(l) = \{0\} \rangle$$

i.e. we take the homogenous ideal generated by degree 1 elements which vanish on all of l. We claim that $\mathfrak{p}_l$ is prime; Fix a $v \in l$, and then let $u_1, \ldots, u_{n-1}$ be a set of vectors such that $\{u_1, \ldots, u_{n-1}, v\}$ is a basis. If we let $\{\mu_1, \ldots, \mu_{n-1}, \nu\}$ be a dual basis basis such that $\mu_i(u_j) = \delta_{ij}$, $\mu_i(v) = 0$, then $\mathfrak{p}_l = \langle \mu_1, \ldots, \mu_{n-1} \rangle$ which is manifestly prime.

We show that $\mathbb{V}(\mathfrak{p}_l) = \{\mathfrak{p}_l\}$. Indeed, suppose that there was a $\mathfrak{q} \in \mathrm{Proj}(\mathrm{Sym}\, V^*)$ such that $\mathfrak{p}_l \subset \mathfrak{q}$. In particular, we have that $V^* \cap \mathfrak{p}_l \subset \mathfrak{q} \cap V^*$, but in the process of showing that $\mathfrak{p}_l$ was prime, we showed that $\mathfrak{p}_l \cap V^*$ is an $n-1$ dimensional vector space, hence $\mathfrak{q} \cap V^* = \mathfrak{p}_l \cap V^*$ or V^*. In the latter case $\mathfrak{q} \notin \mathrm{Proj}(\mathrm{Sym}\, V^*)$, and in the former, we claim this implies that $\mathfrak{q} = \mathfrak{p}_l$. Suppose there was some homogenous $\omega \in \mathfrak{q}$ of degree n, that was not in $\mathfrak{p}_l$. Then, $\langle \mu_1, \ldots, \mu_{n-1}, \omega \rangle \subset \mathfrak{q}$; since $\omega \notin \mathfrak{p}_l$, we can write:

$$\omega = \mu + \alpha \cdot \nu$$

where $\mu \in \mathfrak{p}_l$. It follows that $\alpha \cdot \nu \in \mathfrak{q}$, so $\mathfrak{q}$ is not prime, hence we must have $\mathfrak{q} = \mathfrak{p}_l$. It follows that $\mathfrak{p}_l$ is maximal amongst prime ideals in $\mathrm{Proj}(\mathrm{Sym}\, V^*)$, and is thus a closed point.

Now suppose that $\mathfrak{p}$ is a closed point of $\mathbb{P}(V)$. We first note that since k is algebraically closed, $\mathfrak{p} \cap V^* \neq \{0\}$. Indeed, after choosing a basis, we can identify $\mathrm{Sym}\, V^*$ with $k[x_1, \ldots, x_n]$, and so $\mathfrak{p} \cap V^* = \{0\}$ implies that $\mathfrak{p} = \langle f_1, \ldots, f_m \rangle$ where each f_i is homogenous of degree greater than 1. Since $\mathfrak{p}$ is closed, and thus maximal amongst homogenous prime ideals not containing the irrelevant ideal, we have that by Proposition 2.2.1, the corresponding ideal in $k[x_1/x_i, \ldots, x_n/x_i]$ for some i is maximal.[27] The generating set of this ideal must contain a polynomial with leading term of degree greater than 1 as other wise x_i divides each f_i. However, this then implies the existence of a maximal ideal of $k[x_1/x_i, \ldots, x_n/x_i]$ which is not generated by linear factors, which contradicts Hilbert's Nullstellensatz. It follows that $\mathfrak{p} \cap V^* \neq \{0\}$, and thus must have dimension $n-1$ as otherwise it is not maximal. We send $\mathfrak{p} \cap V^*$ to the linear subspace:

$$l_{\mathfrak{p}} = \{v \in V : \omega(v) = 0, \forall \omega \in \mathfrak{p} \cap V^*\}$$

This is clearly one dimensional, and in particular the maps $l \mapsto \mathfrak{p}_l$, and $\mathfrak{p} \mapsto l_{\mathfrak{p}}$ are clear inverse of each other. We thus have the following obvious bijections:

$$|\mathbb{P}(V)| \longleftrightarrow \{\text{one dimensional linear subspaces of } V\}$$

We at times denote one dimensional linear subspaces by equivalence classes $[v]$, such that $[v] = [w]$ if and only if there exists a scalar $\lambda \in k^\times$ satisfying $v = \lambda w$. Note that if $k \neq \bar{k}$, then not every maximal homogenous prime ideal corresponds to a linear subspace; in particular, we have that if $V = \mathbb{R}^2$, then $\left\langle x^2 + y^2 \right\rangle$ is such an ideal.

[26] Or more generally a free module over a ring A.

[27] The value i is clearly dependent on which open set U_{x_i} $\mathfrak{p}$ lives in.

Example 2.2.5. Fix $k = \bar{k}$; we wish to construct a closed subscheme of $G_k(d,n) \subset \mathbb{P}(W)$ for some k-linear vector space W, such that the closed points $|G_k(d,n)|$, can be identified with d dimensional linear subspaces of $V = k^n$. In other words, we wish to define a scheme which is the algebraic geometry analogue of the Grassmannian from differential geometry.

We claim that the correct W is given by $W = \Lambda^d V$, then:

$$\mathbb{P}(W) = \operatorname{Proj} \operatorname{Sym}(\Lambda^d V^*)$$

We define $D \subset \Lambda^d V$ as:

$$D = \{v_1 \wedge \cdots \wedge v_d \in \Lambda^d V : v_i \in V\}$$

Note that we are not taking this as a linear subspace or span, we are simply considering all elements in $\Lambda^d V$ which can be written in this form, i.e. alternating tensors which are simple or pure. We define an ideal via:

$$I = \{\omega \in \operatorname{Sym}(\Lambda^d V^*) : \omega(D) = \{0\}\}$$

and immediately note that $I \cap \Lambda^d V^* = 0$. We need to check that this ideal is homogenous; let $\omega \in I$, and write:

$$\omega = \sum_i \omega_i$$

where each ω_i has degree i. It suffices to check that if $\omega \in I$, then $\omega_i \in I$ for each I. Since $\omega(D) = 0$, we see that for any $\lambda \in k^\times$ that $\omega(\lambda \cdot D) = 0$. It follows that for all $v_1 \wedge \cdots \wedge v_d$, and all λ we have that:

$$\omega(\lambda v_1 \wedge \cdots \wedge v_d) = \sum_i \lambda^i \omega_i(v_1 \wedge \cdots \wedge v_d) = 0$$

Fixing $v_1 \wedge \cdots \wedge v_d$, and writing $a_i = \omega_i(v_1 \wedge \cdots \wedge v_d)$, we thus have a sequence of elements $(a_1, \ldots, a_m)$ for some m, such that:

$$\sum_i \lambda^i a_i = 0$$

for all non zero $\lambda \in k$. In particular, this means that the polynomial $p(x) \in k[x]$ given by:

$$p(x) = \sum_i x^i a_i$$

is the zero polynomial, hence each $a_i = 0$[28]. . Since this hold for all $v_1 \wedge \cdots v_d$, it follows that each ω_i is identically zero on D, and thus I is generated by homogenous elements.

We claim that $G_k(d,n) = \mathbb{V}(I)$ is the desired subscheme. Given a d dimensional linear subspace $W \subset V$, we choose a basis $\{v_1, \ldots, v_d\}$ and send it to $[v_1 \wedge \cdots \wedge v_d] \in \mathbb{P}(\Lambda^d V)$. Note that $[v_1 \wedge \cdots \wedge v_d] \in \mathbb{V}(I)$, as every element in I vanishes on $l = \operatorname{span}\{v_1 \wedge \cdots \wedge v_d\}$, hence $I \subset \mathfrak{p}_l$. Note that this independent of the chosen basis, as another basis $\{w_1, \ldots, w_n\}$, yields an automorphism $g : W \to W$, such that

$$v_1 \wedge \cdots \wedge v_d = \det(g) \cdot w_1 \wedge \cdots \wedge w_d$$

which both determine the same $l \in |\mathbb{P}(\Lambda^d V)|$. Now let $\mathfrak{p}$ be a closed point of $\mathbb{P}(\Lambda^d V)$, and suppose that $\mathfrak{p} \in \mathbb{V}(I)$. Then we can uniquely identify $\mathfrak{p}$ with a linear subspace of $\Lambda^d V$, and since $\mathfrak{p} \in \mathbb{V}(I)$, this linear subspace must be spanned by some $v_1 \wedge \cdots \wedge v_d$ for some $v_i \in V$. We then send $\mathfrak{p}$ to the vector subspace spanned by $v_1, \ldots, v_d$. These operations are inverses of one another and thus we have obtained a bijection:

$$|G_k(d,n)| \longleftrightarrow \{d \text{ dimensional linear subspaces of } V\}$$

[28]Note, that this argument only works if k has infinitely many elements, as then the ideal $\bigcap_{\lambda \in k} \langle x - \lambda \rangle = \langle 0 \rangle$. We will fix this later, when we give a better definition of the Grassmanian.

2.3 Fibre Products

Just as the coproduct does not generally exist in the category of rings, and is replaced with the more general notion of the tensor product of rings (which becomes the coproduct in the category of A algebras), we have a similar situation regarding direct products in the category of schemes. In particular, the direct product does not generally exist in the category of schemes, but is instead replaced with the more general notion of a fibre product.

Definition 2.3.1. Let X, Y and Z be objects in an arbitrary category, with morphisms $f : X \to Z$ and $g : Y \to Z$. The **fibre product of** X **and** Y **over** Z is the triplet $(X \times_Z Y, \pi_X, \pi_Y)$ such that the following hold:

i) $X \times_Z Y$ is an object in the aforementioned category.

ii) π_X and π_Y are morphisms $X \times_Z Y$ to X and Y respectively.

iii) If Q is any other object with morphisms $p_X : Q \to X$ and $p_Y : Q \to Y$ such that $f \circ p_X = g \circ \pi_Y$ then there exists a unique morphism $\psi : Q \to X \times_Z Y$ such that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[dr, "\exists ! \psi"] \arrow[drr, bend left, "p_Y"] \arrow[ddr, bend right, "p_X"'] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

We call $X \times_Z Y$ the **fibre products** and the morphisms π_X and π_Y **projection maps**.=

Note that this is the diagram defining a tensor product in the category of rings with the arrows reversed. Before we prove that fiber products of schemes exist, we will first prove some very general properties of fibre products. We will state most of our results in terms of schemes, but we alert the reader to the fact that the following results will hold in any category where fibre products exist. For now suppose we have already proven that fibre products exist in the category of schemes. First we employ the following definition:

Definition 2.3.2. Let Z be a scheme; a pair (X, f) where X is a scheme, and $f : X \to Z$ is a morphism of schemes is called **scheme over Z** or a **Z-scheme**. If (X, f) and (Y, g) are Z schemes, then a **morphism of Z schemes** $F : X \to Y$ is a morphism of schemes such that $f = g \circ F$.

One easily verifies the that collection of all Z schemes and their morphisms is a category which contains fibered products (assuming fibered products exist in the category of schemes.).

Lemma 2.3.1. *Let (X, f), (Y, g), (W, h) be Z schemes, then there are canonical isomorphisms:*

$$X \times_Z Y \cong Y \times_Z X \qquad \text{and} \qquad (X \times_Z Y) \times_Z W \cong X \times_Z (Y \times_Z W)$$

Proof. For notation purposes, we will denote projection maps on the left hand side of the first isomorphism with a superscript 1, and those on the right hand side with a superscript 2. Now note that we trivially have that the projection maps satisfy $f \circ \pi^i_X = g \circ \pi^i_Y$, so there exists unique morphisms $\psi : X \times_Z Y \to Y \times_Z X$ and $\phi : Y \times_Z X \to X \times_Z Y$. We thus have a morphism $\phi \circ \psi : X \times_Z Y \to X \times_Z Y$ which makes the following diagram commute:

$$\begin{tikzcd}
X \times_Z Y \arrow[dr, "\phi \circ \psi"] \arrow[drr, bend left, "\pi^1_Y"] \arrow[ddr, bend right, "\pi^1_X"'] & & \\
& X \times_Z Y \arrow[r, "\pi^1_Y"] \arrow[d, "\pi^1_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

However, the identity map also clearly satisfies this, so by uniqueness $\phi \circ \psi = \mathrm{Id}$. A similar argument shows that $\psi \circ \phi = \mathrm{Id}$, hence ψ (and ϕ) is a unique isomorphism.

Now note that $(X \times_Z Y) \times_Z W$ comes equipped with morphisms to $(X \times_Z Y)$ and W, given by $\pi^1_{X \times_Z Y}$ and π_W. We thus have a morphism from $(X \times_Z Y) \times_Z W$ to X and Y given by $\pi_X \circ \pi_{X \times_Z Y}$ and $\pi_Y \circ \pi_{X \times_Z Y}$. We see that $X \times_Z Y$ is a Z scheme when equipped with the morphism $f \circ \pi_X$ (equivalently $g \circ \pi_Y$), so we have morphisms:

$$\pi_Y \circ \pi_{X \times_Z Y} : (X \times_Z Y) \times_Z W \longrightarrow Y$$

and:

$$\pi_W : (X \times_Z Y) \times_Z W \longrightarrow W$$

which satisfy:

$$\begin{aligned} g \circ (\pi_Y \circ \pi_{X \times_Z Y}) =& (f \circ \pi_X) \circ \pi_{X \times_Z Y} \\ =& h \circ \pi_W \end{aligned}$$

so we have a unique morphism $\xi : (X \times_Z Y) \times_Z W \to Y \times_Z W$. Now we $Y \times_Z W$ is a Z scheme when equipped with the morphism $g \circ \pi_Y$ (or equivalently $h \circ \pi_W$). We see that:

$$\begin{aligned} (g \circ \pi_Y) \circ \xi =& g \circ \pi_Y \circ \pi_{X \times_Z Y} \\ =& f \circ \pi_X \circ \pi_{X \times_Z Y} \end{aligned}$$

so there is a unique morphism:

$$\psi : (X \times_Z Y) \times_Z W \longrightarrow X \times_Z (Y \times_Z W)$$

and the same argument gives a unique morphism:

$$\phi : X \times_Z (Y \times_Z W) \longrightarrow (X \times_Z Y) \times_Z W$$

which make similar diagrams commute. We see that the composition $\phi \circ \psi$ makes the following diagram commute:

$$\begin{tikzcd}
(X \times_Z Y) \times_Z W \arrow[dr, "\phi \circ \psi"] \arrow[drr, bend left, "\pi_W"] \arrow[ddr, bend right, "\pi_{X \times_Z Y}"'] & & \\
& (X \times_Z Y) \times_Z W \arrow[r, "\pi_W"] \arrow[d, "\pi_{X \times_Z Y}"] & W \arrow[d, "h"] \\
& X \times_Z Y \arrow[r, "f \circ \pi_X"] & Z
\end{tikzcd}$$

so $\phi \circ \psi = \mathrm{Id}$. The same argument then shows that $\psi \circ \phi = \mathrm{Id}$, so ϕ and ψ are isomorphisms as desired. □

We also have the following analogue of the fact that for commutative rings $A \otimes_B B \cong A$:

Lemma 2.3.2. *Let X be a Z-scheme, then there is a natural isomorphism $X \times_Z Z \cong X$.*

Proof. We will show that (X, Id_X, f) satisfies the universal property of $X \times_Z Z$. Indeed, note that Z is naturally a Z-scheme when equipped with the identity morphism $\mathrm{Id}_Z : Z \to Z$. Trivially, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{Id}_X"] & Z \arrow[d, "\mathrm{Id}_Z"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

Suppose Q is another scheme with morphisms $p_X : Q \to X$ and $p_Z : Q \to Z$, such that $f \circ p_X = \mathrm{Id}_Z \circ p_Z$, then the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "p_Z"] \arrow[ddr, bend right, "p_X"'] & & \\
& X \arrow[r, "f"] \arrow[d, "\mathrm{Id}_X"] & Z \arrow[d, "\mathrm{Id}_Z"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

We see that putting $p_X : Q \to X$ in the empty diagonal makes the diagram commute, and that any other morphism $\phi : Q \to X$ must satisfy $\mathrm{Id}_X \circ \phi = p_X$, so $\phi = p_X$ and the morphism is unique. It follows that X satisfies the universal property of the fibre product and is thus naturally isomorphic to $X \times_Z Z$. □

We have the following extension of the previous results:

Lemma 2.3.3. *Let X and Y be Z-schemes, and S a Y schemes viewed as an Z scheme via the composition $S \to Y \to Z$, then there is a canonical isomorphism of Z schemes:*

$$(X \times_Z Y) \times_Y S \cong X \times_Z S$$

where $(X \times_Z Y)$ is viewed as Y scheme via the second projection π_Y.

Proof. Let $f : X \to Z$, $g : Y \to Z$, and $h : S \to Y$ be the various morphisms that make X and Y Z-schemes, and S an Y scheme. We first know that by Lemma 2.3.1 and Lemma 2.3.2, as Y schemes $S \cong Y \times_Y S$; we claim these are also isomorphic as Z-schemes. There is then a unique isomorphism which makes the following diagram commute:

$$\begin{tikzcd}
S \arrow[dr, "\phi"] \arrow[drr, bend left, "\mathrm{Id}_S"] \arrow[ddr, bend right, "h"'] & & \\
& Y \times_Y S \arrow[r, "\pi_S"] \arrow[d, "\pi_Y"] & S \arrow[d, "h"] \\
& Y \arrow[r, "\mathrm{Id}_Y"] & Y
\end{tikzcd} \qquad \star$$

Now $Y \times_Y S$ is a Z scheme in one of two ways, via $g \circ \pi_Y$, or via $g \circ h \circ \pi_S$, however, $h \circ \pi_S = \mathrm{Id}_Y \circ \pi_Y = \pi_Y$, so these are actually equivalent Z-scheme structures and $Y \times_Y Z$ has a natural Z-scheme structure independent of choice. We thus see that:

$$g \circ \pi_Y \circ \phi = g \circ h$$

so ϕ is a Z scheme isomorphism as well. We now claim that:

$$X \times_Z S \cong X \times_Z (Y \times_Y S)$$

We have a morphism $\phi \circ \pi_S : X \times_Z S \to Y \times_Y S$ and a morphism $\pi_X : X \times_Z S \to X$ which satisfy:

$$g \circ \pi_Y \circ \phi \circ \pi_S = g \circ \pi_Y = f \circ \pi_X$$

so there is a unique morphism ψ which makes the following diagram commute:

$$\begin{tikzcd}
X \times_Z S \arrow[dr, "\psi"] \arrow[drr, bend left, "\phi\circ\pi_S"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z (Y \times_Y S) \arrow[r, "\pi_{Y\times_Y S}"] \arrow[d, "\pi_X"] & Y \times_Y S \arrow[d, "g\circ\pi_Y"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

In a similar vein, we have a Z scheme isomorphism $\phi^{-1} : Y \times_Y S \to Y$, which by the same argument induces a unique Z-scheme morphism $\xi : X \times_Z (Y \times_Y S) \to X \times_Z S$ which makes the following diagram commute:

$$\begin{tikzcd}
X \times_Z (Y \times_Y S) \arrow[dr, "\xi"] \arrow[drr, bend left, "\pi_S\circ\pi_{Y\times_Y S}"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z S \arrow[r, "\pi_S"] \arrow[d, "\pi_X"] & S \arrow[d, "g\circ h"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

The composition $\xi \circ \psi : X \times_Z S \to X \times_Z S$ then satisfies:

$$\pi_X \circ \xi \circ \psi = \pi_X \circ \psi = \pi_X$$

and:

$$\begin{aligned}
\pi_S \circ \xi \circ \psi =& \pi_S \circ \pi_{Y\times_Y S} \circ \psi \\
=& \pi_S \circ \phi \circ \pi_S \\
=& \pi_S \circ \mathrm{Id}_Y \\
=& \pi_S
\end{aligned}$$

So $\xi \circ \psi$ is the unique map making the following diagram commute:

$$\begin{tikzcd}
X \times_Z S \arrow[dr, "\xi\circ\psi"] \arrow[drr, bend left, "\pi_S"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z S \arrow[r, "\pi_S"] \arrow[d, "\pi_X"] & S \arrow[d, "g\circ h"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

however, as before the identity map satisfies this as well so by uniqueness $\xi \circ \psi$ is the identity. Similarly, $\psi \circ \xi$ is the identity map as well, so $X \times_Z S \cong X \times_Z (Y \times_Y S)$ as desired.

It now suffices to show that as Z-schemes:

$$(X \times_Z Y) \times_Y S \cong X \times_Z (Y \times_Y S)$$

We first note that as a Y-scheme we have the following commutative diagram:

$$\begin{tikzcd}
(X \times_Z Y) \times_Y S \arrow[r, "\pi_S"] \arrow[d, "\pi_{X\times_Z Y}"] & S \arrow[d, "h"] \\
X \times_Z Y \arrow[r, "\pi_Y"] & Y
\end{tikzcd}$$

Now note that that $f \circ \pi_X = g \circ \pi_Y$, so we obtain that:

$$\begin{aligned} f \circ \pi_X \circ \pi_{X \times_Z Y} =& g \circ \pi_Y \circ \pi_{X \times_Z Y} \\ =& g \circ h \circ \pi_S \end{aligned}$$

Hence we have the following commutative diagram:

$$\begin{tikzcd} (X \times_Z Y) \times_Y S \arrow[r, "\pi_S"] \arrow[d, "\pi_X \circ \pi_{X \times_Z Y}"] & S \arrow[d, "g \circ h"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

We have a morphism $\pi_X \circ \pi_{X \times_Z Y} : (X \times_Z Y) \times_Y S \to X$, and a morphism $\phi \circ \pi_S : (X \times_Z Y) \times_Y S \to Y \times_Y S$ such that:

$$\begin{aligned} (g \circ h \circ \pi_S) \circ \phi \circ \pi_S =& g \circ h \circ \pi_S \\ =& f \circ \pi_X \circ \pi_{X \times_Z Y} \end{aligned}$$

hence there exists a unique morphism $\psi : (X \times_Z Y) \times_Y S \to X \times_Z (Y \times_Y S)$ such that the following diagram commutes:

$$\begin{tikzcd} (X \times_Z Y) \times_Y S \arrow[dr, "\psi"] \arrow[drr, bend left, "\phi \circ \pi_S"] \arrow[ddr, bend right, "\pi_X \circ \pi_{X \times_Z Y}"'] & & \\ & X \times_Z (Y \times_Y S) \arrow[r, "\pi_{Y \times_Y S}"] \arrow[d, "\pi_X"] & Y \times_Y S \arrow[d, "g \circ \pi_Y"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

Now we go the other direction; we already have a morphism $\pi_S \circ \pi_{Y \times_Y S} : X \times_Z (Y \times_Y S) \to Y$, so we need to construct a morphism $\alpha : X \times_Z (Y \times_S Y) \to X \times_Z Y$. We have a morphism to X, and we have a morphism to Y given by $\pi_Y \circ \pi_{Y \times_Y S}$. We have that:

$$g \circ \pi_Y \circ \pi_{Y \times_Y S} = f \circ \pi_X$$

by the Z scheme structure on $X \times_Z (Y \times_S Y)$ so α is then the unique map that makes the following diagram commute:

$$\begin{tikzcd} X \times_Z (Y \times_Y S) \arrow[dr, "\alpha"] \arrow[drr, bend left, "\pi_Y \circ \pi_{Y \times_Y S}"] \arrow[ddr, bend right, "\pi_X"'] & & \\ & X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

We now see that:

$$f \circ \pi_X \circ \alpha = f \circ \pi_X = g \circ \pi_Y \circ \pi_{Y \times_Y S}$$

so we have a unique map $\xi : X \times_Z (Y \times_Y S) \to (X \times_Z Y) \times_Y S$ that makes the following diagram commute:

$$
\begin{tikzcd}
X \times_Z (Y \times_Y S) \arrow[dr, "\xi"] \arrow[drr, bend left, "\pi_Y \circ \pi_{Y \times_Y S}"] \arrow[ddr, bend right, "\alpha"'] & & \\
& (X \times_Z Y) \times_Y S \arrow[r, "\pi_S"] \arrow[d, "\pi_{X \times_Z Y}"] & S \arrow[d, "g \circ h"] \\
& X \times_Z Y \arrow[r, "f \circ \pi_X"] & Z
\end{tikzcd}
$$

Now note that $\xi \circ \psi$ satisfies:

$$\pi_{X \times_Z Y} \circ \xi \circ \psi = \alpha \circ \psi$$

And moreover, see that:

$$\pi_X \circ \alpha \circ \psi = \pi_X \circ \psi = \pi_X \circ \pi_{X \times_Z Y}$$

as well as:

$$
\begin{aligned}
\pi_Y \circ \alpha \circ \psi =& \pi_Y \circ \pi_{Y \times_Y S} \circ \psi \\
=& \pi_Y \circ \phi \circ \pi_S \\
=& h \circ \pi_S \\
=& \mathrm{Id}_Y \circ \pi_Y \\
=& \pi_Y
\end{aligned}
$$

So $\alpha \circ \psi$ makes the following diagram commute:

$$
\begin{tikzcd}
(X \times_Z Y) \times_Y S \arrow[dr, "\alpha \circ \psi"] \arrow[drr, bend left, "\pi_Y \circ \pi_{Y \times_Y S}"] \arrow[ddr, bend right, "\pi_X \circ \pi_{X \times_Z Y}"'] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}
$$

However, replacing $\pi_{X \times_Z Y}$ also makes this diagram commute, so $\pi_{X \times_Z Y} = \alpha \circ \psi$, and we have that:

$$\pi_{X \times_Z Y} \circ \xi \circ \psi = \pi_{X \times_Z Y}$$

We also see that:

$$
\begin{aligned}
\pi_S \circ \xi \circ \psi =& \psi_S \circ \pi_{Y \times_Y S} \circ \psi \\
=& \pi_S \circ \phi \circ \pi_S \\
=& \pi_S
\end{aligned}
$$

It follows that $\xi \circ \psi$ makes the following diagram commute:

$$
\begin{tikzcd}
(X \times_Z Y) \times_Y S \arrow[dr, "\alpha \circ \psi"] \arrow[drr, bend left, "\pi_Y \circ \pi_{Y \times_Y S}"] \arrow[ddr, bend right, "\pi_X \circ \pi_{X \times_Z Y}"'] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}
$$

since the identity map makes this diagram commute as well we have that $\xi \circ \psi = \mathrm{Id}$. We now see that:

$$\begin{aligned}\pi_X \circ \psi \circ \xi &= \pi_X \circ \pi_{X\times_Z Y} \circ \xi \\ &= \pi_X \circ \alpha \\ &= \alpha\end{aligned}$$

while:

$$\begin{aligned}\pi_{Y\times_Y S} \circ \psi \circ \xi &= \phi \circ \pi_S \circ \xi \\ &= \phi \circ \pi_S \circ \pi_{Y\times_Y S}\end{aligned}$$

We claim that $\phi \circ \pi_S$ is the identity map; indeed note that we have:

$$\pi_S \circ \phi \circ \pi_S = \mathrm{Id}_S \circ \pi_S = \pi_S$$

while:

$$\pi_Y \circ \phi \circ \pi_S = h \circ \pi_S = \mathrm{Id}_Y \circ \pi_Y = \pi_Y$$

so $\phi \circ \pi_S$ makes the following diagram commute:

$$\begin{tikzcd}
Y \times_Y S \arrow[dr, "\phi\circ\pi_S"] \arrow[drr, bend left, "\pi_S"] \arrow[ddr, bend right, "\pi_Y"'] & & \\
& Y \times_Y S \arrow[r, "\pi_S"] \arrow[d, "\pi_Y"] & S \arrow[d, "g\circ h"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

However, so does the identity map, hence $\phi \circ \pi_S = \mathrm{Id}_{Y\times_Y S}$, and we have that:

$$\begin{aligned}\pi_{Y\times_Y S} \circ \psi \circ \xi &= \phi \circ \pi_S \circ \pi_{Y\times_Y S} \\ &= \mathrm{Id}_{Y\times_Y S} \circ \pi_{Y\times_Y S} \\ &= \pi_{Y\times_Y S}\end{aligned}$$

So it follows that $\psi \circ \xi$ makes the following diagram commute:

$$\begin{tikzcd}
X \times_Z (Y \times_Y S) \arrow[dr, "\psi\circ\xi"] \arrow[drr, bend left, "\pi_{Y\times_Y S}"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z (Y \times_Y S) \arrow[r, "\pi_{Y\times_Y S}"] \arrow[d, "\pi_X"] & Y \times_Y S \arrow[d, "g\circ\pi_Y"] \\
& X \arrow[r, "f\circ\pi_X"] & Z
\end{tikzcd}$$

but again so does the identity so $\psi \circ \xi = \mathrm{Id}$. It follows that:

$$(X \times_Z Y) \times_Y S \cong X \times_Z (Y \times_Y S) \cong X \times_Z S$$

implying the claim. □

The following lemmas are extremely helpful in identifying schemes as fibre products, as well as morphisms between them. They will be crucial in our existence proof of the fibre product.

Definition 2.3.3. Let Q, X, Y and Z, be schemes which fit into the following commutative square:

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & Z
\end{tikzcd}$$

If the induced map $Q \to X \times_Z Y$ is an isomorphism, then we call the above diagram a **cartesian square**.

Lemma 2.3.4. *Consider the following commutative diagram of schemes:*

$$\begin{tikzcd}
X'' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_{S''}"] & X' \arrow[r, "\pi_X"] \arrow[d, "\pi_{S'}"] & X \arrow[d, "\pi_S"] \\
S'' \arrow[r, "f_{S'}"] & S' \arrow[r, "f_S"] & S
\end{tikzcd}$$

If the left and right squares are cartesian then the outer square is cartesian. Moreover, if the outer square and the right square are cartesian, then the left is as well.

Proof. We need to show that the following square:

$$\begin{tikzcd}
X'' \arrow[r, "\pi_X \circ \pi_{X'}"] \arrow[d, "\pi_{S''}"] & X \arrow[d, "\pi_S"] \\
S'' \arrow[r, "f_S \circ f_{S'}"] & S
\end{tikzcd}$$

is cartesian. We do so by showing that $(X'', \pi_{S''}, \pi_X \circ \pi_{X'})$ satisfies the universal property of the fibre product. Suppose that Q is a scheme equipped with morphisms $p_{S''}$ and p_X such that $\pi_S \circ p_X = f_S \circ f_{S'} \circ p_{S''}$, then we have the following commutative diagram:

$$\begin{tikzcd}
Q \arrow[drrr, bend left, "p_X"] \arrow[ddr, bend right, "p_{S''}"'] & & & \\
& X'' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_{S''}"] & X' \arrow[r, "\pi_X"] \arrow[d, "\pi_{S'}"] & X \arrow[d, "\pi_S"] \\
& S'' \arrow[r, "f_{S'}"] & S' \arrow[r, "f_S"] & S
\end{tikzcd}$$

In particular, since the outer square is cartesian we have a unique morphism $p_{X'}$ such that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[drrr, bend left, "p_X"] \arrow[drr, "p_{X'}"] \arrow[ddr, bend right, "p_{S''}"'] & & & \\
& X'' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_{S''}"] & X' \arrow[r, "\pi_X"] \arrow[d, "\pi_{S'}"] & X \arrow[d, "\pi_S"] \\
& S'' \arrow[r, "f_{S'}"] & S' \arrow[r, "f_S"] & S
\end{tikzcd}$$

So now Q comes equipped with maps $p_{X'} : Q \to X'$ and $p_{S'} : Q \to S''$ such that $f_{S'} \circ p_{S''} = \pi_{S'} \circ p_{X'}$. By hypothesis there is then a unique map $\phi : Q \to X''$ such that:

$$\pi_{S''} \circ \phi = p_{S''} \qquad \text{and} \qquad \pi_{X'} \circ \phi = p_{X'} \tag{2.3.1}$$

We thus need only show that:

$$\pi_X \circ \pi_{X'} \circ \phi = p_X$$

However, we know that $p_X = \pi_X \circ p_{X'}$ so by (2.6) we have that:

$$\pi_X \circ \pi_{X'} \circ \phi = \pi_X \circ p_{X'} = p_X$$

hence X'' satisfies the universal property of the fibre product and thus the outer square is a cartesian.

Now suppose the outer square and the right square are cartesian, and let Q be scheme equipped with morphisms $p_{S''} : Q \to S''$ and $p_{X'} : Q \to X'$ such that $f_{S'} \circ p_{S''} = \pi_{S'} \circ p_{X'}$. We thus have the following diagram:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "p_{X'}"] \arrow[ddr, bend right, "p_{S''}"'] & & & & \\
& X'' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_{S''}"] & X' \arrow[r, "\pi_X"] \arrow[d, "\pi_{S'}"] & X \arrow[d, "\pi_S"] \\
& S'' \arrow[r, "f_{S'}"] & S' \arrow[r, "f_S"] & S
\end{tikzcd}$$

Now note that the map $\pi_X \circ p_{X'} : Q \to X$ makes the following diagram a commute:

$$\begin{tikzcd}
Q \arrow[dr, "p_{X'}"] \arrow[drr, bend left, "\pi_X \circ p_{X'}"] \arrow[ddr, bend right, "f_S \circ p_{S''}"'] & & \\
& X' \arrow[r, "\pi_X"] \arrow[d, "\pi_{S'}"] & X \arrow[d, "\pi_{S'}"] \\
& S' \arrow[r, "f_S"] & S
\end{tikzcd} \qquad (\star)$$

and since X' is a fibre product, we have that $p_{X'}$ and $\pi_{X'} \circ p_{X'}$ are the unique maps that make this diagram commute. We then obtain the following commutative diagram:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "p_{X'}"] \arrow[drrr, bend left, "\pi_X \circ p_{X'}"] \arrow[ddr, bend right, "p_{S''}"'] & & & \\
& X'' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_{S''}"] & X' \arrow[r, "\pi_X"] \arrow[d, "\pi_{S'}"] & X \arrow[d, "\pi_S"] \\
& S'' \arrow[r, "f_{S'}"] & S' \arrow[r, "f_S"] & S
\end{tikzcd}$$

Clearly we have that $f_S \circ f_{S'} \circ p_{S''} = \pi_S \circ \pi_X \circ p_{X'}$, so since the outer square is cartesian we have a unique map $\phi : Q \to X''$ such that:

$$\pi_{S''} \circ \phi = p_{S''} \qquad \text{and} \qquad \pi_X \circ \pi_{X'} \circ \phi = \pi_X \circ p_{X'}$$

So we need only show that:

$$\pi_{X'} \circ \phi = p_{X'}$$

However, this is clear as $\pi_{X'} \circ \phi$ satisfies:

$$\begin{aligned} f_{S'} \circ p_{S''} &= f_{S'} \circ \pi_{S''} \circ \phi \\ &= \pi_{S'} \circ \pi_{X'} \circ \phi \end{aligned}$$

and trivially:

$$\pi_X \circ \pi_{X'} \circ \phi = \pi_X \circ p_{X'}$$

It follows that replacing $p_{X'}$ with $\pi_{X'} \circ \phi$ in $(\star)$ makes the diagram commute, so by the uniqueness of $p_{X'}$ we have that $\pi_{X'} \circ \phi = p_{X'}$. Therefore, X'' satisfies the universal property of the fibre product $S'' \times_{S'} X'$, and the left square is cartesian. ☐

We continue with our litany of lemmas regarding fibre products:

Lemma 2.3.5. *Let $F : X \to X'$ and $G : Y \to Y'$ be morphisms of Z-schemes. Then there is a morphism $F \times G : X \times_Z X \to Y \times_Z Y'$ which makes the following diagram commute:*

$$\begin{tikzcd}
X \arrow[r, "F"] & Y \\
X \times_Z Y \arrow[u, "\pi_X"] \arrow[r, "F \times G"] \arrow[d, "\pi_Y"] & X' \times_Z Y' \arrow[u, "\pi_{X'}"] \arrow[d, "\pi_{Y'}"] \\
Y \arrow[r, "G"] & Y'
\end{tikzcd}$$

Proof. Note that since F and G are Z scheme morphisms, we have morphisms $F \circ \pi_X : X \times_Z Y \to X'$ and $G \circ \pi_Y : X \times_Z Y \to Y'$ which satisfy:

$$f' \circ F \circ \pi_X = f \circ \pi_X = g \circ \pi_Y = g' \circ G \circ \pi_Y$$

so we have a unique map $F \times G$ which makes the following diagram commute:

$$\begin{tikzcd}
X \times_Z Y \arrow[dr, "F \times G"] \arrow[drr, bend left, "G \circ \pi_Y"] \arrow[ddr, bend right, "F \circ \pi_X"'] & & \\
& X' \times_Z Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "\pi_{X'}"] & Y' \arrow[d, "g'"] \\
& X' \arrow[r, "f'"] & Z
\end{tikzcd}$$

We then see that:

$$G \circ \pi_Y = \pi_{Y'} \circ F \times G \qquad \text{and} \qquad F \circ \pi_X = \pi_{X'} \circ F \times G$$

so the diagram commutes as desired. ☐

We now come upon, and end our category theoretic results with, the first statement worthy of being called a theorem. We adopt Vakil's terminology and call this the magic square theorem, or the diagonal base change theorem.

Theorem 2.3.1. *Let X and X' be Y-schemes, and Y a Z-scheme; then the following square is cartesian:*

$$\begin{tikzcd}
X \times_Y X' \arrow[r] \arrow[d] & X \times_Z X' \arrow[d] \\
Y \arrow[r] & Y \times_Z Y
\end{tikzcd}$$

Before we prove this theorem, let us actually check that the above square is commutative, and construct the maps. First, let f, f' and g be morphisms making X and X' Y-schemes, and Y a Z-scheme.

The left vertical map is then given by $f \circ \pi_X$ (or equivalently $f' \circ \pi_{X'}$), and the right vertical map is the map $f \times f'$ constructed as in Lemma 2.3.5. Now, note that in the top right corner X and X' are Z-schemes with the morphisms $g \circ f$ and $g \circ f'$. Clearly, since $f \circ \pi_X = f' \circ \pi_{X'}$, we have that $(g \circ f) \circ \pi_X = (g \circ f') \circ \pi_{X'}$. It follows that there is then a map $\psi : X \times_Y X' \to X \times_Z X'$ such that the following diagram commutes[29]

$$\begin{tikzcd}
X \times_Y X' \arrow[dr, "\psi"] \arrow[drr, bend left, "\pi_{X'}"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z X' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_X"] & X' \arrow[d, "g\circ f'"] \\
& X \arrow[r, "g\circ f"] & Z
\end{tikzcd}$$

Finally, the bottom map is what we call the diagonal map $\Delta : Y \to Y \times_Z Y$, and is the unique map which makes the following diagram commute:

$$\begin{tikzcd}
Y \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{Id}_Y"] \arrow[ddr, bend right, "\mathrm{Id}_Y"'] & & \\
& Y \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_Y"] & Y \arrow[d, "g"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Now, we want to show that $\Delta \circ f \circ \pi_X = (f \times f') \circ \psi$, and we do so by showing that the both make the following diagram commute:

$$\begin{tikzcd}
X \times_Y X' \arrow[dr] \arrow[drr, bend left, "f'\circ\pi_{X'}"] \arrow[ddr, bend right, "f\circ\pi_X"'] & & \\
& Y \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_Y"] & Y \arrow[d, "g"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

We see that:

$$\pi_Y \circ \Delta \circ f \circ \pi_X = f \circ \pi_X = f \circ \pi_{X'}$$

so $\Delta \circ f \circ \pi_X$ makes the diagram commute. Moreover:

$$\pi_Y \circ (f \times f') \circ \psi = f \circ \pi_X \circ \psi = f \circ \pi_X = f' \circ \pi_X$$

so the two are equal by the uniqueness of the morphism which makes the diagram commute. We thus have that the square in Theorem 2.3.1 commutes and is:

$$\begin{tikzcd}
X \times_Y X' \arrow[r, "\psi"] \arrow[d, "f\circ\pi_X"] & X \times_Z X' \arrow[d, "f\times f'"] \\
Y \arrow[r, "\Delta"] & Y \times_Z Y
\end{tikzcd}$$

We now begin with actually proving the statement:

[29]Abuse of notation alert! We are again denoting different projection maps in the same way. We hope our judicious inclusion of diagrams helps the reader parse through this poor choice.

Proof. We will show that $X \times_Y X'$ satisfies the universal property of the fibre product. Let Q be another scheme with morphisms $\alpha : Q \to Y$ and $\beta : Q \to X \times_Z X'$ such that:

$$\Delta \circ \alpha = (f \times f') \circ \beta \tag{2.3.2}$$

Now first note that β is the unique map the makes the following diagram commute:

$$\begin{tikzcd}
Q \arrow[dr, "\beta"] \arrow[drr, bend left, "p_{X'}"] \arrow[ddr, bend right, "p_X"'] & & \\
& X \times_Z X' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_X"] & X' \arrow[d, "g \circ f'"] \\
& X \arrow[r, "g \circ f"] & Z
\end{tikzcd}$$

where $p_X = \pi_X \circ \beta$ and $p_{X'} = \pi_{X'} \circ \beta$. The maps satisfy $g \circ f \circ p_X = g \circ f' \circ p_{X'}$, however we want to show that the maps satisfy $f \circ p_X = f' \circ p_{X'}$. Applying π_Y to both sides of (2.3.2) yields:

$$\begin{aligned}
\alpha =& \pi_Y \circ (f \times f') \circ \beta \\
=& f \circ \pi_X \circ \beta \\
=& f \circ p_X
\end{aligned}$$

However, $f \circ \pi_X = f \circ \pi_{X'}$ so we also have that:

$$\begin{aligned}
\alpha =& f' \circ \pi_X \circ \beta \\
=& f \circ p_{X'}
\end{aligned}$$

so $f \circ p_X = f \circ p_{X'}$. There is then a unique morphism $\phi : Q \to X \times_Y X'$ such that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[dr, "\phi"] \arrow[drr, bend left, "p_{X'}"] \arrow[ddr, bend right, "p_X"'] & & \\
& X \times_Y X' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_X"] & X' \arrow[d, "f'"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

Now note that:

$$\begin{aligned}
f \circ \pi_X \circ \phi =& f \circ p_X \\
=& \alpha
\end{aligned}$$

so we need to that:

$$\psi \circ \phi = \beta$$

and it suffices to show that:

$$\pi_X \circ \psi \circ \phi = p_X \qquad \text{and} \qquad \pi_{X'} \circ \psi \circ \phi = p_{X'}$$

We have that:

$$\pi_X \circ \psi \circ \phi = \pi_X \circ \phi = p_X$$

and that:

$$\pi_{X'} \circ \psi \circ \phi = \pi_{X'} \circ \phi = p_{X'}$$

so $\psi \circ \phi = \beta$ by the uniqueness of β. We thus have that ϕ is the unique map which makes the following diagram commute:

$$\begin{tikzcd}
Q \arrow[dr, "\phi"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& X \times_Y X' \arrow[r, "\psi"] \arrow[d, "f \circ \pi_X"] & X \times_Z X' \arrow[d, "f \times f'"] \\
& Y \arrow[r, "\Delta"] & Y \times_Z Y
\end{tikzcd}$$

Therefore, $X \times_Y X'$ is isomorphic to the fibre product $Y \times_Z X \times_Z X'$ and the square is cartesian as desired. ☐

Now that we hav sufficiently established our results regarding fibre products that have nothing to do with algebraic geometry, it is time to actually prove that the fibre product of schemes indeed exist. We will prove this in varying steps, slowly building up to the general case. We begin where all schemes are affine:

Lemma 2.3.6. *Let X and Y be Z-schemes, and let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $Z = \operatorname{Spec} C$. The fibre product $X \times_Z Y$ is then the affine scheme $\operatorname{Spec}(A \otimes_C B)$.*

Proof. Since X and Y are Z-schemes, there are ring morphisms $f_X^\sharp : C \to A$ and $g_Y^\sharp : C \to B$ which turn A and B into C algebras so we can construct the tensor product $A \otimes_B C$. The tensor product comes equipped with maps $\iota_A : A \to A \otimes_C B$, and $\iota_B : B \to A \otimes_C B$ given by $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$, and satisfies the universal property that for any two maps $\phi_A : A \to R$ and $\phi_B : B \to R$ such that:

$$\phi_A \circ f_X^\sharp = \phi_B \circ g_Y^\sharp$$

then there is a unique ring homomorphism $\alpha : A \otimes_B C \to R$ such that the following diagram commutes:

$$\begin{tikzcd}
R & & \\
& A \otimes_C B \arrow[ul, "\alpha"] & B \arrow[l, "\iota_B"'] \arrow[ull, bend right, "\phi_B"'] \\
& A \arrow[u, "\iota_A"] \arrow[uul, bend left, "\phi_A"] & C \arrow[l, "f_X^\sharp"] \arrow[u, "g_Y^\sharp"']
\end{tikzcd}$$

Via the anti equivalence between the category of commutative rings and affine schemes, have that $\operatorname{Spec}(A \otimes_C B)$ comes equipped with projection maps $\pi_X : \operatorname{Spec}(A \otimes_C B) \to X$, $\pi_Y : \operatorname{Spec}(A \otimes_C B) \to Y$ which make the obvious square commute. If Q is any scheme with maps $p_X : Q \to X$ and $p_Y : Q \to Y$ satisfying $f \circ p_X = g \circ p_Y$, then the induced ring homomorphisms satisfy the conditions of the universal property of the tensor product of commutative rings. It follows there is a unique ring homomorphism $A \otimes_B C \to \mathscr{O}_Q(Q)$ which by Proposition 2.1.2 induces a unique scheme morphism $\psi : Q \to \operatorname{Spec}(A \otimes_C B)$ which makes the following diagram commute:

$$\begin{tikzcd}
Q \arrow[dr, "\psi"] \arrow[drr, bend left, "p_Y"] \arrow[ddr, bend right, "p_X"'] & & \\
& \operatorname{Spec}(A \otimes_C B) \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

The affine scheme $\operatorname{Spec}(A \otimes_C B)$ then satisfies the universal property of the fibre product, implying the claim. ☐

When the base scheme is affine $Z = \operatorname{Spec} C$, we often denote the fibre product $X \times_Z Y$ by $X \times_C Y$. We thus immediately have that:

$$\mathbb{A}^n_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^m_{\mathbb{C}} \cong \mathbb{A}^{m+n}_{\mathbb{C}}$$

via the isomorphism:

$$\mathbb{C}[x_1, \ldots, x_n] \otimes_{\mathbb{C}} \mathbb{C}[y_1, \ldots, y_m] \cong \mathbb{C}[x_1, \ldots, x_{n+m}]$$

Clearly the same statement holds for any commutative ring. Before we continue with our construction, we need the following result, where we note that we make no assumption on U, Z, or Y being affine:

Lemma 2.3.7. *Let $f : U \to Z$ be an open embedding, and $g : Y \to Z$ be any morphism. Then $U \times_Z Y$ exists, and there is an induced open embedding $U \times_Z Y \to Y$.*

Proof. Let $V = f(U)$, then we claim that the open subscheme $(g^{-1}(V), \mathscr{O}_Y|_{g^{-1}(V)})$ is the fibre product $U \times_Z Y$. We first note that we have an inclusion map $\iota : g^{-1}(V) \hookrightarrow Y$, as well as an isomorphism $f^{-1} : V \to U$, so since $g|_{g^{-1}(V)}$ is a morphism $g^{-1}(V) \to V$, we have that $f^{-1} \circ g|_{g^{-1}(V)}$ is a morphism $g^{-1}(V) \to U$. Now note that:

$$f \circ f^{-1} \circ g|_{g^{-1}(V)} = g|_{g^{-1}(V)} = g \circ \iota$$

so we have the following commutative square which we wish to show is cartesian:

$$\begin{tikzcd} g^{-1}(V) \arrow[r, "\iota"] \arrow[d, "f^{-1} \circ g|_{g^{-1}(V)}"'] & Y \arrow[d, "g"] \\ U \arrow[r, "f"] & Z \end{tikzcd}$$

Let Q be a scheme and $p_U : Q \to U$, $p_Y : Q \to Y$ be morphisms such that $f \circ p_U = g \circ p_Y$, then we want to find a morphism $\phi : Q \to g^{-1}(V)$ such that $f^{-1} \circ g|_{g^{-1}(V)} \circ \phi = p_U$ and $\iota \circ \phi = p_Y$. Since $f \circ p_U = g \circ p_Y$, we must have p_Y maps into $g^{-1}(V)$, so there is a unique morphism $\phi : Q \to g^{-1}(V)$ such that $\iota \circ \phi = p_Y$. Now note that:

$$\begin{aligned} f^{-1} \circ g|_{g^{-1}(V)} \circ \phi =& f^{-1} \circ g \circ \iota \circ \phi \\ =& f^{-1} \circ g \circ p_Y \\ =& f^{-1} \circ f \circ p_U \\ =& p_U \end{aligned}$$

It follows that $g^{-1}(V)$ satisfies the universal property of the fibre product $U \times_Z Y$, so there is a unique isomorphism $\psi : U \times_Z Y \to g^{-1}(V)$, and thus an open embedding $U \times_Z Y \to Y$ given by $\iota \circ \psi$ as desired. ☐

Note that the morphism $\iota \circ \psi$ is equal to the canonical projection $\pi_Y : U \times_Z Y \to Y$. In particular, if $U \to Z$ and $V \to Z$ are two inclusion maps, then we have that by the lemma above $U \times_Z V \cong U \cap V$. This matches up with the fact $A_f \otimes_A A_g \cong A_{fg}$. We now have the following result:

Lemma 2.3.8. *Let X and Y be Z-schemes, with $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $Z = \operatorname{Spec} C$. Moreover, let the morphism $\alpha : Y' \to Y$ be an open embedding. Then the fibre product $X \times_Z Y'$ exists, and the induced map $X \times_Z Y' \to X \times_Z Y$ is an open embedding.*

Proof. Note that $X \times_Z Y$ is a fibre product, and so by the previous lemma $(X \times_Z Y) \times_Y Y'$ is a fibre product as $Y' \to Y$ is an open embedding and we have a morphism $\pi_Y : X \times_Z Y \to Y$. It follows that the following diagram is commutative:

$$\begin{tikzcd} (X \times_Z Y) \times_Y Y' \arrow[r, "\pi_{X \times_Z Y}"] \arrow[d, "\pi_{Y'}"'] & X \times_Z Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \arrow[d, "f"] \\ Y' \arrow[r, "\alpha"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Since the right square and left square are both cartesian the outer square is cartesian, and we have by Lemma 2.3.5 that $(X \times_Z Y) \times_Y Y' \cong X \times_Z Y'$. By the preceding lemma we have that $(X \times_Z Y) \times_Y Y' \to X \times_Z Y$ is an open embedding, so the induced map $X \times_Z Y' \to X \times_Z Y$ is an open embedding as well. □

Lemma 2.3.9. *Let X and Y be Z-schemes, with $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} C$, and Y arbitrary. Then the fibre product $X \times_Z Y$ exists.*

Proof. Let $\{U_i\}$ be an open affine cover of Y. For each U_i, we have scheme morphisms $g|_{U_i} : U_i \to Z$ making each U_i a Z-scheme, hence the fibre product $X \times_Z U_i$ exists by Lemma 2.3.6. Let $U_{ij} = U_i \cap U_j$[30], then the scheme obtained by gluing each affine open along U_{ij} via the identity map is trivially isomorphic to Y. For each i let $V_i = X \times_Z U_i$, and moreover we have a morphism $g|_{U_{ij}} : U_{ij} \to Z$ which factors as $\iota \circ g : U_{ij} \to U_i \to Z$ where $\iota : U_{ij} \to U_i$. By Lemma 2.3.8 we have that the fibre product $X \times_Z U_{ij}$ exists and that there is an open embedding $\alpha_{ij} : X \times_Z U_{ij} \to V_i$.

We define $V_{ij} \subset V_i$ to then be the open subscheme $\alpha_{ij}(X \times_Z U_{ij})$. Now note that $U_{ij} = U_{ji} \subset Y$, so we have an equality $X \times_Z U_{ij} = X \times_Z U_{ji}$. Denoting by α_{ij}^{-1} the isomorphism $V_{ij} \to X \times_Z U_{ij}$, we obtain scheme isomorphisms $\phi_{ij} : V_{ij} \to V_{ji}$ given by $\alpha_{ji} \circ \alpha_{ij}^{-1}$.

We want to glue the schemes V_i together along the open subschemes V_{ij} via these scheme isomorphisms. Clearly we have that $\phi_{ij} = \phi_{ji}^{-1}$, so we need to check that $\phi_{ij}(V_{ij} \cap V_{ik}) = V_{ji} \cap V_{jk}$ and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $V_{ij} \cap V_{ik}$. Note that $V_{ij} \cap V_{ik}$ is the fibre product $V_{ij} \times_{V_i} V_{ik}$[31], and similarly we have that $V_{ji} \cap V_{jk}$ is the fibre product $V_{ji} \times_{V_j} V_{jk}$. Now note that:

$$V_{ji} \cong X \times_Z U_{ij} \cong X \times_Z U_i \times_Y U_j \cong V_{ij}$$

while:

$$V_{ik} \cong X \times_Z U_i \times_Y U_k \qquad \text{and} \qquad V_{jk} \cong X \times_Z U_j \times_Y U_k$$

so:

$$\begin{aligned} V_{ij} \cap V_{ik} \cong V_{ij} \times_{V_i} V_{ik} &\cong (X \times_Z U_i \times_Y U_j) \times_{V_i} (X \times_Z U_i \times_Y U_k) \\ &\cong X \times_Z U_i \times_Y U_j \times_Y U_k \\ &\cong X \times_Z U_{ijk} \end{aligned}$$

where $U_{ijk} = U_i \cap U_j \cap U_k$. Similarly, we have that:

$$\begin{aligned} V_{ji} \cap V_{jk} \cong V_{ji} \times_{V_j} V_{jk} &\cong (X \times_Z U_j \times_Y U_i) \times_{V_j} (X \times_Z U_j \times_Y U_k) \\ &\cong X \times_Z U_j \times_Y U_i \times_Y U_k \\ &\cong X \times_Z U_{ijk} \end{aligned}$$

It follows that $V_{ij} \cap V_{ik}$ is uniquely isomorphic to $V_{ji} \cap V_{jk}$. We need to show that this isomorphism is precisely ϕ_{ij} restricted to $V_{ij} \cap V_{ik}$. We note that the embedding $\alpha_{ij} : X \times_Z U_{ij} \to V_i$ comes from the cartesian square:

$$\begin{tikzcd} X \times_Z U_{ij} \arrow[r, "\alpha_{ij}"] \arrow[d, "\pi_{U_{ij}}"] & V_i \arrow[d, "\pi_{U_i}"] \\ U_{ij} \arrow[r, "\iota"] & U_i \end{tikzcd}$$

and since $U_{ijk} \hookrightarrow U_{ij}$ we have an open embedding $\beta_{ijk} : X \times_Z U_{ijk} \to X \times_Z U_{ij}$. Let ψ_{ijk} be the isomorphism $V_{ij} \cap V_{ik} \to X \times_Z U_{ijk}$, then we obtain the following diagram of cartesian squares:

$$\begin{tikzcd} V_{ij} \cap V_{ik} \arrow[r, "\psi_{ijk}"] \arrow[d, "\pi_{U_{ijk}}"] & X \times_Z U_{ijk} \arrow[r, "\beta_{ijk}"] \arrow[d, "\pi_{U_{ijk}}"] & X \times_Z U_{ij} \arrow[r, "\alpha_{ij}"] \arrow[d, "\pi_{U_{ij}}"] & V_i \arrow[d, "\pi_{U_i}"] \\ U_{ijk} \arrow[r, "\mathrm{Id}"] & U_{ijk} \arrow[r, "\iota"] & U_{ij} \arrow[r, "\iota"] & U_i \end{tikzcd}$$

[30] Note that the intersection of two affine opens is not necessarily affine.
[31] Which exists by Lemma 2.3.7 as both V_{ij} and V_{ik} are open subschemes of V_i

However, the inclusion map $\iota : V_{ij} \cap V_{ik} \to V_i$ also makes this diagram commute, so we have that:

$$\alpha_{ij} \circ \beta_{ijk} \circ \psi_{ijk} = \iota$$

similarly we have that:

$$\alpha_{ji} \circ \beta_{jik} \circ \psi_{jik} : V_{ji} \cap V_{jk} \to V_j$$

is the inclusion map. It follows that these maps are isomorphisms onto their images hence we have that:

$$\alpha_{ij}|_{\beta_{ijk}(X \times_Z U_{ijk})} \circ \beta_{ijk} \circ \psi_{ijk} = \mathrm{Id}_{V_{ij} \cap V_{ik}}$$

and:

$$\alpha_{ij}|_{\beta_{jik}(X \times_Z U_{jik})} \circ \beta_{jik} \circ \psi_{jik} = \mathrm{Id}_{V_{ji} \cap V_{jk}}$$

so in particular,

$$\alpha_{ij}^{-1}|_{V_{ij} \cap V_{ik}} = \beta_{ijk} \circ \psi_{ijk}$$

Moreover, note that $\beta_{ijk} = \beta_{jik}$, and that the unique isomorphism $V_{ij} \cap V_{ik} \to V_{ji} \cap V_{jk}$ is given by $\psi_{jik}^{-1} \circ \psi_{ijk}$. We see that:

$$\psi_{jik}^{-1} = \alpha_{ji}|_{\beta_{jik}(X \times_Z U_{jik})} \circ \beta_{jik}$$

hence:

$$\begin{aligned}
\psi_{jik}^{-1} \circ \psi_{ijk} =& \alpha_{ji}|_{\beta_{jik}(X \times_Z U_{jik})} \circ \beta_{jik} \circ \psi_{ijk} \\
=& \alpha_{ji}|_{\beta_{jik}(X \times_Z U_{ijk})} \circ \alpha_{ij}^{-1}|_{V_{ij} \cap V_{ik}} \\
=& (\alpha_{ji} \circ \alpha_{ij}^{-1})|_{V_{ij} \cap V_{ik}} \\
=& \phi_{ij}|_{V_{ij} \cap V_{ik}}
\end{aligned}$$

implying that $\phi_{ij}(V_{ij} \cap V_{ik}) = V_{ji} \cap V_{jk}$ as desired. It follows that $\phi_{ik}(V_{ij} \cap V_{ik}) = V_{kj} \cap V_{ki}$ while:

$$\phi_{jk} \circ \phi_{ij}(V_{ij} \cap V_{ik}) = \phi_{jk}(V_{ji} \cap V_{jk}) = V_{ki} \cap V_{kj}$$

so $\phi_{jk} \circ \phi_{ij}$ is the unique isomorphism $V_{ij} \cap V_{ik} \to V_{ki} \cap V_{kj}$, and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$. We thus have that the schemes V_i and V_j glue together along V_{ij} for all i and j and are locally isomorphic to $X \times_Z U_i$.

We denote this scheme by S and show that it satisfies the universal property of the fibre product. We first construct projection maps $\pi_X : S \to X$ and $\pi_Y : S \to Y$. We see that the isomorphisms ϕ_{ij} fit into the diagram:

$$\begin{tikzcd}
V_{ij} \arrow[dr, "\phi_{ij}"] \arrow[drr, bend left, "\pi_{U_{ij}}"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& V_{ji} \arrow[r, "\pi_{U_{ji}}"] \arrow[d, "\pi_X"] & U_{ij} \subset Y \arrow[d, "g|_{U_{ij}}"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

Note that here both π_X and $\pi_{U_{ij}}$ are the restrictions of the projection maps $\pi_X : V_i \to X$ and $\pi_{U_i} : V_i \to U_{ij} \subset Y$ to V_{ij} and similarly for V_j and V_{ji}. We thus have induced morphisms $\pi_X : S \to X$ and $\pi_Y : S \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

We want to show that this square is cartesian, so let Q be any other scheme, with projection maps $p_X : Q \to X$ and $p_Y : Q \to Y$ which make the relevant diagram commute. We have an open covering of Q by $\{\pi_Y^{-1}(U_i)\}$, and for each open we have a unique map ξ_i such that the following diagram commutes:

$$\begin{tikzcd}
\pi_Y^{-1}(U_i) \arrow[dr, "\xi_i"] \arrow[drr, bend left, "p_Y|_{\pi_Y^{-1}(U_i)}"] \arrow[ddr, bend right, "p_X"'] & & \\
& V_i \arrow[r, "\pi_{U_i}"] \arrow[d, "\pi_X"] & U_i \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

We need to show that:

$$\xi_j|_{\pi_Y^{-1}(U_{ij})} = \phi_{ij} \circ \xi_i|_{\pi_Y^{-1}(U_{ij})}$$

We need only check that $\phi_{ij} \circ \xi_i|_{\pi_Y^{-1}(U_{ij})}$ makes the relevant diagram commute. Note that:

$$\pi_X \circ \phi_{ij} \circ \xi_i|_{\pi_Y^{-1}(U_{ij})} = \pi_X \circ \xi_i|_{\pi_Y^{-1}(U_{ij})} = p_X$$

and:

$$\pi_{U_{ji}} \circ \phi_{ij} \circ \xi_i|_{\pi_Y^{-1}(U_{ij})} = \pi_{U_{ij}} \circ \xi_i|_{\pi_Y^{-1}(U_{ij})} = p_Y|_{\pi_Y^{-1}(U_{ji})}$$

so the two are equal, and we thus have a unique morphism $\xi : Q \to S$ such that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[dr, "\xi"] \arrow[drr, bend left, "p_Y"] \arrow[ddr, bend right, "p_X"'] & & \\
& S \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & U_i \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

so S satisfies the universal property of $X \times_Z Y$ implying the claim. ☐

We now move to the next case:

Lemma 2.3.10. *Let X and Y be Z-schemes, with $Z = \operatorname{Spec} C$. Then the fibre product $X \times_Z Y$ exists.*

Proof. Let $\{U_i\}$ be an open affine covering of X, then by Lemma 2.3.9 the fibre products $U_i \times_Z Y$ exist. We have open embeddings $U_{ij} = U_i \cap U_j \to U_i$ given by the inclusion map. We have that the scheme $U_{ij} \times_{U_i} (U_i \times_Z Y)$ exists, so we have the following commutative diagram:

$$\begin{tikzcd}
U_{ij} \times_{U_i} (U_i \times_Z Y) \arrow[r, "\pi_{U_i \times_Z Y}"] \arrow[d, "\pi_{U_{ij}}"] & U_i \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_{U_i}"] & Y \arrow[d] \\
U_{ij} \arrow[r, "\iota"] & U_i \arrow[r, "f|_{U_i}"] & Z
\end{tikzcd}$$

where the left and right squares are cartesian, so the outer square is cartesian. It follows that the fibre product $U_{ij} \times_Z Y$ exists and comes with open embeddings $\alpha_{ij} : U_{ij} \times_Z Y \to U_i \times_Z Y$. These open embeddings satisfy the same properties as the ones in Lemma 2.3.9, so if $V_{ij} = \alpha_{ij}(U_{ij} \times_Z Y)$, we have isomorphisms $\alpha_{ji}^{-1} \circ \alpha_{ij} : V_{ij} \to V_{ji}$ which agree on triple overlaps. It follows that the V_i's glue together along V_{ij} for all i and j, hence we obtain a scheme S which is locally isomorphic to $U_i \times_Z Y$. The same argument as in Lemma 2.3.9 shows that S satisfies universal property of $X \times_Z Y$, implying the claim. ☐

We now repeat the same result as in Lemma 2.3.8

Lemma 2.3.11. *Let X and Y be Z schemes, and suppose that there is an open embedding $Z \to Z'$, with Z' affine. Then the fibre product $X \times_Z Y$ exists.*

Proof. Let $\alpha : Z \to Z'$ be the open embedding, and $f : X \to Z$, $g : Y \to Z$ the morphisms making X and Y Z-schemes. Then we have by Lemma 2.3.10 that the following square is cartesian:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "\alpha\circ g"] \\ X \arrow[r, "\alpha\circ f"] & Z' \end{tikzcd}$$

In particular, since α is a monomorphism, we have that $\pi_X \circ f = \pi_Y \circ g$, so the following square is commutative:

$$\begin{tikzcd} X \times_{Z'} Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Let Q be any scheme, with morphisms $p_X : Q \to X$ and $p_Y : Q \to Y$ such that the relevant diagram commutes. Then we have that $\alpha \circ f \circ p_X = \alpha \circ g \circ p_Y$ so there is a unique map $Q \to X \times_Z Y$ such that the fibre product diagram commutes. However, note that this same morphism makes the following diagram commute:

$$\begin{tikzcd} Q \arrow[dr, "\phi"] \arrow[drr, bend left, "p_Y"] \arrow[ddr, bend right, "p_X"'] & & \\ & X \times_{Z'} Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

so $X \times_{Z'} Y$ satisfies the universal property of $X \times_Z Y$, implying the claim. $\square$

We now prove the statement in generality:

Theorem 2.3.2. *Let X, and Y be Z-schemes, then the fibre product $X \times_Z Y$ exists.*

Proof. Let $\{U_i\}$ be an affine open cover of Z, then for all i, set $X_i = f^{-1}(U_i)$ and $Y_i = g^{-1}(U_i)$, then by Lemma 2.3.10 the fibre product $W_i = X_i \times_{U_i} Y_i$ exists for all i. Set $U_{ij} = U_i \cap U_j$, $X_{ij} = f^{-1}(U_{ij})$, and $Y_{ij} = g^{-1}(U_{ij})$, then by the preceding lemma $W_{ij} = X_{ij} \times_{U_{ij}} Y_{ij}$ exists for all i and j, and is isomorphic to $X_{ij} \times_{U_i} Y_{ij}$. There are then open embeddings W_{ij} into W_i and W_j by Lemma 2.3.8.

We now show that W_i satisfies the universal property of $X \times_Z Y_i$. Indeed, we have the following cartesian square:

$$\begin{tikzcd} W_i \arrow[r, "\pi_{Y_i}"] \arrow[d, "\pi_{X_i}"] & Y_i \arrow[d, "g|_{Y_i}"] \\ X_i \arrow[r, "f|_{X_i}"] & U_i \end{tikzcd}$$

Since $f|_{X_i} = f \circ \iota$, where ι is the inclusion map $X_i \hookrightarrow X$, and since we have inclusion maps $\iota : U_i \to Z$, we have the following commutative square:

$$\begin{tikzcd} W_i \arrow[r, "\pi_{Y_i}"] \arrow[d, "\iota\circ\pi_{X_i}"] & Y_i \arrow[d, "\iota\circ g|_{Y_i}"] \\ X \arrow[r, "\iota\circ f"] & Z \end{tikzcd}$$

Now suppose we are given a scheme Q, and morphisms $p_X : Q \to X$, $p_{Y_i} : Q \to Y_i$ such that:

$$\iota \circ f \circ p_X = \iota \circ g|_{Y_i} \circ p_{Y_i}$$

implying that $f \circ p_X = g|_{Y_i} \circ p_{Y_i}$. We see that p_X has image contained in X_i, and thus factors uniquely as $p_X = \iota \circ \alpha_{X_i}$ where $\iota : X_i \hookrightarrow X$ is the inclusion map. We hus have that $f \circ \iota \circ \alpha_{X_i} = f|_{X_i} \circ \alpha_{X_i} = g|_{Y_i} \circ p_{Y_i}$, so there is a unique morphism $\phi : Q \to W_i$ such that the following diagram commutes:

$$\begin{tikzcd} Q \arrow[dr, "\phi"] \arrow[drr, bend left, "p_{Y_i}"] \arrow[ddr, bend right, "\alpha_{X_i}"'] & & \\ & W_i \arrow[r, "\pi_{Y_i}"] \arrow[d, "\pi_{X_i}"] & Y_i \arrow[d, "g|_{Y_i}"] \\ & X_i \arrow[r, "f|_{X_i}"] & U_i \end{tikzcd}$$

However, since $\pi_{X_i} \circ \phi = \alpha_{X_i}$, we have that:

$$\iota \circ \pi_{X_i} = \iota \circ \alpha_{X_i} = p_X$$

so the following diagram commutes as well:

$$\begin{tikzcd} Q \arrow[dr, "\phi"] \arrow[drr, bend left, "p_{Y_i}"] \arrow[ddr, bend right, "p_X"'] & & \\ & W_i \arrow[r, "\pi_{Y_i}"] \arrow[d, "\iota \circ \pi_{X_i}"] & Y_i \arrow[d, "\iota \circ g|_{Y_i}"] \\ & X \arrow[r, "\iota \circ f"] & Z \end{tikzcd}$$

So $W_i \cong X \times_Z Y_i$ as desired, and similarly that $W_{ij} \cong X \times_Z Y_{ij}$. However, we are now in the same situation as Lemma 2.3.9 as the only point where we used that X and Z were affine was for the existence of the schemes $X \times_Z Y_i$ and $X \times_Z Y_{ij}$. We can thus glue the schemes W_i along W_{ij} as before, and the same argument shows that this scheme satisfies the universal property of $X \times_Z Y$, implying the claim. □

We now point out the following fact: fibre products in general have more points than naive cartesian products. Indeed, consider the scheme $X = \operatorname{Spec} \mathbb{C}[t]$, then the $X \times_{\mathbb{C}} X$, is the spectrum of the ring $\mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[t] \cong \mathbb{C}[u, t]$. The prime ideals of this ring are then certainly not of the form $(\mathfrak{p}, \mathfrak{q})$ for primes $\mathfrak{p}, \mathfrak{q} \subset \mathbb{C}[t]$. However, note that we that the closed points of $\operatorname{Spec} \mathbb{C}[t, u]$ are in bijection with $\mathbb{C}^2$, which is the naive set product of the closed points of $\operatorname{Spec} \mathbb{C}[t]$ with itself (all points save the zero ideal are closed in $\operatorname{Spec} \mathbb{C}[t]$ though). We wish to extend this discussion to arbitrary, but first we need the following definition, which we will explore more in the subsequent chapter:

Definition 2.3.4. Let k be a field, and X a scheme over $\operatorname{Spec} k$. Then X is **locally of finite type over k** if there exists an affine open cover $\{U_i\}$ such that $\mathscr{O}_X(U_i)$ is a finitely generated k algebra.

We will need the following lemma:

Lemma 2.3.12. *Let X be a scheme locally of finite type over k, then $x \in X$ is a closed point if and only if there exists an affine open $U = \operatorname{Spec} A$ containing x such that x corresponds to a maximal ideal of A. In particular:*

$$|X| = \bigcup_i |U_i|$$

for any affine open cover $\{U_i\}$

Proof. Let $x \in U \subset X$, with $U = \operatorname{Spec} A$, then x corresponds to a prime ideal $\mathfrak{p} \subset A$. We first claim that $\overline{\{\mathfrak{p}\}} = \mathbb{V}(\mathfrak{p})$. Indeed, suppose $\mathbb{V}(I)$ is any closed set containing $\{\mathfrak{p}\}$, then we have that $I \subset \mathfrak{p}$, so $\mathbb{V}(\mathfrak{p}) \subset \mathbb{V}(I)$ implying that $\overline{\{\mathfrak{p}\}} = \mathbb{V}(\mathfrak{p})$. Now suppose that x is closed in the subspace topology, then we have that $\{\mathfrak{p}\} = \mathbb{V}(\mathfrak{p})$; if $\mathfrak{p} \subset I$ for some ideal $I \subset A$, we have that $\mathbb{V}(I) \subset \mathbb{V}(\mathfrak{p})$ so $\mathbb{V}(I) = \{\mathfrak{p}\}$ or $\mathbb{V}(I) = \emptyset$. If $\mathbb{V}(I) = \emptyset$, then $I = A$, and if $\mathbb{V}(I) = \{\mathfrak{p}\}$ then $\mathbb{V}(I) = \mathbb{V}(\mathfrak{p})$, so we have that $\sqrt{I} = \mathfrak{p}$, but $I \subset \sqrt{I}$ so $I \subset \mathfrak{p}$, implying that $I = \mathfrak{p}$. It follows that points which are closed in the subspace topology of U are precisely the maximal ideals of A.

Now the stalk at x is the localization $A_{\mathfrak{p}}$, and the residue field k_x is given by:

$$k_x = A_{\mathfrak{p}}/\mathfrak{m}_x$$

where:

$$\mathfrak{m}_x = \left\{\frac{p}{a} : p \in \mathfrak{p}\right\}$$

We claim that[32]:

$$A_{\mathfrak{p}}/\mathfrak{m}_x \cong A/\mathfrak{p}$$

Note that we have map $A \to A_{\mathfrak{p}}/\mathfrak{m}_x$ by combining the localization morphism with the projection morphism to the quotient. If $p \in \mathfrak{p}$, then $p/1 \in \mathfrak{m}_x$ so this map factors through the quotient hence we have a unique homomorphism:

$$\begin{aligned} \psi : A/\mathfrak{p} &\longrightarrow A_{\mathfrak{p}}/\mathfrak{m}_x \\ [a] &\longmapsto [a/1] \end{aligned}$$

We claim this map is an isomorphism; indeed $A/\mathfrak{p}$ is a field, so if $[a] \mapsto 0$ then $[a]$ is not invertible and thus must be the zero element. Now suppose that $[a/b] \in A_{\mathfrak{p}}/\mathfrak{m}_x$, then since $A/\mathfrak{p}$ is a field, there must be an element $h \in A$ such that $b \cdot h - 1 \in \mathfrak{p}$. We claim that $\psi([ah]) = [a/b]$; indeed note that:

$$\frac{ah}{1} - \frac{a}{b} = \frac{a(hb-1)}{b}$$

but $hb - 1 \in \mathfrak{p}$, so this element lies in $\mathfrak{m}_x$ and thus $[ah/1] - [a/b] = 0$ and ψ is an isomorphism.

It follows that k_x is a field extension of k, and is a finitely generated k algebra, so by Zariski's lemma [33] is a finite field extension of k. Now let $V = \operatorname{Spec} B$ be another open affine containing x, and suppose that $\mathfrak{q} \subset B$ is the prime ideal associated to x. We have that $B_{\mathfrak{q}}/\mathfrak{m}'_x \cong k_x$, and we now want to show that $B/\mathfrak{q}$ is a field. First note that there is a morphism $B \to B_{\mathfrak{q}}/\mathfrak{m}'_x$ which again sends any element in $\mathfrak{q}$ to zero, so we have a unique morphism $B/\mathfrak{q} \to B_{\mathfrak{q}}/\mathfrak{m}'_x$. This morphism is injective as if $[a] \mapsto 0$, then this implies that $a/1 \in \mathfrak{m}'_x$, but for this to be true a must lie in $\mathfrak{q}$. It follows that we can identify $B/\mathfrak{q}$ as (a priori) a sub k algebra of $B_{\mathfrak{q}}/\mathfrak{m}_x$, which is a finite dimensional k-vector space, so $B/\mathfrak{q}$ must also be a finite dimensional k-vector space. However, $B/\mathfrak{q}$ is prime so $B/\mathfrak{q}$ is an integral domain and the linear map of k vector spaces:

$$\begin{aligned} M_{[b]} : B/\mathfrak{q} &\longrightarrow B/\mathfrak{q} \\ [a] &\longmapsto [a] \cdot [b] \end{aligned}$$

is thus injective for all nonzero $[b] \in B$. Indeed, if $[a] \cdot [b] = 0$, then $[a] = 0$ so the map is injective. By rank nullity the map is an isomorphism, so there must exist an $[a]$ such that $[b] \cdot [a] = 1$ implying that $B/\mathfrak{q}$ is a field, so $\mathfrak{q}$ is a maximal ideal.

We have thus shown that if $x \in U$ is closed in the subspace topology, then x corresponds to a maximal ideal in every affine open containing x, and is thus closed in every such open affine. Now let $\{U_i\}$ be an open affine cover of X, then:

$$X \setminus \{x\} = \bigcup_i (U_i \setminus \{x\})$$

We see that $U_i \setminus \{x\}$ is open in X for all i, as either $U_i \setminus \{x\}$ is U_i since $x \notin U_i$, or $U_i \setminus \{x\}$ is open in U_i as $\{x\}$ is closed in U_i, so it is open in X. It follows that $X \setminus \{x\}$ is open, so $\{x\}$ is a closed point. If

[32] This is only true as we are supposing that $\mathfrak{p}$ is a maximal ideal!

[33] See Theorem 6.1.3

$x \in X$ corresponds to a maximal ideal $\mathfrak{p} \in U = \operatorname{Spec} A$, then x is closed. Conversely, if x is closed, and $U = \operatorname{Spec} A$. is an open affine of X containing x, then we have that:

$$U \smallsetminus \{x\} = U \cap (X \smallsetminus \{x\})$$

so $U \smallsetminus \{x\}$ is open in the subspace topology, implying that $\{x\}$ is closed in the subspace topology and thus corresponds to a maximal ideal of A, as desired.

The second claim is now clear, because every closed point of x is a closed point of every affine open, and vice versa. $\square$

We now turn to the main result:

Theorem 2.3.3. *Let X and Y be schemes locally of finite type over k with k algebraically closed. Then there exists a bijection:*

$$\begin{aligned} \phi : |X \times_k Y| &\longrightarrow |X| \times |Y| \\ z &\longmapsto (\pi_X(z), \pi_Y(z)) \end{aligned} \tag{2.3.3}$$

Proof. Let $\{U_i = \operatorname{Spec} A_i\}$ and $\{V_i = \operatorname{Spec} B_i\}$ be affine open covers of X and Y respectively. We then have that $\{U_i \times_k V_j = \operatorname{Spec} A_i \otimes_k B_i\}$ is an affine open cover of $X \times_k Y$. We see that each $A_i \otimes_k B_j$ is a a finite generated k-algebra. We first determine a bijection

$$|U_i \times_k V_j| \longleftrightarrow |U_i| \times |V_j|$$

for all i and j. We suppress the the indices going forward. The projection map π_X is locally induced by the inclusion $\iota_A : A \to A \otimes_k B$. Let $\mathfrak{m} \subset A \otimes_k B$ be a maximal ideal, then we have a morphism $\psi : A \to A \otimes_k B/\mathfrak{m}$ by composing with the projection onto the quotient. We have that $A \otimes_k B/\mathfrak{m}$ is a field, and a finitely generated k algebra so $A \otimes_k B/\mathfrak{m}$ is a finite field extension of k by Zariski's lemma. Note that if $a \in \ker \psi$, then we have that $\iota_A(a) \in \mathfrak{m}$, so $a \in \iota_A^{-1}(\mathfrak{m})$, and if $a \in \iota_A^{-1}(\mathfrak{m})$ then $\psi(a) = 0$, so $\ker \psi = \iota_A^{-1}(\mathfrak{m})$. We thus have an injective morphism $\psi' : A/\iota_A^{-1}(\mathfrak{m}) \to A \otimes_k B/\mathfrak{m}$, which is an isomorphism onto its image. We want to show that $\psi'(A/\iota_A^{-1}(\mathfrak{m}))$ is a subfield of $A \otimes_k B$. However, since $\mathfrak{m}$ is maximal, we have that $\iota_A^{-1}(\mathfrak{m})$ is prime so $\psi'(A/\iota_A^{-1}(\mathfrak{m}))$ is an integral domain. It follows that $A/\iota_A^{-1}(\mathfrak{m})$ is an integral domain. The argument in Lemma 2.3.12 then demonstrates that since $A \otimes_k B/\mathfrak{m}$ is a finite field extension of k, and $A/\iota_A^{-1}(\mathfrak{m})$ is a finite k-algebra as well as an integral domain, that $A/\iota_A^{-1}(\mathfrak{m})$ must be a field. Therefore, the morphisms π_X and π_Y take closed points to closed points.

We thus define our morphism $\phi : |U \times_k V| \longrightarrow |U| \times |V|$ by (2.8) restricted to $U \times_k V$. We define an inverse map by taking the pair $(\mathfrak{m}, \mathfrak{n}) \in |U| \times |V|$ and mapping it to the ideal $I = \langle \iota_A(\mathfrak{m}), \iota_B(\mathfrak{n}) \rangle$. Now we claim that $A \otimes_k B/I$ is a field; indeed we have the following canonical isomorphism:

$$A \otimes_k B/I \cong A/\mathfrak{m} \otimes_k B/\mathfrak{n}$$

which is a finitely generated k algebra, and is finite as both $A/\mathfrak{m}$ and $B/\mathfrak{n}$ are finite field extensions of k. Since k is algebraically closed both fields are isomorphic to k as the only finite field extension of an algebraically closed field is k. We check that this is indeed an inverse, let $(\mathfrak{m}, \mathfrak{n}) \in |U| \times |V|$, then we have that $\phi \circ \phi^{-1}(\mathfrak{m}, \mathfrak{n}) = (\iota_A^{-1}(I), \iota_B^{-1}(I))$. We see that by definition $\iota_A(\mathfrak{m}) \subset I$, hence $\iota_A^{-1}(\iota_A(\mathfrak{m})) \subset \iota_A^{-1}(I)$, but $\mathfrak{m} \subset \iota_A^{-1}(\iota_A(\mathfrak{m}))$, so $\mathfrak{m} \subset \iota_A^{-1}(I)$ implying that $\mathfrak{m} = \iota_A^{-1}(I)$ as $\mathfrak{m}$ is maximal. The same argument holds for $\mathfrak{n}$, so we have that $\phi \circ \phi^{-1} = \mathrm{Id}$. Now suppose that $\mathfrak{m} \subset A \otimes_k B$, then $\mathfrak{m}$ is the kernel of a morphism $\psi : A \otimes_k B \to k$, and such a morphism induces morphisms $\psi_A : A \to k$ and $\psi_B : B \to k$ such that the following diagram commutes:

$$\begin{tikzcd}
k & & \\
& A \otimes_k B \arrow[ul, "\psi"] & B \arrow[l, "\iota_B"'] \arrow[ull, bend right, "\psi_B"'] \\
& A \arrow[u, "\iota_A"] \arrow[uul, bend left, "\psi_A"] & k \arrow[l] \arrow[u]
\end{tikzcd}$$

If $a \in \iota_A^{-1}(\mathfrak{m})$, then $a \in \ker(\iota_A \circ \psi) = \ker \psi_A$, so we have that $\iota_A^{-1}(\mathfrak{m}) = \ker \psi_A$, and similarly that $\iota_B^{-1}(\mathfrak{m}) = \ker \psi_B$. It therefore suffices to show that $\ker \psi = \langle \iota_A(\ker \psi_A), \iota_B(\ker \psi_B) \rangle$. Note that by the same argument we know that $\langle \iota_A(\ker \psi_A), \iota_B \ker \psi_B \rangle$ is maximal, so let $\omega \in \langle \iota_A(\ker \psi_A), \iota_B \ker \psi_B \rangle$, then we see that $\omega = \beta \cdot \iota_A(a) + \xi \cdot \iota_B(b)$ for some $a \in \ker \psi_A$, $b \in \ker \psi_B$, and some $\beta, \xi \in A \otimes_k B$. Clearly $\psi(\omega) = 0$, so we have that $\langle \iota_A(\ker \psi_A), \iota_B(\ker \psi_B) \rangle \subset \ker \psi$. We thus have that

$$\mathfrak{m} = \ker \psi = \langle \iota_A(\ker \psi_A), \iota_B(\ker \psi_B) \rangle = \left\langle \iota_A(\iota_A^{-1}(\mathfrak{m})), \iota_B(\iota_B^{-1}(\mathfrak{m})) \right\rangle$$

so $\phi^{-1} \circ \phi = \mathrm{Id}$.

Now by the preceding lemma we have that:

$$|X \times_k Y| = \bigcup_{ij} |U_i \times_k V_j| \qquad \text{and} \qquad |X| \times |Y| = \bigcup_{ij} |U_i| \times |V_j|$$

Since our projection maps agree on all overlapping open sets, they must agree on overlapping closed points, hence the local bijection induced by the inclusion homomorphisms described above also agrees on overlapping closed points. It follows that the bijections $|U_i \times_k V_j| \to |U_i| \times |V_j|$ glue together to yield the desired set bijection, implying the claim. $\square$

We will use fibre products in the following section when we further discuss the topological and algebraic properties of schemes and their morphisms. For now, we end with the following examples:

Example 2.3.1. We claim that $\mathbb{P}^n_{\mathbb{C}} \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{C}$, where here the fibre product is taken over $\operatorname{Spec} \mathbb{Z}$. Note that we have a morphism $g : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Z}$ induced by the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{C}$, and a morphism $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ induced locally by the inclusion map:

$$\mathbb{Z} \hookrightarrow \mathbb{Z}[\{x_l/x_i\}_{l \neq i}]$$

Indeed, for each i, the above morphism of rings induces morphisms of affine schemes $f_i : U_{x_i} \subset \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$. We have that

$$U_{x_i} \cap U_{x_j} = U_{x_i x_j} \cong \operatorname{Spec} \mathbb{Z}[\{x_l/x_i\}_{l \neq i}, x_i/x_j]$$

It follows that the morphisms $f_i|_{U_{x_i x_j}} : U_{x_i x_j} \to \operatorname{Spec} \mathbb{Z}$ and $f_j|_{U_{x_i x_j}} : U_{x_i x_j} \to \operatorname{Spec} \mathbb{Z}$ are induced by the inclusion map:

$$\mathbb{Z} \hookrightarrow \mathbb{Z}[\{x_l/x_i\}_{l \neq i}, x_i/x_j]$$

so they trivially agree. It follows that we have a morphism $f : \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$. Now we wish to define morphisms $p_Y : \mathbb{P}^n_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$, and $p_X : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{Z}}$. We define the first morphisms as we did in the case of $\mathbb{P}^n_{\mathbb{Z}}$, and we define the second morphism by first defining ring morphisms:

$$\mathbb{Z}[\{x_l/x_i\}_{l \neq i}] \hookrightarrow \mathbb{C}[\{x_l/x_i\}_{l \neq i}]$$

induced by the map $\mathbb{Z} \hookrightarrow \mathbb{C}$, and then noting that these give scheme morphisms $U_{x_i} \subset \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{Z}}$ which have image contained in $U_{x_i} \subset \mathbb{P}^n_{\mathbb{Z}}$. These scheme morphisms then trivially agree on overlaps so we have a morphism $p_X : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{Z}}$. We claim that:

$$f \circ p_X = g \circ p_Y$$

and it suffices to check this on affine opens. Indeed, if we restrict to $U_{x_i} \subset \mathbb{P}^n_{\mathbb{C}}$, then these are morphisms of affine schemes, so it suffices to check that the corresponding ring morphisms agree. We see that the first ring homomorphism is given by:

$$\begin{aligned} (f \circ p_X)|^\sharp_{U_{x_i}} : \mathbb{Z} &\longrightarrow \mathbb{C}[\{x_l/x_i\}_{l \neq i}] \\ z &\longmapsto z \end{aligned}$$

while the second is given by:

$$\begin{aligned} (g \circ p_X)|^\sharp_{U_{x_i}} : \mathbb{Z} &\longrightarrow \mathbb{C}[\{x_l/x_i\}_{l \neq i}] \\ z &\longmapsto z \end{aligned}$$

so the two agree. There is thus a unique morphism $\phi : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{C}$, which we wish to check is an isomorphism. We have the following diagram:

$$\begin{tikzcd}
\mathbb{P}^n_{\mathbb{C}} \arrow[dr, "\phi"] \arrow[drrr, bend left, "p_Y"] \arrow[ddr, bend right, "p_X"'] & & & \\
& \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{C} \arrow[rr, "\pi_Y"] \arrow[d, "\pi_X"] & & \operatorname{Spec} \mathbb{C} \arrow[d, "g"] \\
& \mathbb{P}^n_{\mathbb{Z}} \arrow[rr, "f"] & & \mathbb{Z}
\end{tikzcd}$$

We see that $\pi_X \circ \phi|_{U_{x_i}} = p_X|_{U_{x_i}}$, and the $p_X|_{U_{x_i}}$ has image in $U_{x_i} \subset \mathbb{P}^n_{\mathbb{Z}}$, so $\phi|_{U_{x_i}}$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd}
U_{x_i} \arrow[dr, "\phi"] \arrow[drrr, bend left, "p_Y"] \arrow[ddr, bend right, "p_X|_{U_{x_i}}"'] & & & \\
& U_{x_i} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{C} \arrow[rr, "\pi_Y"] \arrow[d, "\pi_{U_{x_i}}"] & & \operatorname{Spec} \mathbb{C} \arrow[d, "g"] \\
& U_{x_i} \arrow[rr, "f"] & & \mathbb{Z}
\end{tikzcd}$$

where we have identified $U_{x_i} \times_Z \operatorname{Spec} \mathbb{C}$ with the open subset of $\mathbb{P}^n_{\mathbb{Z}} \times \operatorname{Spec} \mathbb{C}$ which satisfies the same universal property[34]. Since all these schemes are affine, we now go to the ring picture, and see that we have the following diagram:

$$\begin{tikzcd}
\mathbb{C}[\{x_l/x_i\}_{l \neq i}] & & & \\
& \mathbb{Z}[\{x_l/x_i\}_{l \neq j}] \otimes_{\mathbb{Z}} \mathbb{C} \arrow[ul, "\phi|^\sharp_{U_{x_i}}"'] & & \mathbb{C} \arrow[ll, "\pi_Y^\sharp"'] \arrow[ulll, bend right, "p_Y^\sharp"'] \\
& \mathbb{Z}[\{x_l/x_i\}_{l \neq j}] \arrow[u, "\pi_X|^\sharp_{U_{x_i}}"] \arrow[uul, bend left, "p_X|^\sharp_{U_{x_i}}"] & & \mathbb{Z} \arrow[ll] \arrow[u]
\end{tikzcd}$$

We note that these projections must be given by $p \mapsto p \otimes 1$, and $w \mapsto 1 \otimes m$, so the map $h : p \otimes w \mapsto p \cdot w$ makes the diagram commute. It follows that $\phi|^\sharp_{U_{x_i}} = h$, so $\phi|^\sharp_{U_{x_i}}$ is an isomorphism, implying that $\phi|_{U_{x_i}}$ is an isomorphism. Since $\phi|_{U_{x_i}}$ is an isomorphism for all x_i, we have that:

$$\phi(\mathbb{P}^n_{\mathbb{C}}) = \bigcup_{i=0}^{n} \phi(U_{x_i}) = \bigcup_{i=0}^{n} U_{x_i} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{C} = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{C}$$

so ϕ is surjective, and is clearly injective. Moreover, we see that if $U \subset \mathbb{P}^n_{\mathbb{C}}$ is any open set, then we can write:

$$\phi(U) = \bigcup_{i=0}^{n} \phi(U \cap U_{x_i})$$

which is a finite union of open sets, so ϕ is a bijective open continuos map implying that ϕ is a homeomorphism. Moreover, the map $\phi^\sharp : \mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{C}} \to \phi_* \mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}$ restricts to isomorphisms $\phi^\sharp|_{U_{x_i}} : \mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{C}}|_{U_{x_i}} \to$

[34] All is well because this how we explicitly constructed the fibre product!

$(\phi_*\mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}})|_{U_{x_i}}$ as:

$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}\times_{\mathbb{Z}}\operatorname{Spec}\mathbb{C}}(U_{x_i}) = \mathbb{Z}[\{x_l/x_i\}_{l\neq i}]\otimes_{\mathbb{Z}}\mathbb{C} \qquad \text{and} \qquad (\phi_*\mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}})(U_{x_i}) = \mathbb{C}[\{x_l/x_i\}]$$

and $\phi^\sharp|_{U_{x_i}}$ is then given by the isomorphism h. By Corollary 1.2.4, it follows that $\phi^\sharp$ is indeed an isomorphism of sheaves, so $(\phi,\phi^\sharp)$ is an isomorphism of schemes as desired, implying the claim.

Though we have proved this in the case of $\mathbb{C}$ and $\mathbb{Z}$, the same proof shows that $\mathbb{P}^n_B \cong \mathbb{P}^n_A \times_A \operatorname{Spec} B$, whenever B is an A algebra.

Example 2.3.2. Let A be any commutative ring, and I and J be ideals of A. We then claim that:

$$\mathbb{V}(I)\times_A \mathbb{V}(J) \cong \operatorname{Spec}(A/\langle I+J\rangle)$$

where $\mathbb{V}(I)$ and $\mathbb{V}(J)$ have the natural induced reduced subscheme structure. However, this follows from the easily verifiable fact that:

$$A/I\otimes_A A/J \cong A/\langle I+J\rangle$$

Moreover, since the scheme $\operatorname{Spec}(A/\langle I+j\rangle)$ is isomorphic to $\mathbb{V}(I+J)$, we have that:

$$\mathbb{V}(I)\times_A\mathbb{V}(J)\cong\mathbb{V}(I+J)=\mathbb{V}(I)\cap\mathbb{V}(J)$$

In particular, if X and Y are closed subsets of Z equipped with induced reduced subscheme structure, we have that:

$$X\times_Z Y\cong X\cap Y$$

where $X\cap Y$ is equipped with the induced reduced subscheme structure.

Example 2.3.3. Recall from Example 2.2.3 where we showed that $|X=\operatorname{Proj}\mathbb{C}[x,y,z]|\cong|\mathbb{A}^1_{\mathbb{C}}|\times|\mathbb{P}^1_{\mathbb{C}}|$ when $\mathbb{C}[x,y,z]$ is equipped the grading induced by $\deg x=0$, and $\deg y=\deg z=1$. We now claim that as schemes $X\cong\mathbb{A}^1_{\mathbb{C}}\times_{\mathbb{C}}\mathbb{P}^1_{\mathbb{C}}$. Let U_y and U_z be the distinguished open sets of X, and ξ_{zy} the ring isomorphism $\mathbb{C}[x,y/z,z/y]\to\mathbb{C}[x,z/y,y/z]$ sending $x\mapsto x$, $y/z\mapsto z/y$, which induces the gluing isomorphism along $U_x\cap U_y$. Then we have ring homomorphisms:

$$\begin{aligned}\iota_{yx}:\mathbb{C}[x]&\longrightarrow(\mathbb{C}[x,y,z]_y)_0\cong\mathbb{C}[x,z/y]\\ x&\longmapsto x\end{aligned}$$

and similarly a ring homomorphism ι_{zx} for $\mathbb{C}[x,y/z]$ which clearly satisfies $\xi_{zy}\circ\iota'_{yx}=\iota'_{zx}$, where the primed morphisms are the ones composed with the inclusions $\mathbb{C}[x,z/y],\mathbb{C}[x,y/z]\to\mathbb{C}[x,y/z,z/y]$. It follows that the induced scheme morphisms agree on U_{xy} so we get a unique morphism $p_{\mathbb{A}^1_{\mathbb{C}}}:X\to\mathbb{A}^1_{\mathbb{C}}$. Now we set $\mathbb{P}^1_{\mathbb{C}}=\operatorname{Proj}\mathbb{C}[u,v]$ with the standard grading, and note that the ring homomorphism:

$$\begin{aligned}\iota_{yu}:\mathbb{C}[v/u]&\longrightarrow\mathbb{C}[x,z/y]\\ v/u&\longmapsto z/y\end{aligned}$$

induces a morphism of affine schemes:

$$p_{yu}:U_y\longrightarrow U_u\subset\mathbb{P}^1_{\mathbb{C}}$$

Similarly, the morphism $\iota_{zv}:\mathbb{C}[u/v]\to\mathbb{C}[x,y/z]$ given by $u/v\mapsto y/z$ gives a morphism of affine schemes $p_{zv}:U_z\to U_v\subset\mathbb{P}^1_{\mathbb{C}}$. We see that on the overlap $p_{yu}|_{U_{zy}}$ and $p_{zv}|_{U_{zy}}$ are induced by the ring homomorphisms:

$$v/u\in\mathbb{C}[v/u,u/v]\longmapsto z/y\in\mathbb{C}[x,z/y,y/z]\text{ and }u/v\in\mathbb{C}[u/v,v/u]\longmapsto y/z\in\mathbb{C}[x,z/y,y/z]$$

Clearly we have that $\xi_{zy}(z/y)=y/z$ so we have that $\xi_{zy}\circ\iota_{yu}=\iota_{vu}$, so $p_{yu}|_{U_{zy}}=p_{zv}|_{U_{zy}}$ implying that the morphisms glue together to yield our second map $p_{\mathbb{P}^1_{\mathbb{C}}}:X\to\mathbb{P}^1_{\mathbb{C}}$. If f and g are the morphisms making $\mathbb{A}^1_{\mathbb{C}}$ and $\mathbb{P}^1_{\mathbb{C}}$ $\mathbb{C}$-schemes[35], then we clearly have that $f\circ p_{\mathbb{A}^1_{\mathbb{C}}}=g\circ p_{\mathbb{P}^1_{\mathbb{C}}}$, hence there is a unique

[35] The constructions are essentially the same as in Example 2.3.2

morphism of schemes making the following diagram commute:

$$\begin{tikzcd}
X \arrow[dr, "\psi"] \arrow[drr, bend left, "p_{\mathbb{P}^1_{\mathbb{C}}}"] \arrow[ddr, bend right, "p_{\mathbb{A}^1_{\mathbb{C}}}"'] & & \\
& \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \arrow[r, "\pi_{\mathbb{P}^1_{\mathbb{C}}}"] \arrow[d, "\pi_{\mathbb{A}^1_{\mathbb{C}}}"] & \mathbb{P}^1_{\mathbb{C}} \arrow[d, "g"] \\
& \mathbb{A}^1_{\mathbb{C}} \arrow[r, "f"] & \mathbb{C}
\end{tikzcd}$$

We claim this morphism is an isomorphism. Indeed, we have that $\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} U_u$ and $\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} U_v$, cover $\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$, and that U_z and U_y cover X. We see that by the constructions of the maps $p_{\mathbb{P}^1_{\mathbb{C}}}$ and $p_{\mathbb{A}^1_{\mathbb{C}}}$ that $\psi|_{U_y}$ must make the following the diagram commute:

$$\begin{tikzcd}
U_y \arrow[dr, "\psi|_{U_u}"] \arrow[drr, bend left, "p_{\mathbb{P}^1_{\mathbb{C}}}|_{U_u}"] \arrow[ddr, bend right, "p_{\mathbb{A}^1_{\mathbb{C}}}"'] & & \\
& \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} U_u \arrow[r, "\pi_{U_u}"] \arrow[d, "\pi_{\mathbb{A}^1_{\mathbb{C}}}"] & U_u \arrow[d, "g"] \\
& \mathbb{A}^1_{\mathbb{C}} \arrow[r, "f"] & \mathbb{C}
\end{tikzcd}$$

so we have the following commutative diagram in the category of rings:

$$\begin{tikzcd}
\mathbb{C}[x, y/z] & & \\
& \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[u/v] \arrow[ul, "\psi|^{\sharp}_{U_u}"'] & \mathbb{C}[u/v] \arrow[l] \arrow[ull, bend right, "\iota_{yu}"'] \\
& \mathbb{C}[x] \arrow[u] \arrow[uul, bend left, "\iota_{yx}"] & \mathbb{C} \arrow[l] \arrow[u]
\end{tikzcd}$$

But the isomorphism $x \otimes (u/v) \mapsto x \cdot (u/v)$ makes this diagram commute so $\psi|_{U_y} : U_y \to \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} U_u \subset \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ is an isomorphism, and similarly for $\psi|_{U_z}$. Since $\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_C = (\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} U_u) \cup (\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} U_v)$ it follows that ψ itself is an isomorphism, implying the claim.

Example 2.3.4. We claim that there exists a morphism:

$$\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^3_{\mathbb{C}}$$

which on closed points satisfies:

$$([w_0, w_1], [z_0, z_1]) \longmapsto [w_0 z_0, w_1 z_0, w_0 z_1, w_1 z_1]$$

Set the first copy of $\mathbb{P}^1_{\mathbb{C}}$ to $\operatorname{Proj} \mathbb{C}[x_0, x_1]$ and the second to be $\operatorname{Proj} \mathbb{C}[y_0, y_1]$, also set $\mathbb{P}^3_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[v_0, v_1, v_2, v_3]$. Now we have have an affine open cover of $\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ given by $\{U_{x_i} \times_{\mathbb{C}} U_{y_j}\}_{ij}$, meanwhile $\mathbb{P}^3_{\mathbb{C}}$ is covered by $\{U_{v_k}\}_k$. We can write $[w_0, w_1]$ as the homogenous prime ideal:

$$[w_0, w_1] = \langle x_0 w_1 - x_1 w_0 \rangle \subset \mathbb{C}[x_0, x_1]$$

If $[w_0, w_1] \in U_{x_0}$, then this corresponds to the prime ideal:

$$[w_0, w_1] = \langle x_1/x_0 - w_1/w_0 \rangle \subset \mathbb{C}[x_1/x_0]$$

and similarly if $[z_0, z_1] \in U_{y_0}$, then:

$$[z_0, z_1] = \langle y_1/y_0 - z_1/z_0 \rangle \subset \mathbb{C}[y_1/y_0]$$

We can thus rewrite $[w_0, w_1]$ and $[z_0, z_1]$ as $[1, w_1/w_0]$ and $[1, z_1/z_0]$. Our desired morphism will then send these two pairs of points to $[1, w_1/z_0, z_1/z_0, w_1z_1/w_0z_0]$ which lies in U_{v_0}. We thus need a morphism of affine schemes $U_{x_0} \times_{\mathbb{C}} U_{y_0} \to U_{v_0}$ which satisfies:

$$I = \left\langle \frac{x_1}{x_0} - \frac{w_1}{w_0}, \frac{y_1}{y_0} - \frac{z_1}{z_0} \right\rangle \longmapsto \left\langle \frac{v_1}{v_0} - \frac{w_1}{w_0}, \frac{v_2}{v_0} - \frac{z_1}{z_0}, \frac{v_3}{v_0} - \frac{w_1z_1}{w_0w_1} \right\rangle$$

and we claim this is given by the ring homomorphism:

$$\begin{aligned} \phi_0 : \mathbb{C}[v_1/v_0, v_2/v_0, v_3/v_0] &\longrightarrow \mathbb{C}[x_1/x_0, y_1/y_0] \\ v_i/v_0 &\longmapsto \begin{cases} x_1/x_0 & \text{if } i = 1 \\ y_1/y_0 & \text{if } i = 2 \\ (x_1/x_0) \cdot (y_1/y_0) & \text{if } i = 3 \end{cases} \end{aligned}$$

It is then clear that:

$$\left\langle \frac{v_1}{v_0} - \frac{w_1}{w_0}, \frac{v_2}{v_0} - \frac{z_1}{z_0}, \frac{v_3}{v_0} - \frac{w_1z_1}{w_0w_1} \right\rangle \subset \phi_0^{-1}\left(\left\langle \frac{x_1}{x_0} - \frac{w_1}{w_0}, \frac{y_1}{y_0} - \frac{z_1}{z_0} \right\rangle \right)$$

as the first two generators of the left hand side trivially map into I, and the third generator satisfies :

$$\begin{aligned} \phi_0(v_3/v_0 - w_1z_1/z_0z_1) =& (x_1/x_0) \cdot (y_1/y_0) - w_1z_1/z_0z_1 \\ =& y_1/y_0\,(x_1/x_0 - w_1/w_0) + w_1/w_0\,(y_1/y_0 - z_1/z_0) \end{aligned}$$

Since the left hand ideal is maximal, we have equality, and thus our ring homomorphism ϕ_{00} induce scheme morphisms which satisfy the desired property on closed points. By the same logic we define $\phi_i : \mathbb{C}[\{v_j/v_i\}_{j \neq i}] \to \mathbb{C}[x_k/x_l, y_m/y_n]$ where we have that:

$$(k, l, m, n) = \begin{cases} (1, 0, 1, 0) & \text{if } i = 0 \\ (0, 1, 1, 0) & \text{if } i = 1 \\ (1, 0, 0, 1) & \text{if } i = 2 \\ (0, 1, 0, 1) & \text{if } i = 3 \end{cases}$$

by:

$$\begin{aligned} \phi_1 : \mathbb{C}[v_0/v_1, v_2/v_1, v_3/v_1] &\longrightarrow \mathbb{C}[x_0/x_1, y_1/y_0] \\ v_i/v_1 &\longmapsto \begin{cases} x_0/x_1 & \text{if } i = 0 \\ (x_0/x_1) \cdot (y_1/y_0) & \text{if } i = 2 \\ y_1/y_0 & \text{if } i = 3 \end{cases} \\ \phi_2 : \mathbb{C}[v_0/v_2, v_1/v_2, v_3/v_2] &\longrightarrow \mathbb{C}[x_1/x_0, y_0/y_1] \\ v_i/v_2 &\longmapsto \begin{cases} y_0/y_1 & \text{if } i = 0 \\ (x_1/x_0) \cdot (y_0/y_1) & \text{if } i = 1 \\ x_1/x_0 & \text{if } i = 3 \end{cases} \\ \phi_3 : \mathbb{C}[v_0/v_3, v_1/v_3, v_2/v_3] &\longrightarrow \mathbb{C}[x_0/x_1, y_0/y_1] \\ v_i/v_3 &\longmapsto \begin{cases} (x_0/x_1) \cdot (y_0/y_1) & \text{if } i = 0 \\ y_0/y_1 & \text{if } i = 1 \\ x_0/x_1 & \text{if } i = 2 \end{cases} \end{aligned}$$

which then induce the morphisms:

$$\psi_i : U_{x_l} \times_{\mathbb{C}} U_{y_n} \longrightarrow U_{v_i}$$

We will show that these maps glue together in the specific case of $U_{x_0} \times_{\mathbb{C}} U_{y_1} \cap U_{x_1} \times_{\mathbb{C}} U_{y_0} \cong U_{x_0x_1} \times_{\mathbb{C}} U_{y_0y_1}$ which is isomorphic to the affine scheme $X = \operatorname{Spec} \mathbb{C}[x_1/x_0, x_0/x_1, y_1/y_0, y_0/y_1]$. When identifying these affine open subsets with this affine scheme, we see that the isomorphism gluing $U_{x_0} \times_{\mathbb{C}} U_{y_1}$ with $U_{x_1} \times_{\mathbb{C}} U_{y_0}$ along $U_{x_0 x_1} \times_{\mathbb{C}} U_{y_0 y_1}$ is given by the tensors product morphism induced by the gluing U_{x_0} and U_{x_1} along $U_{x_0 x_1}$ and similarly for U_{y_0} and U_{y_1}. It follows that the gluing isomorphism $\xi : X \subset U_{x_1} \times_{\mathbb{C}} U_{y_0} \longrightarrow X \subset U_{x_0} \times_{\mathbb{C}} U_{y_1}$ is induced by the ring automorphism:

$$\xi^\sharp : \mathbb{C}[x_1/x_0, x_0/x_1, y_1/y_0, y_0/y_1] \longrightarrow \mathbb{C}[x_1/x_0, x_0/x_1, y_1/y_0, y_0/y_1]$$

which sends each generator to itself. The morphisms we wish to glue are clearly ψ_1 and ψ_2, and we see that $\psi_1|_X$ and $\psi_2|_X$ now clearly have image in $U_{v_1 v_2}$ which as a subset of U_{v_1} we identify with $\operatorname{Spec}\mathbb{C}[v_0/v_1, v_2/v_1, v_3/v_1, v_1/v_2]$, and as a subset of U_{v_2} we identify of $\operatorname{Spec}\mathbb{C}[v_0/v_2, v_1/v_2, v_3/v_2, v_2/v_1]$. Let $\eta : U_{v_1 v_2} \subset U_{v_1} \to U_{v_1 v_2} \subset U_{v_2}$ be the gluing isomorphism, then to show that these agree, we have to show that:

$$\eta \circ \psi_1|_X = \psi_2|_X$$

so it suffices to show that:

$$(\psi_1|_X)^\sharp \circ \eta^\sharp = (\psi_2|_X)^\sharp$$

Recall that $\eta^\sharp$ is given by:

$$\begin{aligned} \eta^\sharp : \mathbb{C}[v_0/v_2, v_1/v_2, v_3/v_2, v_2/v_1] &\longrightarrow \mathbb{C}[v_0/v_1, v_2/v_1, v_3/v_1, v_1/v_2] \\ v_i/v_j &\longmapsto \begin{cases} (v_i/v_1) \cdot (v_1/v_2) & \text{if } i \neq 2 \text{ and } j = 2 \\ v_1/v_2 & \text{if } i = 1 \text{ and } j = 2 \\ v_2/v_1 & \text{if } i = 2 \text{ and } j = 1 \end{cases} \end{aligned}$$

while the maps $(\psi_1|_X)^\sharp$ and $(\psi_2|_X)^\sharp$ are the maps induced by localization. We now calculate the image of each generator beginning with v_0/v_2:

$$\begin{aligned} (\psi_1|_X)^\sharp \circ \eta^\sharp(v_0/v_2) =& (\psi_1|_X)^\sharp(v_0/v_1 \cdot v_1/v_2) \\ =& x_0/x_1 \cdot x_1/x_0 \cdot y_0/y_1 \\ =& y_0/y_1 \end{aligned}$$

while:

$$(\psi_2|_X)^\sharp(v_0/v_2) = y_0/y_1$$

For the next generator we have that:

$$\begin{aligned} (\psi_1|_X)^\sharp \circ \eta^\sharp(v_1/v_2) =& (\psi_1|_X)^\sharp(v_1/v_2) \\ =& x_1/x_0 \cdot y_0/y_1 \end{aligned}$$

while:

$$(\psi_2|_X)^\sharp(v_1/v_2) = x_1/x_0 \cdot y_0/y_1$$

For v_3/v_2 we have that:

$$\begin{aligned} (\psi_1|_X)^\sharp \circ \eta^\sharp(v_3/v_2) =& \psi_1^\sharp(v_3/v_1 \cdot v_1/v_2) \\ =& y_1/y_0 \cdot x_1/x_0 \cdot y_0/y_1 \\ =& x_1/x_0 \end{aligned}$$

while:

$$(\psi_2|_X)^\sharp(v_3/v_2) = x_2/x_0$$

Finally, we have that:

$$(\psi_1|_X)^\sharp \circ \eta^\sharp(v_2/v_1) = (\psi_1|_X)^\sharp(v_2/v_1) = x_0/x_1 \cdot y_1/y_0$$

while:

$$(\psi_2|_X)^\sharp(v_2/v_1) = x_0/x_1 \cdot y_1/y_0$$

2.4 Some Category Theory: Representable Functors

Over the next two sections we wish to develop an alternative but equivalent view of schemes, which will at times prove more convenient to work with. To do so, we first must take a detour through some abstract nonsense. Recall that a category $\mathscr{C}$ is locally small[36] if the Hom 'sets' are actually sets. We begin with the following lemma/notation:

Lemma 2.4.1. *Let $\mathscr{C}$ be a locally small category, and Y an object. Then there exists a contravariant functor $h_Y : \mathscr{C} \to \mathrm{Set}$ which sends an object X to* Set *via:*

$$X \longmapsto \mathrm{Hom}_{\mathscr{C}}(X, Y)$$

and sends $f \in \mathrm{Hom}_{\mathscr{C}}(X, Z)$ to the morphism:

$$\begin{aligned} h_Y(f) : \mathrm{Hom}_{\mathscr{C}}(Z, Y) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(X, Y) \\ \alpha &\longmapsto f^*\alpha = \alpha \circ f \end{aligned}$$

Proof. This is all essentially obvious, but we spell it out to fix our notation. Clearly if h_Y defines a functor then it is contravariant. Moreover, for $\mathrm{Id} \in \mathrm{Hom}_{\mathscr{C}}(X, X)$, we have that $h_Y(\mathrm{Id})$ is clearly the identity morphism on $\mathrm{Hom}_{\mathscr{C}}(X, X)$. Now let $f \in \mathrm{Hom}_{\mathscr{C}}(X, Z)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Z, W)$, then $g \circ f \in \mathrm{Hom}_{\mathscr{C}}(X, W)$. Let $\alpha \in \mathrm{Hom}_{\mathscr{C}}(W, Y)$, then:

$$h_Y(g \circ f)(\alpha) = (g \circ f)^*\alpha = \alpha \circ (g \circ f) = (g^*\alpha) \circ f = f^*(g^*\alpha) = (h_Y(f) \circ h_Y(g))(\alpha)$$

Since α was arbitrary we have that:

$$h_Y(g \circ f) = h_Y(f) \circ h_Y(g)$$

implying the claim. $\square$

Definition 2.4.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. The **product category** is the category where objects are pairs $(X_{\mathscr{C}}, X_{\mathscr{D}})$, and morphisms are pairs of morphisms $(f_{\mathscr{C}}, f_{\mathscr{D}})$, where $f_C : X_{\mathscr{C}} \to Y_{\mathscr{C}} \in \mathrm{Hom}_{\mathscr{C}}(X_{\mathscr{C}}, Y_{\mathscr{C}})$ and $g : X_{\mathscr{D}} \to Y_{\mathscr{D}} \in \mathrm{Hom}_{\mathscr{D}}(X_{\mathscr{D}}, Y_{\mathscr{D}})$.

One easily checks that the above is a category.

Example 2.4.1. Let $\mathscr{C}$ be a locally small category, and $\mathscr{D} = \mathscr{C}^{\mathrm{op}}$, i.e. the object of $\mathscr{D}$ are the objects of $\mathscr{C}$ but 'morphisms go the other way', so a morphism $X \to Y$ in $\mathscr{D}$ is given by $f \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$. The product category $\mathscr{C} \times \mathscr{C}^{\mathrm{op}}$ is then of interest as we have a contravariant $\mathrm{Hom}(\cdot, \cdot)$ functor given by $(X, Y) \mapsto \mathrm{Hom}(X, Y)$, which sends a morphism $(f, g) : (X, Y) \to (W, Z)$ to the morphism:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(W, Z) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(X, Y) \\ \alpha &\longmapsto g \circ \alpha \circ f \end{aligned}$$

as $g : Y \to Z$ is an element of $\mathrm{Hom}_{\mathscr{C}}(Z, Y)$. One can make this covariant by considering $\mathrm{Hom}(\cdot, \cdot)$ as a functor $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$, then if $(f, g) : (X, Y) \to (W, Z)$, we have that the natural set map is given by:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(W, Z) \\ \alpha &\longmapsto g \circ \alpha \circ f \end{aligned}$$

since in this case $f : X \to W$ is an element of of $\mathrm{Hom}_{\mathscr{C}}(W, X)$. The above is also an example of the fact that any contravariant functor can be viewed as a covariant functor from the opposite category.

Let $\mathscr{C}^{\mathscr{D}}$ denote the category of covariant functors from $\mathscr{C}$ to $\mathscr{D}$, and $\mathscr{C}_{\mathscr{D}}$ the category of contravariant functors from $\mathscr{C}$ to $\mathscr{D}$, where the objects in both are covariant/contravariant functors, and the morphisms are natural transformations. We denote the class[37] of natural formations between covariant/contravariant functors $\mathscr{F}$ and $\mathscr{G}$ by $\mathrm{Nat}(\mathscr{F}, \mathscr{G})$, and note that $\mathrm{Nat}(\cdot, \cdot)$ can be viewed as a covariant, or contravariant functor from a suitable product category to the category of classes.

[36] This is a borderline technicality that we honor here for the sake of being precise. In reality, we will almost never deal with a category which is not locally small. Moreover, its not exactly important that categories are locally small, the most vital results of this section, such as Yoneda's lemma, will still hold, the functor h_A will just have a different target category, namely the category of all classes.

[37] Generally the collection of all natural transformations do not form a set, but a class.

We also have the notion of an evaluation functor. That is given categories $\mathscr{C}$ and $\mathscr{D}$, we have a contravariant functor $\mathrm{ev} : \mathscr{C} \times \mathscr{C}_{\mathscr{D}}^{\mathrm{op}} \to \mathscr{D}$ given on objects by $(Y, \mathscr{F}) \mapsto \mathscr{F}(Y)$. Letting $(f, F) : (Y, \mathscr{F}) \to (X, \mathscr{G})$, we obtain the following the following commutative diagram:

$$\begin{tikzcd} \mathscr{F}(Y) & \mathscr{F}(X) \arrow[l, "\mathscr{F}(f)"'] \\ \mathscr{G}(Y) \arrow[u, "F_Y"] & \mathscr{G}(X) \arrow[l, "\mathscr{G}(f)"'] \arrow[u, "F_X"'] \end{tikzcd}$$

so we send (f, F) to $\mathscr{F}(f) \circ F_X$, or equivalently $F_Y \circ \mathscr{G}(f)$. It is then clear that ev is a contravariant functor $\mathscr{C} \times \mathscr{C}_{\mathscr{D}}^{\mathrm{op}}$ to $\mathscr{D}$.

We also term the following contravariant functor from $\mathscr{Y} : \mathscr{C} \times \mathscr{C}_{\mathrm{Set}}^{\mathrm{op}}$ to Class[38], the category of classes, given on objects by:

$$(Y, \mathscr{F}) \longmapsto \mathrm{Nat}(h_Y, \mathscr{F})$$

If $(f, F) : (Y, \mathscr{F}) \longrightarrow (X, \mathscr{G})$ is a morphism, then note that we have a natural transformation $\tilde{f} : h_Y \to h_X$ defined by:

$$\begin{aligned} \tilde{f}_Z : h_Y(Z) &\longrightarrow h_X(Z) \\ \alpha &\longmapsto f \circ \alpha \end{aligned}$$

It follows that $G \circ \tilde{f}$ is a natural transformation $h_y \to \mathscr{G}$, while F is a natural transformation $\mathscr{G} \to \mathscr{F}$. Hence, we send (f, F) to the morphism:

$$G \in \mathrm{Nat}(h_X, \mathscr{G}) \longmapsto F \circ G \circ \tilde{f} \in \mathrm{Nat}(h_Y, \mathscr{F})$$

We call $\mathscr{Y}$ the *Yoneda functor*[39], and note that by reversing arrows, this can be entirely formulated covariantly. The following famous result is known as Yoneda's lemma:

Lemma 2.4.2. *Let $\mathscr{C}$ be a locally small category, and consider the evaluation functor* $\mathrm{ev} : \mathscr{C} \times \mathscr{C}_{\mathrm{Set}} \longrightarrow \mathrm{Set}$. *There is a natural isomorphism:*

$$\mathscr{Y} \cong \mathrm{ev}$$

In particular, for all $Y \in \mathscr{C}$, and $\mathscr{F} \in \mathscr{C}_{\mathrm{Set}}$, we have that:

$$\mathrm{Nat}(h_Y, \mathscr{F}) \cong \mathscr{F}(Y)$$

Proof. Fixing an object $(Y, \mathscr{F})$, we first determine a morphism:

$$T_{Y,\mathscr{F}} : \mathrm{Nat}(h_Y, \mathscr{F}) \longrightarrow \mathscr{F}(Y)$$

Let G be a natural transformation, then this is the data of a morphism $G_Z : h_Y(Z) \to \mathscr{F}(Z)$ for all objects Z of $\mathscr{C}$ such that if $f : Z \to W$ is a morphism in $\mathscr{C}$ the following diagram commutes:

$$\begin{tikzcd} h_Y(Z) \arrow[r, "G_Z"] & \mathscr{F}(Z) \\ h_Y(W) \arrow[u, "h_Y(f)"] \arrow[r, "G_W"] & \mathscr{F}(W) \arrow[u, "\mathscr{F}(f)"'] \end{tikzcd}$$

In particular, G_Y is a map:

$$\mathrm{Hom}_{\mathscr{C}}(Y, Y) \longrightarrow \mathscr{F}(Y)$$

[38] As we are about to see, this functor will actually have target in Set. We stress again that we do not really care that much about classes, and are simply paying heed for the moment out of necessity.

[39] This is not standard terminology.

so we send $G \mapsto G_Y(\mathrm{Id}_A)$ for all F. We need to check that this is actually a natural transformation, i.e that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Nat}(Y, \mathscr{F}) \arrow[r, "T_{Y,\mathscr{F}}"] & \mathscr{F}(Y) \\ \mathrm{Nat}(X, \mathscr{G}) \arrow[u, "\mathscr{Y}(f,F)"] \arrow[r, "T_{X,\mathscr{G}}"] & \mathscr{G}(X) \arrow[u, "\mathrm{ev}(f,F)"'] \end{tikzcd}$$

Let G be a natural transformation $h_X \to \mathscr{G}$, then we have that:

$$\begin{aligned} \mathrm{ev}(f, F) \circ T_{X,\mathscr{G}}(G) =& \mathrm{ev}(f, F)(G_X(\mathrm{Id}_X)) \\ =& F_Y \circ \mathscr{G}(f) \circ G_X(\mathrm{Id}_X) \end{aligned}$$

However, G is a natural transformation $h_X \to \mathscr{G}$, so the following diagram commutes:

$$\begin{tikzcd} h_X(X) \arrow[r, "G_X"] \arrow[d, "h_X(f)"'] & \mathscr{G}(X) \arrow[d, "\mathscr{G}(f)"] \\ h_X(Y) \arrow[r, "G_Y"] & \mathscr{G}(Y) \end{tikzcd}$$

hence:

$$\mathrm{ev}(f, F) \circ T_{X,\mathscr{G}}(G) = F_Y \circ G_Y \circ h_X(f)(\mathrm{Id}_X)$$

Now, $h_X(f)$ is the morphism:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, X) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(Y, X) \\ \alpha &\longmapsto \alpha \circ f \end{aligned}$$

hence $h_X(f) = f \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$ so:

$$\mathrm{ev}(f, F) \circ T_{X,\mathscr{G}}(G) = F_Y \circ G_Y(f)$$

Similarly, we have that:

$$\begin{aligned} T_{Y,\mathscr{F}} \circ \mathscr{Y}(f, F)(G) =& (F \circ G \circ \tilde{f})_Y(\mathrm{Id}_Y) \\ =& F_Y \circ G_Y \circ \tilde{f}_Y(\mathrm{Id}_Y) \\ =& F_Y \circ G_Y(f) \end{aligned}$$

so the diagram is commutative and T defines a natural transformation $Y \to \mathrm{ev}$.

Now $\mathscr{F}(Y)$ is a set by assumption; let $x \in \mathscr{F}(Y)$, then we want to define a natural transformation $G_x \in \mathrm{Nat}(h_Y, \mathscr{F})$. Let Z be any object in $\mathscr{C}$, and define a morphism:

$$\begin{aligned} (G_x)_Z : h_Y(Z) &\longrightarrow \mathscr{F}(Z) \\ f &\longmapsto \mathscr{F}(f)(x) \end{aligned}$$

as $\mathscr{F}(f) : \mathscr{F}(Y) \to \mathscr{F}(Z)$. We need to show the following diagram commutes:

$$\begin{tikzcd} h_Y(Z) \arrow[r, "(G_x)_Z"] & \mathscr{F}(Z) \\ h_Y(W) \arrow[u, "h_Y(g)"] \arrow[r, "(G_x)_W"] & \mathscr{F}(W) \arrow[u, "\mathscr{F}(g)"'] \end{tikzcd}$$

for any $g : Z \to W$. Let $f \in h_Y(W)$, then $h_Y(g)(f) = f \circ g$, and $(G_x)_Z(f \circ g) = \mathscr{F}(f \circ g)(x)$. Meanwhile, $(G_x)_W(f) = \mathscr{F}(f)(x)$, and $\mathscr{F}(g) \circ \mathscr{F}(f)(x) = \mathscr{F}(f \circ g)(x)$ as $\mathscr{F}$ is contravariant. It follows that G_x determines a natural transformation $h_Y \to \mathscr{F}$. Define $S_{Y,\mathscr{F}}$ by:

$$\begin{aligned} S_{Y,\mathscr{F}} : \mathscr{F}(Y) &\longrightarrow \mathrm{Nat}(h_Y, \mathscr{F}) \\ x &\longmapsto G_x \end{aligned}$$

then we need to show that this determines a natural transformation as well, so once again consider the diagram:

$$\begin{tikzcd} \mathscr{F}(Y) \arrow[r, "S_{Y,\mathscr{F}}"] & \mathrm{Nat}(h_Y, \mathscr{F}) \\ \mathscr{G}(X) \arrow[r, "S_{X,\mathscr{G}}"] \arrow[u, "\mathrm{ev}(f,F)"] & \mathrm{Nat}(h_X, \mathscr{G}) \arrow[u, "\mathscr{Y}(f,F)"'] \end{tikzcd}$$

for all morphisms $(f, F) : (Y, \mathscr{F}) \to (X, \mathscr{G})$. Let $x \in \mathscr{G}(X)$, and set:

$$z = F_Y \circ \mathscr{G}(f)(x) = \mathscr{F}(f) \circ F_X(x)$$

then:

$$S_{Y,\mathscr{F}} \circ \mathrm{ev}(f, F)(x)$$

is the natural transformation G_z. We then need to show the following equality of natural transformations:

$$F \circ G_x \circ \tilde{f} = G_z$$

Let W be any object in $\mathscr{C}$, then $(G_z)_W$ send $g \in h_Y(W)$ to $\mathscr{F}(g)(z)$ so:

$$(G_z)_W(g) = \mathscr{F}(g)(\mathscr{F}(f) \circ F_X(x)) = \mathscr{F}(f \circ g) \circ F_X(x)$$

Meanwhile,

$$\begin{aligned}(F \circ G_x \circ \tilde{f})_W(g) =& F_W \circ G_x(f \circ g) \\ =& F_W \circ \mathscr{G}(f \circ g)(x)\end{aligned}$$

Note that $f \circ g : W \to X$, so by the naturality of F, we have that:

$$(F \circ G_x \circ \tilde{f})_W(g) = \mathscr{F}(f \circ g) \circ F_X(x)$$

Therefore S determines a natural transformation $\mathrm{ev} \to \mathscr{Y}$ as desired.

It remains to show that $S \circ T = \mathrm{Id}_{\mathscr{Y}}$ and $T \circ S = \mathrm{Id}_{\mathrm{ev}}$. We can do this object wise, let $(Y, \mathscr{F})$ be a pair, then:

$$S_{Y,\mathscr{F}} \circ T_{Y,\mathscr{F}} : \mathrm{Nat}(h_Y, \mathscr{F}) \longrightarrow \mathrm{Nat}(h_Y, \mathscr{F})$$

Let $G \in \mathrm{Nat}(h_Y, \mathscr{F})$, then $T_{Y,\mathscr{F}}(G) = G_Y(\mathrm{Id}_Y) \in \mathscr{F}(Y)$. We need to show that the natural transformation corresponding to $x = G_Y(\mathrm{Id}_Y)$, G_x is equal to G. Let W be an object of $\mathscr{C}$, and consider $g \in h_Y(W)$, then:

$$(G_x)_W(g) = \mathscr{F}(g)(x) = \mathscr{F}(g)(G_Y(\mathrm{Id}_Y)) = G_W \circ h_Y(g)(\mathrm{Id}_Y) = G_W(g)$$

It follows that $G_x = G$, hence $S_{Y,\mathscr{F}} \circ T_{Y,\mathscr{F}} = \mathrm{Id}_{\mathscr{Y}}$.

For the other direction we have:

$$T_{Y,\mathscr{F}} \circ S_{Y,\mathscr{F}} : \mathscr{F}(Y) \longrightarrow \mathscr{F}(Y)$$

Taking a point $x \in \mathscr{F}(Y)$, we need to show that $(G_x)_Y(\mathrm{Id}_Y) = x$. However, $(G_x)_Y(\mathrm{Id}_Y) = \mathscr{F}(\mathrm{Id}_Y)(x) = \mathrm{Id}_{\mathscr{F}(Y)}(x) = x$, implying the claim. □

The following corollary, known as the Yoneda embedding, is immediate:

Corollary 2.4.1. *Let X and Y be objects in a locally small category $\mathscr{C}$, then there is a natural bijection:*

$$\mathrm{Nat}(h_Y, h_X) \cong h_X(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X)$$

Note that if we denote by h^Y and h^X the covariant analogues of h_X and h_Y, then almost verbatim the same proof shows that:

$$\mathrm{Nat}(h^Y, \mathscr{F}) \cong \mathscr{F}(Y)$$

where $\mathscr{F}$ is now a covariant functor. In particular,

$$\mathrm{Nat}(h^Y, h^X) \cong \mathrm{Hom}_{\mathscr{C}}(X, Y)$$

Example 2.4.2. We explore the implications of the Yoneda lemma in a concrete algebraic category. Let $\mathscr{C} = \mathrm{Mod}_A$ be the category of A modules, and M and N modules. Then in particular, every natural transformation from $\mathrm{Hom}_{\mathrm{Mod}_A}(\cdot, M)$ to $\mathrm{Hom}_{\mathrm{Mod}_A}(\cdot, N)$ is uniquely determined by an A-module homomorphism $M \to N$.

Definition 2.4.2. Let $\mathscr{F}$ be contravariant functor $\mathscr{C} \to \mathrm{Set}$, then $\mathscr{F}$ is **representable**, if there exists a natural isomorphism $F \cong h_Y$ for some object Y.

Lemma 2.4.3. *Let $\mathscr{F}$ be a representable functor, represented by Y. Then the pair $(Y, F : h_Y \to \mathscr{F})$ is unique up to unique isomorphism.*

Proof. Let $F : h_Y \to \mathscr{F}$ be a natural isomorphism, and suppose that $G : h_X \to \mathscr{F}$ is another natural isomorphism. It follows that $G^{-1} \circ F : h_Y \to h_X$ is a natural isomorphism, and thus corresponds to a unique morphism in $\mathrm{Hom}_{\mathscr{C}}(Y, X)$. This morphism is given by $\alpha = G_Y^{-1} \circ F_Y(\mathrm{Id}_Y)$, and similarly we have a morphism $\beta = F_X^{-1} \circ G_X(\mathrm{Id}_X) \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. We see that we have the following diagrams:

$$\begin{tikzcd}
h_Y(Y) \arrow[r, "F_Y"] & \mathscr{F}(Y) \\
h_Y(X) \arrow[u, "h_Y(\alpha)"] \arrow[r, "F_X"] & \mathscr{F}(X) \arrow[u, "\mathscr{F}(\alpha)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
h_X(Y) \arrow[r, "G_Y"] \arrow[d, "h_X(\beta)"'] & \mathscr{F}(Y) \arrow[d, "\mathscr{F}(\beta)"] \\
h_X(X) \arrow[r, "G_X"] & \mathscr{F}(X)
\end{tikzcd}$$

It follows that for $\beta \in h_Y(X)$, we have that $h_Y(\alpha)(\beta) = \beta \circ \alpha$, and that $F_Y(\beta \circ \alpha) = \mathscr{F}(\alpha) \circ F_X(\beta)$. Since $\beta = F_X^{-1} \circ G_X(\mathrm{Id}_X)$:

$$\begin{aligned}
\mathscr{F}(\alpha) \circ F_X(\beta) = \mathscr{F}(\alpha) \circ G_X(\mathrm{Id}_X) = G_Y \circ h_X(\alpha)(\mathrm{Id}_X) =& G_Y(\alpha) \\
=& G_Y(G_Y^{-1} \circ F_Y(\mathrm{Id}_Y)) \\
=& F_Y(\mathrm{Id}_Y)
\end{aligned}$$

Since F_Y is an isomorphism it follows that $\beta \circ \alpha = \mathrm{Id}_Y$. Similarly, for $\alpha \in h_X(Y)$, we have that $G_X(\alpha \circ \beta) = \mathscr{F}(\beta) \circ G_Y(\alpha)$. The same argument shows that:

$$\begin{aligned}
\mathscr{F}(\beta) \circ G_Y(\alpha) =& \mathscr{F}(\beta) \circ F_Y(\mathrm{Id}_Y) \\
=& F_X \circ h_Y(\beta)(\mathrm{Id}_Y) \\
=& F_X(\beta) \\
=& F_X(F_X^{-1} \circ G_X(\mathrm{Id}_X)) \\
=& G_X(\mathrm{Id}_X)
\end{aligned}$$

Since G is an isomorphism, it follows that $\alpha \circ \beta = \mathrm{Id}_X$, so α and β are unique isomorphisms as desired. ☐

Example 2.4.3. Let Vec be the category of vector spaces over some field k. Consider the functor $D : \mathrm{Vec} \to \mathrm{Vec}$ given by $V \mapsto V^*$, and $A : V \to W$ maps to $A^* : W^* \to V^*$. This is a contravariant functor, is easily seen to be represented by k, essentially by definition.

Consider again the category of A modules Mod_A, and fix an object N. Define a functor by $\mathscr{F} : \mathrm{Mod}_A \to \mathrm{Set}$ by:

$$\mathscr{F}(M) = \{A - \text{bilinear forms on } M \oplus N\}$$

If $\phi : M \to M'$ is a morphism of A-modules, then we define a morphism:

$$\begin{aligned}
\mathscr{F}(\phi) : \mathscr{F}(M') &\longrightarrow \mathscr{F}(M) \\
\omega &\longmapsto \phi^*\omega
\end{aligned}$$

where $\phi^*\omega$ is the form on $M \oplus N$ given by $(\phi^*\omega)(m, n) = \omega(\phi(m), n)$. This clearly defines a contravariant functor, and in particular we claim is represented by $N^* := \mathrm{Hom}_{\mathrm{Mod}_A}(N, A)$. Indeed, define:

$$\begin{aligned}
F_M : \mathscr{F}(M) &\longrightarrow h_{N^*}(M) \\
\beta &\longmapsto f_\beta
\end{aligned}$$

where $f_\beta : M \to N^*$ is the morphism given by $m \mapsto \beta(m, \cdot) \in N^*$. This is clearly A-linear for each M, and given $\phi : M \to M'$ makes the relevant diagram commute, hence the assignment $M \mapsto F_M$ defines a natural transformation.

We define:

$$\begin{aligned} G_M : h_{N^*}(M) &\longrightarrow \mathscr{F}(M) \\ f &\longmapsto \beta_f \end{aligned}$$

by $\beta_f(m, n) = f(m)(n)$, as $f(m) \in N^*$. This is also clearly A-linear, an defines a natural transformation. We see that $F_M \circ G_M$ sends f to f_{β_f}, which is the morphism given by $f_{\beta_f}(m)(n) = \beta_f(m, n) = f(m)(n)$, hence $f_{\beta_f} = f$, so $F \circ G = \mathrm{Id}_{h_{N^*}}$. Similarly $G_M \circ F_M$ sends β to β_{f_β}, which is the bilinear form given by $\beta(m, n) = f_\beta(m)(n) = \beta(m, n)$, so $G \circ F = \mathrm{Id}_{\mathscr{F}}$, implying the claim.

We also have an example of a similar phenomenon happening in the covariant case:

Example 2.4.4. The forgetful functor $\mathscr{F} : \mathrm{Ring} \to \mathrm{Set}$ is represented by $\mathbb{Z}[x]$, by which we mean $h^{\mathbb{Z}[x]} \cong \mathscr{F}$. For any $A \in \mathrm{Ring}$ we construct the following map:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x], A) &\longmapsto \mathscr{F}(A) \\ f &\longmapsto f(x) \end{aligned}$$

which lies in the set A. This is a bijection because $\mathbb{Z}[x]$ is the free object on one generator in Ring, hence each element in $\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x], A)$ is determined precisely by where x is sent. The relevant diagram then obviously commutes implying the natural isomorphism.

We end this section by briefly exploring the notion of universal objects and how this is related to the representability of a functor. We provide no examples of this phenomenon, but this is extremely relevant in the study of moduli spaces.

Definition 2.4.3. Let $\mathscr{F} : \mathbb{C} \to \mathrm{Set}$ be a contravariant functor, then (X, ξ) is a **universal object of** $\mathscr{F}$ if $X \in \mathbb{C}$, $\xi \in \mathscr{F}(X)$, and for all $Y \in \mathbb{C}$, $\alpha \in \mathscr{F}(Y)$ there is a unique morphism $f : Y \to X$ such that $\mathscr{F}(f)(\xi) = \alpha$.

We now have prove the following:

Lemma 2.4.4. *Let $\mathscr{F} : \mathbb{C} \to \mathrm{Set}$ be a contravariant functor, then $\mathscr{F}$ is representable if and only if there exists a universal object (X, ξ) of $\mathscr{F}$.*

Proof. Suppose that (X, ξ) is a universal object of $\mathscr{F}$, then we construct a natural isomorphism:

$$\Psi : h_X \longrightarrow \mathscr{F}$$

on objects via:

$$\begin{aligned} \Psi_Y : \mathrm{Hom}_{\mathbb{C}}(Y, X) &\longrightarrow \mathscr{F}(Y) \\ f &\longmapsto F(f)(\xi) \end{aligned}$$

By definition this is injective and surjective. Take $g : Y \to Z$, and consider the following diagram:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{C}}(Y, X) \arrow[r, "\Psi_Y"] & \mathscr{F}(Y) \\ \mathrm{Hom}(Z, X) \arrow[u, "h_X(g)"] \arrow[r, "\Psi_Z"] & \mathscr{F}(Z) \arrow[u, "\mathscr{F}(g)"'] \end{tikzcd}$$

For any $f \in \mathrm{Hom}_{\mathbb{C}}(Z, X)$, we have that going up and to the right gives:

$$\mathscr{F}(h_X(f))(\xi) = \mathscr{F}(f \circ g)(\Xi) = (\mathscr{F}(g) \circ \mathscr{F}(f))(\xi)$$

which is precisely going right and then up. It follows that $h_X \cong \mathscr{F}$ as desired.

Now suppose that $\mathscr{F} \cong h_X$, and let $\Psi : h_X \to \mathscr{F}$ be the isomorphism. We set $\Psi_X(\mathrm{Id}) = \xi$, and claim that (X, ξ) is the universal object. Let $Y \in \mathbb{C}$, and $\alpha \in \mathscr{F}(Y)$, then $\Psi_Y^{-1}(\alpha) \in \mathrm{Hom}_{\mathbb{C}}(Y, X)$. The following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{C}}(Y,X) \arrow[r, "\Psi_Y"] & \mathscr{F}(Y) \\ \mathrm{Hom}(X,X) \arrow[u, "h_X(\Psi_Y^{-1}(\alpha))"] & \mathscr{F}(X) \arrow[l, "\Psi_X^{-1}"] \arrow[u, "\mathscr{F}(\Psi_Y^{-1}(\alpha))"'] \end{tikzcd}$$

implying that:

$$\mathscr{F}(\Psi_Y^{-1}(\alpha)) = \Psi_Y \circ h_X(\Psi_Y^{-1}(\alpha)) \circ \Psi_X^{-1}$$

hence:

$$\begin{aligned} \mathscr{F}(\Psi_Y^{-1}(\alpha))(\xi) =& \Psi_Y \circ h_X(\Psi_Y^{-1}(\alpha)) \circ \Psi_X^{-1}(\xi) \\ =& \Psi_Y \circ h_X(\Psi_Y^{-1}(\alpha))(\mathrm{Id}) \\ =& \Psi_Y(\Psi_Y^{-1}(\alpha)) \\ =& \alpha \end{aligned}$$

Clearly our choice of $\Psi_Y^{-1}(\alpha)$ is unique, so (X, ξ) is a universal object. □

Note that universal objects (X, ξ) are also clearly unique up to unique isomorphism.

2.5 Schemes are Functors and (Some) Functors are Schemes

Properties of Schemes and their Morphisms

3.1 Closed Embeddings

In this chapter we will broadly discuss some topological, and algebraic properties of schemes and subschemes, along with their morphisms. Reader be warned: this chapter may feel like whiplash. Recall that in Definition 1.3.7 we defined what an open embedding is; we now define a similar class of morphisms:

Definition 3.1.1. Let $f : X \to Y$ be a morphism of schemes, then f is a **closed embedding**[40] if $f(X) \subset Y$ is closed, f is a homeomorphism onto it's image, and $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is surjective.

Example 3.1.1. Let A be a ring and $I \subset A$ be an ideal. We claim that the natural map $g : \operatorname{Spec} A/I \to \operatorname{Spec} A$ induced by the projection map $\pi : A \to A/I$ is a closed embedding. First note that if $\mathfrak{p} \subset A/I$ is a prime ideal, then we have that $I \subset \pi^{-1}(\mathfrak{p})$. Indeed, we have that $\ker \pi = I$, so $\pi^{-1}(0) = I$, and $\pi^{-1}(0) \subset \pi^{-1}(\mathfrak{p})$. It follows that we get a induced continuous map $g : \operatorname{Spec} A/I \to \mathbb{V}(I)$. However, we have already shown in Proposition 2.1.3 that there is a homeomorphism $f : \mathbb{V}(I) \to \operatorname{Spec} A/I$ given by $\mathfrak{p} \mapsto \pi(\mathfrak{p})$. We see that $f \circ g(\mathfrak{p}) = \pi(\pi^{-1}(\mathfrak{p})) = \mathfrak{p}$, so $f \circ g = \mathrm{Id}$. We want to show that $\pi^{-1}(\pi(\mathfrak{p})) = \mathfrak{p}$ as well. Note that:

$$\pi^{-1}(\pi(\mathfrak{p})) = \{a \in A : [a] \in \pi(\mathfrak{p})\}$$

while:

$$\pi(\mathfrak{p}) = \{[a] \in A/I : a \in \mathfrak{p}\}$$

If $a \in \mathfrak{p}$, then clearly we have that $[a] \in \pi(\mathfrak{p})$ so $a \in \pi^{-1}(\pi(\mathfrak{p}))$ implying that $\mathfrak{p} \subset \pi^{-1}(\pi(\mathfrak{p}))$. If $a \in \pi^{-1}(\pi(\mathfrak{p}))$ then $[a] \in \pi(\mathfrak{p})$, so $a + i \in \mathfrak{p}$ for some $i \in I$. We have that $I \subset \mathfrak{p}$, so $i \in \mathfrak{p}$, hence $a + i - i = a \in \mathfrak{p}$, implying that $\pi^{-1}(\pi(\mathfrak{p})) \subset \mathfrak{p}$. It follows that $g \circ f(\mathfrak{p}) = \pi^{-1}(\pi(\mathfrak{p})) = \mathfrak{p}$ so $g \circ f = \mathrm{Id}$ as well. We thus have that g is a homeomorphism onto the closed subspace A/I.

We now check that the morphism $g^\sharp : \mathscr{O}_{\operatorname{Spec} A} \longrightarrow g_*\mathscr{O}_{\operatorname{Spec} A/I}$ is surjective, and it suffices to check that $g^\sharp_{U_h}$ is surjective for every distinguished open U_h, as then the induced morphism on stalks will always be surjective. Note that:

$$g_*\mathscr{O}_{\operatorname{Spec} A/I}(U_h) = \mathscr{O}_{\operatorname{Spec} A/I}(U_{[h]}) \cong (A/I)_{[h]}$$

Note that that $g^\sharp_{U_h}$ is given by:

$$\begin{aligned} g^\sharp_{U_h} : A_h &\longrightarrow (A/I)_{[h]} \\ a/h^k &\longmapsto [a]/[h]^k \end{aligned}$$

which is clearly surjective so $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is a closed embedding as desired.

With this example in mind, we wish to show that every closed embedding is locally of this form.

Lemma 3.1.1. *Let $f : X \to Y$ be a morphism of schemes. Then f is a closed embedding if and only if for every open affine $U = \operatorname{Spec} A \subset Y$ there exists an ideal $I \subset A$ such that $f^{-1}(U) = \operatorname{Spec} A/I \subset X$, and $f|_{f^{-1}(U)}$ comes from the projection (up to isomorphism).*

[40] This is sometimes referred to in the literature as a closed immersion.

Proof. Let $f : X \to Y$ be a closed immersion, and let $I_{X/Y}$ be the sheaf of ideals on Y given by $\ker f^\sharp$. If $U = \operatorname{Spec} A \subset Y$ is an affine open then $I = I_{X/Y}(U)$ is an ideal of A and thus determines a closed subset $\mathbb{V}(I) \subset U$. Let $V = f^{-1}(U)$ then we have an induced morphism of schemes $f|_V : V \to U$ which must be a homeomorphism onto its image, so we simply need to show that $f(V) = \mathbb{V}(I)$. By Proposition 2.1.2, we have this morphism of schemes is uniquely determined by the morphism $(f|_V)^\sharp_U : \mathscr{O}_U(U) = A \to \mathscr{O}_V(V)$, which we denote by ψ going forward. If $x \in V$, then we have that:

$$f|_V(x) = \psi^{-1}(\pi_x^{-1}(\mathfrak{m}_x))$$

where π_x is the morphism $\mathscr{O}_V(V) \to (\mathscr{O}_V)_x$. We have that I is the kernel of ψ, and so $I \subset f(x)$ as $0 \in \pi_x^{-1}(\mathfrak{m}_x)$ hence $\psi^{-1}(0) \subset \psi^{-1}(\pi_x^{-1}(x))$. It follows that $f|_V : V \to U$ has image in $\mathbb{V}(I)$. Now suppose that $\mathfrak{p} \in \mathbb{V}(I)$, we want to show that $\mathfrak{p} \in f(V)$; since $f|_V$ is a closed embedding, we have that the stalk map:

$$(f|_V)^\sharp_\mathfrak{p} : A_\mathfrak{p} \longrightarrow ((f|_V)_*\mathscr{O}_V)_\mathfrak{p}$$

is surjective with kernel $I_\mathfrak{p}$. If $\mathfrak{p} \notin f(V)$ then we clearly have that $((f|_V)_*\mathscr{O}_V)_\mathfrak{p}$ is zero, implying that $I_\mathfrak{p} = A_\mathfrak{p}$. However, $I \subset \mathfrak{p}$, so this means that $\mathfrak{m}_\mathfrak{p} = A_\mathfrak{p}$ as $I_\mathfrak{p} \subset \mathfrak{m}_\mathfrak{p}$. This is clearly a contradiction, so we have that if $I \subset \mathfrak{p}$ then $\mathfrak{p} \in f(V)$ as desired. It follows that $f|_V : V \to U$ is a homeomorphism onto $\mathbb{V}(I)$.

Note that $\mathbb{V}(I) \cong \operatorname{Spec} A/I$, so we can freely identify the two. Let $g : V \to \operatorname{Spec} A/I$ be the homeomorphism induced by $f|_V : V \to U$. We note that for all $x \in V$, we have that $f|_V(x) = g(x)$. If $W \subset U$ is open, we have that $W \cap \mathbb{V}(I)$ is open in $\operatorname{Spec} A/I$, and we thus have that:

$$(f|_V)^{-1}(W) = (f|_V^{-1})(W) \cap (f|_V)^{-1}(\mathbb{V}(I)) = f|_V^{-1}(W \cap \mathbb{V}(I)) = g^{-1}(W \cap \mathbb{V}(I))$$

It follows that for any open set $Z = W \cap \mathbb{V}(I) \subset \mathbb{V}(I)$:

$$g_*\mathscr{O}_V(Z) = (f|_V)_*\mathscr{O}_V(W)$$

In particular, if U_g is an affine open of $\operatorname{Spec} A$, then:

$$g_*\mathscr{O}_V(U_{[g]}) = (f|_V)_*\mathscr{O}_V(U_g)$$

We thus define a morphism $g^\sharp : \mathscr{O}_{\operatorname{Spec} A/I} \to g_*\mathscr{O}_V$ on a basis of affine opens by noting that for each U_g we have a morphism:

$$(f|_V)^\sharp_{U_g} : A_g \longrightarrow g_*\mathscr{O}_V(U_{[g]})$$

whose kernel is precisely I_g. It follows that we get a unique morphism:

$$g^\sharp_{U_{[g]}} : \mathscr{O}_{\operatorname{Spec} A/I}(U_{[g]}) = A_g/I_g \longrightarrow g_*\mathscr{O}_V(U_{[g]})$$

which is trivially injective on each distinguished open. Moreover, these maps then clearly commute with the restriction maps, since localization commutes with taking quotients, as we have shown earlier. It follows that $g^\sharp : \mathscr{O}_{\operatorname{Spec} A/I} \to g_*\mathscr{O}_V$ is an injective morphism of sheaves, and is surjective on stalks because $(f|_V)^\sharp$ is. Since it is injective and surjective on stalks, we have that $g^\sharp$ is an isomorphism, implying that $f^{-1}(U) \cong \operatorname{Spec} A/I$ as schemes as desired. It follows that $f|_V : V \to U$ is now a morphism of affine schemes $\operatorname{Spec} A/I \to \operatorname{Spec} A$, such that the kernel of $\psi : A \to A/I$ is precisely I, hence up to isomorphism ψ is the projection map as desired.

Now suppose that for every affine open $U = \operatorname{Spec} A \subset Y$ we have that $f^{-1}(U) \cong \operatorname{Spec} A/I$, for some ideal I. Then with $V = f^{-1}(U)$, we have that $f|_V : V \to U$ is a morphism of affine schemes $\operatorname{Spec} A/I \to \operatorname{Spec} A$ induced by the projection. By Example 3.1.1, we have that $f|_V$ is a a closed immersion for all V. Since locally we have that $f^\sharp$ comes from the projection, we have that the stalk map $(f^\sharp)_y : (\mathscr{O}_Y)_y \to (f_*\mathscr{O}_X)_y$, is surjective. It follows that $f^\sharp$ is surjective by Proposition 1.2.8. Moreover, since each $f|_V$ is a homeomorphism onto it's image for all U, we have that $f : X \to Y$ must also be a homeomorphism onto it's image. Let $\{U_i\}$ be an open cover of Y, and $V_i = f^{-1}(U_i)$ then $f(X) \cap U_i = f|_{V_i}(V_i)$ which is closed in U_i. It follows that $U_i \smallsetminus f|_{V_i}(V_i)$ is open in Y. We claim that:

$$Y \smallsetminus f(X) = \bigcup_i U_i \smallsetminus f|_{V_i}(V_i)$$

Indeed, suppose that $y \in Y \setminus f(X)$, then for all i, we have that there is no $x \in V_i$ such that $f|_{V_i}(x) = y$. It follows that $y \in U_i \setminus f|_{V_i}(V_i)$ for all i, hence $Y \setminus f(X) \subset \bigcup_i U_i \setminus f|_{V_i}(V_i)$. Now suppose that:

$$y \in \bigcup_i U_i \setminus f|_{V_i}(V_i)$$

then for all i we have that there so no x such that $f|_{V_i}(x) = y$, hence there is no $x \in X$ such that $f(x) = y$ so $y \in Y \setminus f(X)$ giving us the other inclusion. Since $Y \setminus f(X)$ is the union of open sets, it is open, implying that $f(X)$ is closed, f is a homeomorphism onto its image, and $f^\sharp$ is surjective, hence f is a closed embedding implying the claim. $\square$

We have the following obvious corollaries:

Corollary 3.1.1. *If $X \to \operatorname{Spec} A$ is a closed embedding then $X \cong A/I$ for some I.*

Corollary 3.1.2. *A morphism $f : X \to Y$ is a closed embedding if and only if there exists an affine cover $\{U_i\}$ of Y such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is a closed embedding.*

We can properly define closed subschemes now:

Definition 3.1.2. Let X be a scheme, then a **closed subscheme** of X is an equivalence class of closed immersions $f : Z \to X$, where two closed immersions f and g are equivalent if and only if there is an isomorphism $F : Z \to Z$ such that $f \circ F = g$.

The clunky nature of the definition of above can be best explained by noting that for $X = \operatorname{Spec} \mathbb{C}[x]$, we have that $\mathbb{V}(x) = \mathbb{V}(x^2)$ as $\sqrt{\langle x^2 \rangle} = \sqrt{x}$, but $\operatorname{Spec} \mathbb{C}[x]/\langle x \rangle \ncong \operatorname{Spec} \mathbb{C}[x]/\langle x^2 \rangle$. So even though the two topological spaces agree, and both are the same from a topological embedding point of view, the two closed subschemes are not isomorphic. In particular, there are a multitude of scheme structures one can put on a closed subspace of any scheme X, with the induced reduced subscheme structure being just one of many.

Example 3.1.2. Let $X = \operatorname{Proj} A$ for a graded ring A, and Z a closed subscheme of X. Furthermore, suppose that the irrelevant ideal satisfies[41]:

$$A_+ = \sqrt{\langle g_1, \dots, g_n \rangle} \tag{3.1.1}$$

for some $g_i \in A_+^{\hom}$. Note that this condition is equivalent to $\operatorname{Proj} A$ being quasi-compact; indeed, suppose that $\operatorname{Proj} A$ is quasi-compact then there clearly exists a finite covering of X by projective distinguished opens $\{U_{g_i}\}$. Since $\mathbb{V}(A_+) = \emptyset$, we have that:

$$\mathbb{V}(A_+) = \left(\bigcup_{i=1}^n U_{g_i}\right)^c = \bigcap_{i=1}^n \mathbb{V}(\langle g_i \rangle) = \mathbb{V}(\langle g_1, \dots, g_n \rangle)$$

so (3.1.1) follows immediately. Now suppose that (3.1.1) holds, then X is equal to the union of U_{g_i}, which is finite, hence X is a finite union of quasi-compact schemes and is thus quasi-compact[42].

With the quasi-compactness assumption on X, we wish to show that Z is of the form $\operatorname{Proj} A/I$ for some homogenous ideal $I \subset A$. Supposing (2.4.1), we have an open cover of X given by $\{U_{g_i}\}$, and thus we obtain a finite open cover of Z by $\{V_i = f^{-1}(U_{g_i})\}$. Since f is a closed embedding, each $V_i = \operatorname{Spec}(A_{g_i})_0/I_i$; our goal is to construct I out of these I_i. Let $m_i = \deg g_i$, for each i, and define:

$$J_{i,d} = \begin{cases} \{0\} & \text{if } m_i \nmid d \\ \{a \in A_d : a/g_i^{d/m_i} \in I_i\} & \text{if } m_i \mid d \end{cases}$$

Note that $\deg(a/g_i^{d/m_i}) = d - d/m_i \cdot m_i = 0$, so $a/g_i^{d/m_i} \in (A_{g_i})_0$. We set:

$$J_i = \bigoplus_d J_i$$

[41] Note that A_+ is radical, as if $f \in \sqrt{A_+}$, then for some n, $f^n \in A_+$. If f has a degree zero part then f^n has a degree zero part hence $f^n \notin A_+$. It follows that f is a sum of positively graded elements, and thus $f \in A_+$.

[42] In general topology this is the same as say if X is a finite union of compact spaces then X is compact. This setting just feels weird as for Hausdorff spaces compact sets are closed.

It is clear that J_i is a homogenous ideal for each i, hence we set:

$$I = \bigcap_{i=1}^{n} J_i$$

We want to show that $f(Z) = \mathbb{V}(I)$, and it suffices to show that $f|_{V_i}(V_i) = \mathbb{V}(I) \cap U_{g_i}$ for all i. If $\pi_i : A \to A_{g_i}$ is the localization map, and $\iota_i : (A_{g_i})_0 \to A_{g_i}$ is the inclusion, then we set:

$$(I_{g_i})_0 = \iota_i^{-1}(\langle \pi_i(I) \rangle)$$

Let $\phi : U_{g_i} \to (\operatorname{Spec} A_f)_0$ be the homeomorphism from Proposition 2.2.2; we first claim that $\mathbb{V}(I) \cap U_{g_i} = \phi^{-1}(\mathbb{V}((I_{g_i})_0)) \subset U_{g_i}$. Let $\mathfrak{p} \in \mathbb{V}(I) \cap U_{g_i}$, then $\mathfrak{p}$ is a homogenous prime ideal such that $I \subset \mathfrak{p}$, and $g_i \notin \mathfrak{p}$. Since $\mathfrak{p} \in U_{g_i}$, we have that $\phi(\mathfrak{p}) = (\mathfrak{p}_{g_i})_0 \subset (A_{g_i})_0$. Since $I \subset \mathfrak{p}$, we have that $I_{g_i} \subset \mathfrak{p}_{g_i}$, hence $(I_{g_i})_0 \subset (\mathfrak{p}_{g_i})_0$, so $\mathfrak{p} \in \phi^{-1}(\mathbb{V}((I_{g_i})_0))$.

Now suppose that $\mathfrak{p} \in \phi^{-1}(\mathbb{V}(I_{g_0})_0) \subset U_{g_i}$, then $\mathfrak{p} \in U_{g_i}$ vacuously, so we need to show that $\mathfrak{p} \in V(I)$. By definition, $(I_{g_i})_0 \subset (\mathfrak{p}_{g_i})_0$; in A_f, we have that $(\mathfrak{p}_{g_i})_0$ corresponds to $\sqrt{(\mathfrak{p}_{g_i})_0 A_f}$, so we have that $\sqrt{(I_{g_i})_0 A_f} \subset \sqrt{(\mathfrak{p}_{g_i})_0 A_f}$ as well. It thus suffices to show that $I \subset \pi_i^{-1}\left(\sqrt{(I_{g_i})_0 A_f}\right)$, as then:

$$I \subset \pi_i^{-1}\left(\sqrt{(I_{g_i})_0 A_f}\right) \subset \pi_i^{-1}\left(\sqrt{(\mathfrak{p}_{g_i})_0 A_f}\right) = \mathfrak{p}$$

Furthermore, as I is homogenous, we need only check that every homogenous element of I lies in $\pi_i^{-1}\left(\sqrt{(I_{g_i})_0 A_f}\right)$. Let $a \in I$ be homogenous of degree d; if $a \in \ker \pi_i$ then we are done, otherwise, we have that $a^{m_i}/g_i^d \in (I_{g_i})_0$. It follows that $a^{m_i}/1 \in (I_{g_i})_0$, hence $a^{m_i}/1 \in (I_{g_i})_0 A_f$, so $a/1 \in \sqrt{(I_{g_i})_0 A_f}$ by definition[43].

It now suffices to show that $f|_{V_i}(V_i) = \phi^{-1}(\mathbb{V}(I_{g_i}))$. Since $f|_{V_i}(V_i) \subset U_{g_i}$, we have that $f|_{V_i}$ is a homeomorphism onto the closed subset $\mathbb{V}(I_i) \subset \operatorname{Spec}(A_{g_i})_0$. Therefore, it suffices to check that $\mathbb{V}(I_i) = \mathbb{V}((I_{g_i})_0)$, and in particular that $I_i = (I_{g_i})_0$ for all i. Now note that the only elements in A_{g_i} which have degree zero are those of the form a/g_i^n where a is homogenous and satisfying $\deg a = n \cdot m_i$. Let $a/g_i^n \in (I_{g_i})_0$, then $a/g_i^n \in I_{g_i}$, so $a/1 \in I_{g_i}$ as well. It follows that $a \in I \cap A_{n \cdot m_i}$, hence $a/g_i^n \in I_i$ for all i, so $(I_{g_i})_0 \subset I_i$ as desired.

Now let $a/g_i^n \in I_i$, and $l = \operatorname{lcm}(m_1, \ldots, m_n)$. We have that there exists a $k \leq r \in \mathbb{N}$ such that:

$$n = k \cdot l + r \Rightarrow n + (k - r) = (k+1)l$$

so by taking $a/g_i^n = ag^{k-r}/g^{k-r+n}$, we may assume that l divides n. Since $\ker f^\sharp$ is a sheaf of ideals, if $I_{ij} = \ker f^\sharp_{U_{g_i} \cap U_{g_j}}$, we have that $a|_{U_{g_i} \cap U_{g_j}} \in I_{ij}$. Recall that $U_{g_i} \cap U_{g_j} = U_{g_i g_j} = \operatorname{Spec}(A_{g_i g_j})_0$, hence we have that:

$$a/g^n|_{U_{g_i} \cap U_{g_j}} = ag_j^n/(g_i g_j)^n \in I_{ij} \subset (A_{g_i g_j})_0$$

Moreover, we also have that

$$U_{g_i g_j} \cong \operatorname{Spec}((A_{g_i})_0)_h$$

where $h = g_j^{m_i}/g_i^{m_j}$. The ring homomorphism

$$f^\sharp_{U_{g_i}} : (A_{g_i})_0 \to (A_{g_i})_0/I_i$$

determines a morphism of affine schemes which on all distinguished opens of $\operatorname{Spec}(A_{g_i})_0$ of the form U_b, has kernel given by $(I_i)_b$. The morphism determined by $f^\sharp_{U_i}$ must agree with f on all open subsets of U_{g_i}, hence we have that I_{ij} is naturally isomorphic to the ideal $(I_i)_h$, via the unique isomorphism $((A_{g_i})_0)_h \cong (A_{g_i g_j})_0$ from Lemma 2.2.7. Similarly, with $h^{-1} = g_i^{m_j}/g_j^{m_i}$, we must have that $(I_j)_{h^{-1}}$ is naturally isomorphic to I_{ij} via the same isomorphism. Any element in $(I_j)_{h^{-1}}$ can be written as:

$$\frac{b}{g_j^k} \cdot \left(\frac{g_i^{m_j}}{g_j^{m_i}}\right)^{-e} \tag{3.1.2}$$

[43]Note that we have now shown that for any homogenous ideal I, $\mathbb{V}(I) \cap U_h = \mathbb{V}((I_h)_0) \subset U_h$

where $b/g_j^k \in I_j$. Recall that we took n to be divisible by l, so $n = m_i \cdot p$ and $n = m_j \cdot q$ for some p and q. Hence, under the isomorphism $(I_i)_h \cong (I_j)_{h^{-1}}$ we have that:

$$\frac{a}{g_i^{m_j \cdot q}} \longmapsto \frac{a}{g_j^{m_i \cdot q}} \cdot \left(\frac{g_i^n}{g_j^{m_i \cdot q}} \right)^{-1}$$

So for an element of the form (3.1.2) we must have that:

$$\frac{a}{g_i^{m_j \cdot q}} \longmapsto \frac{a}{g_j^{m_i \cdot q}} \cdot \left(\frac{g_i^n}{g_j^{m_i \cdot q}} \right)^{-1} = \frac{b}{g_j^k} \cdot \left(\frac{g_i^{m_j}}{g_j^{m_i}} \right)^{-e}$$

We thus have that by the definition of localization we have that:

$$\frac{g_i^{m_j \cdot e} a}{g_j^{m_i \cdot e + m_i \cdot q}} \in I_j$$

We can take e large enough so that $e_j' = m_j \cdot e$ is divisible by l, hence we can write that:

$$\frac{g_i^{e_j'} a}{g_j^{(e'+n) \cdot (m_i/m_j)}} \in I_j$$

Do this for all j, and let $e' = \max(e_1', \ldots, e_n')$, then $g_i^{e'} a \in J_j$ for all j. It follows that $g_i^{e'} a \in I$, hence:

$$\frac{g_i^{e'} a}{1} \in I_{g_i}$$

so $a/1 \in I_{g_i}$, giving us that $a/g^n \in (I_{g_i})_0$. It follows that $I_i = (I_{g_i})_0$ so $f(Z) = \mathbb{V}(I)$ as desired.

We now show that $\mathbb{V}(I)$ is homeomorphic to $\operatorname{Proj} A/I$. Let $\pi : A \to A/I$ be the projection map, where A/I has the induced grading, and $\mathfrak{p} \in \operatorname{Proj} A/I$. The prime ideal $\pi^{-1}(\mathfrak{p})$ is homogenous, as if $a \in \pi^{-1}(\mathfrak{p})$ then we write a as:

$$a = \sum_d a_d \tag{3.1.3}$$

where $a_d \in A_d$. It follows that $\pi(a) \in \mathfrak{p}$, and since $\mathfrak{p}$ is homogenous each $\pi(a_d)$ is in $\mathfrak{p}$ so each $a_d \in \pi^{-1}(\mathfrak{p})$. Each $\pi^{-1}(\mathfrak{p})$ contains I so this defines a map $F : \operatorname{Proj} A/I \to \mathbb{V}(I)$. Via the bijection between prime ideals of A/I and prime ideals of A containing I it follows that this map is a bijection, so it suffices to check that this is continuous and open.

We can do this on the distinguished basis for $\operatorname{Proj} A/I$ and the basis $\{\mathbb{V}(I) \cap U_g\}_{g \in A_+^{\mathrm{hom}}}$ for $\mathbb{V}(I)$. Let U_g be the projective distinguished open in $\operatorname{Proj} A$, then

$$F^{-1}(V(I) \cap U_g) = F^{-1}(V(I)) \cap F^{-1}(U_g) = F^{-1}(U_g)$$

I claim that this is equal to $U_{[g]}$. Suppose $[g] \notin \mathfrak{p} \subset A/I$, then for all $i \in I$ we must have that $g+i \notin \pi^{-1}(\mathfrak{p})$ hence $g \notin \pi^{-1}(\mathfrak{p})$. It follows that $\mathfrak{p} \in U_g$ so $U_{[g]} \subset U_g$. Now let $\mathfrak{p} \in f^{-1}(U_g)$, then $g \notin \pi^{-1}(\mathfrak{p})$, but this implies that $[g] \notin \pi(\pi^{-1}(\mathfrak{p})) = \mathfrak{p}$ so $\mathfrak{p} \in U_{[g]}$. Therefore $f^{-1}(U_g) = U_{[g]}$ and f is continuous.

To show that F is open we claim that $F(U_{[g]}) = V(I) \cap U_g$, but this is now clear as $F : \operatorname{Proj} A/I \to V(I)$ is bijective, so since $F^{-1}(V(I) \cap U_g) = U_{[g]}$ we get that $F(F^{-1}(V(I) \cap U_g) = V(I) \cap U_g = U_{[g]}$. It follows that f is a continuous open bijective map and thus a homeomorphism.

Now note that the structure sheaf $\mathscr{O}_{\operatorname{Proj} A/I}$ satisfies:

$$\mathscr{O}_{\operatorname{Proj} A/I}(U_{[g]}) = ((A/I)_{[g]})_0$$

However, recall that there is a unique surjective homomorphism

$$\begin{aligned} A_g &\longrightarrow (A/I)_{[g]} \\ a/g^k &\longmapsto [a]/[g]^k \end{aligned}$$

which commutes with localization maps, and clearly preserves grading. It follows, that we have a unique surjective homomorphism commuting with the isomorphisms from Lemma 2.2.7:

$$\begin{aligned}(A_g)_0 &\longrightarrow ((A/I)_{[g]})_0\\ a/g^k &\longrightarrow [a]/[g]^k\end{aligned}$$

where $\deg a = k \cdot \deg g$. Note that clearly $(I_g)_0$ maps to zero under this map, so we have unique surjective homomorphism:

$$\begin{aligned}\phi : (A_g)_0/(I_g)_0 &\longrightarrow ((A/I)_{[g]})_0\\ [a/g^k] &\longrightarrow [a]/[g]^k\end{aligned}$$

Now suppose that $\phi([a/g^k]) = 0$, then we have that $[a]/[g]^k = 0 \in ((A/I)_{[g]})_0 \subset (A/I)_{[g]}$. It follows that there an M such that $[g^M \cdot a] = 0 \in A/I$, hence $g^M a \in I$. We thus have that $g^M a/1 \in I_g$, so $g^M a/g^{M+k} = a/g^k \in (I_g)_0$. By the naturality[44] of these isomorphisms it follows that up to a unique sheaf isomorphism:

$$\mathscr{O}_{\operatorname{Proj} A/I}(U_{[g]}) = (A_g)_0/(I_g)_0$$

Now equip $\mathbb{V}(I)$ with the sheaf $\mathscr{O}_{\mathbb{V}(I)} = F_* \mathscr{O}_{\operatorname{Proj} A/I}$, and note that this endows $\mathbb{V}(I)$ with the structure of a scheme isomorphic to $\operatorname{Proj} A/I$[45].

Let $\tilde{f}$ be restriction of the codomain to $\mathbb{V}(I)$. In particular, we have that:

$$\tilde{f} : Z \longrightarrow \mathbb{V}(I)$$

Since $I_i = (I_{g_i})_0$, we define a sheaf morphism on the open cover $\{\mathbb{V}(I) \cap U_{g_i}\}$ as the identity map:

$$\tilde{f}^\sharp_{\mathbb{V}(I)\cap U_{g_i}} : \mathscr{O}_{\mathbb{V}(I)}(\mathbb{V}(I) \cap U_{g_i}) = (A_{g_i})_0/(I_{g_i})_0 \longrightarrow \mathscr{O}_Z(V_i) = (A_{g_i})_0/I_i$$

These then agrees on overlaps $U_{g_i} \cap U_{g_j}$ as $((I_{g_i})_0)_h \cong I_{ij} \cong ((I_{g_j})_0)_{h^{-1}}$ via the natural isomorphisms which glue $\operatorname{Proj} A$ together. It follows that this defines a sheaf isomorphism:

$$\tilde{f}^\sharp : \mathscr{O}_{\mathbb{V}(I)} \longrightarrow \mathscr{O}_Z$$

hence $(\tilde{f}, \tilde{f}^\sharp)$ determines a scheme isomorphism $Z \to \mathbb{V}(I)$. Since $\mathbb{V}(I) \cong \operatorname{Proj} A/I$ as schemes, we thus have that $Z \cong \operatorname{Proj} A/I$ as desired.

Now note that the condition that the condtion that $\operatorname{Proj} A$ be quasi-compact is extremely necessary. Indeed take:

$$\mathbb{P}^\infty_k = \operatorname{Proj} k[x_1, x_2, \dots]$$

for any field k. Let:

$$Z = \coprod_{i=1}^{\infty} X_i = \operatorname{Spec} k[x_1/x_i, x_2/x_i, \dots, \hat{x}_i/x_i, \dots]/\langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle^i$$

where the ideal:

$$\langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle^i$$

is generated by all ith fold products of $\langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle$. Note that each scheme is is a singleton set as

$$\sqrt{\langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle^i} = \langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle$$

[44]Note that $(A_g)_0/(I_g)_0$ does not depend on the class representative g, as for any homogeneous i of degree equal to g, $[a/(g+i)^k] = [a/g^k]$.

[45]This is not the reduced scheme structure, rather one induced by the sheaf of ideals determined by I itself. If $\mathbb{V}(I)$ was equipped with the reduced structure, then as schemes $\mathbb{V}(I) \cong \operatorname{Proj} A/\sqrt{I}$.

Denote each point by $0^i \in X_i \subset Z$, and define a closed embedding by:

$$\begin{aligned} f : Z &\longrightarrow \mathbb{P}^\infty_k \\ 0^i &\longmapsto [0, \ldots, 0, 1, 0 \ldots, 0, \ldots] \end{aligned}$$

where the 1 is in the ith position. If we take the homogenous ideal $I = \langle x_i x_j : i \neq j \rangle$, then clearly for all k:

$$(I_{x_k})_0 = \langle x_1/x_k, x_2, x_k, \ldots \hat{x}_k/x_k, \ldots, \rangle$$

So under the identification $\mathbb{V}(I) \cap U_{x_i} = \mathbb{V}((I_{x_i})) \subset U_{x_i}$, we discern that $\mathbb{V}(I) \cap U_{x_i}$ contains only the point $[0, \ldots, 0, 1, 0 \ldots, 0, \ldots]$, where the 1 is again in the ith position. Clearly we then have that for all U_{x_i}, $f(Z) \cap U_{x_i} = \mathbb{V}(I) \cap U_{x_i}$, hence $f(Z) = \mathbb{V}(I)$, and f has closed image.

We set:

$$I_i = \langle x_1/x_i, x_2/x_i, \ldots, \hat{x}^i/x_i \rangle^i$$

and define a sheaf morphism on the affine open cover $\{U_{x_i}\}_{i=1}^\infty$ via the canonical projections:

$$\begin{aligned} f^\sharp_{U_{x_i}} : k[\{x_j/x_i\}_{j=1, j\neq i}^\infty] &\longrightarrow k[\{x_j/x_i\}_{j=1, j\neq i}^\infty]/I_i \\ g &\longmapsto [g] \end{aligned}$$

and note that there is nothing to glue as $f^{-1}(U_{x_i} \cap U_{x+j})$ is the empty set. This sheaf homomorphism is clearly surjective on stalks so $Z \hookrightarrow \mathbb{P}^\infty_k$ is a closed embedding.

We claim that there is no homogenous ideal I such that $Z \cong \operatorname{Proj} A/I$. Indeed, suppose there was. Then by the work above we would have that for all x_i,

$$(I_{x_i})_0 = I_i$$

Let $f \in I$ be homogenous of degree d, then $f/1 \in I_{x_i}$, and $f/x_i^d \in (I_{x_i})_0$[46]. For the above to be true, we must then have that $f/x_i^d \in I_i$ for all i as well. However, if $k > d$, then f/x_k^d cannot lie in I_k as every element must a be a sum of at least k-fold product of elements of the form x_j/x_k, while f/x_k^d can only be a sum of d-fold products of said elements. It follows that for some k we must have that:

$$(I_{x_i})_0 \not\subset I_k$$

implying that $Z \ncong \operatorname{Proj} A/I$ for any homogenous ideal I.

With the above example in mind, we can classify all projective schemes over some fixed ring B:

Theorem 3.1.1. *X is a projective scheme over B if and only if it is a closed subscheme of $\mathbb{P}^n_B$ for some n.*

Proof. Suppose that X is a projective scheme over B, then by Definition 2.2.7, we have that:

$$X = \operatorname{Proj} A$$

where A is a graded ring, satisfying $A_0 = B$, and is finitely generated as a B algebra. Since A is finitely generated in degree one, for some n there is a surjection:

$$\phi : B[x_0, \ldots, x_n] \to A$$

which preserves grading. It follows that $\ker \phi$ is a homogenous ideal, and that $A \cong B[x_0, \ldots, x_n]/\ker \phi$, hence:

$$X = \operatorname{Proj}(B[x_0, \ldots x_n]/\ker \phi)$$

As a scheme, X is canonically isomorphic to $\mathbb{V}(\ker \phi) \subset \mathbb{P}^n_B$[47], hence X determines a closed subscheme of $\mathbb{P}^n_B$.

[46] Note that f/x_i^d and not be zero as k is a field, so localization maps are injective.

[47] Note that $\mathbb{V}(\ker \phi)$ is not necessarily equipped with the reduced subscheme structure, but instead equipped with scheme structure determined by the sheaf of ideals induced by $\ker \phi$. This only coincides with the reduced structure if $\ker \phi$ satisfies $\sqrt{\ker \phi} = \ker \phi$.

If X is a closed subscheme of $\mathbb{P}^n_B$, then since $\mathbb{P}^n_B$ is quasicompact, we have that by Example 3.1.2 $X \cong \operatorname{Proj} B[x_0, \ldots, x_n]/I$ for some homogenous ideal I. If I contains the irrelevant ideal, then X is the empty scheme and thus isomorphic to $\operatorname{Proj} B$, where B has the trivial grading, so X is trivially a projective B scheme. If I does not contain the irrelevant ideal, then $B[x_0, \ldots, x_n]/I$ is a graded, finitely generated in degree one, B-algebra, hence X is projective B scheme as desired. □

Example 3.1.3. Recall from Example 2.3.4 that locally the morphism:

$$f : \mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^3_{\mathbb{C}}$$

is given by scheme morphisms:

$$U_{x_l} \times_{\mathbb{C}} U_{y_n} \to U_{v_i}$$

The U_{v_i} cover $\mathbb{P}^3_{\mathbb{C}}$, and $f^{-1}(U_{v_i}) = U_{x_l} \times_{\mathbb{C}} U_{y_n}$. We claim that this morphism is a closed embedding, and by Corollary 3.1.2 it suffices to check that each $U_{x_l} \times_{\mathbb{C}} U_{y_n} \to U_{v_i}$ is a closed embedding. By Corollary 3.1.2, it suffices to check that $U_{x_l} \times_{\mathbb{C}} U_{y_n} \cong \operatorname{Spec} \mathbb{C}[\{v_k/v_i\}_{k \neq i}]/I$ for some ideal I. We check this in case of $i = 0$. Note that the morphism of affine schemes comes from the ring homomorphism:

$$\begin{aligned} \phi_0 : \mathbb{C}[v_1/v_0, v_2/v_0, v_3/v_0] &\longrightarrow \mathbb{C}[x_1/x_0, y_1/y_0] \\ v_i/v_0 &\longmapsto \begin{cases} x_1/x_0 & \text{if } i = 1 \\ y_1/y_0 & \text{if } i = 2 \\ (x_1/x_0) \cdot (y_1/y_0) & \text{if } i = 3 \end{cases} \end{aligned}$$

This is clearly surjective, hence $\mathbb{C}[x_1/x_0, y_1/y_0] \cong \mathbb{C}[v_1/v_0, v_2/v_0, v_3/v_0]/\ker \phi_0$, and it follows that the induced morphism is a closed embedding. The kernel of this homomorphism is:

$$I = \langle v_1/v_0 \cdot v_2/v_0 - v_3/v_0 \rangle$$

and so the homogenous ideal cutting out $f(\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}})$ is given by $J = \langle v_1 v_2 - v_3 v_0 \rangle$. It follows that as schemes,

$$\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1 \cong \operatorname{Proj} \mathbb{C}[v_0, v_1, v_2, v_3]/\langle v_1 v_2 - v_3 v_0 \rangle$$

Example 3.1.4. Let $Z \subset X$ be a closed subset of a scheme X, and equip Z with the induced reduced closed subscheme structure, then we have that the inclusion map $\iota : Z \to X$ is a homeomorphism onto its image. We want to define a sheaf morphism $\iota^\sharp : \mathscr{O}_X \to \iota_* \mathscr{O}_Z$. Recall that if $I_{Z/X}$ is the sheaf of ideals associated to the closed subset Z then $\mathscr{O}_Z = \iota^{-1} \mathscr{O}_X / I_{Z/X}$. As the next proposition shows we have that there is a canonical morphism:

$$\mathscr{O}_X / I_{Z/X} \longrightarrow \iota_* \iota^{-1} \mathscr{O}_X / I_{Z/X}$$

which is surjective. There is a surjective morphism $\mathscr{O}_X \to \mathscr{O}_X / I_{Z/X}$, so we define $\iota^\sharp$ to the be composition of these sheaf morphisms. It follows that $(\iota, \iota^\sharp)$ is a closed embedding as desired.

We now go to our next result regarding closed embeddings which is an analogue of Lemma 2.3.7:

Lemma 3.1.2. *Let $f : X \to Z$ be a closed embedding, and let $g : Y \to Z$ be any morphism. Then the base change $X \times_Z Y \to Y$ is also a closed embedding.*

Proof. We have the following Cartesian square:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Let $\{U_i = \operatorname{Spec} A_i\}$ be an affine open cover of Z, and choose an affine open cover $\{V_{ij} = \operatorname{Spec} B_{ij}\}$ of Y such that $g(V_{ij}) \subset U_i$. Note that $f^{-1}(U_i) \cong \operatorname{Spec} A_i/I_i$ for some ideal I_i. We have that:

$$\pi_Y^{-1}(V_{ij}) \cong X \times_Z V_{ij}$$

We claim that this is isomorphic to $f^{-1}(U_i) \times_{U_i} V_i$. Indeed, we need to show that the following diagram is cartesian:

$$\begin{tikzcd} f^{-1}(U_i) \times_{U_i} V_i \arrow[r, "\pi_Y"] \arrow[d, "\pi_X \circ \iota"'] & V_{ij} \arrow[d, "g|_{V_{ij}}"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

where $\iota : f^{-1}(U_i) \to X$ is the inclusion, is Cartesian. Let Q be any scheme with maps $p_X : Q \to X$ and $p_{V_{ij}} : Q \to V_{ij}$ which make the relevant diagram commute. Since $g(V_{ij}) \subset U_i$, we have that $g \circ p_{V_{ij}}(Q) \subset U_i$. Since $f \circ p_X = g \circ p_{V_{ij}}$, we have that $f \circ p_X(Q) \subset U_i$ as well, and thus $p_X(Q) \subset f^{-1}(U_i)$. Since $X \times_Z V_{ij}$ is a fibre product we have a unique morphism $\phi : Q \to X \times_Z V_{ij}$ such that $\pi_X \circ \iota \circ \phi = p_X$. We thus have that $\pi_X \circ \iota \circ \phi(Q) = p_X(Q) \subset f^{-1}(U_i)$. We see that $(\pi_X \circ \iota)^{-1}(f^{-1}(U_i)) \subset f^{-1}(U_i) \times_Z V_i$, hence $\phi(Q) \subset p_X(Q)$. Since both $f(f^{-1}(U_i)) \subset U_i$, and $g(V_{ij}) \subset U_i$, we have that this $f^{-1}(U_i) \times_Z V_{ij} = f^{-1}(U_i) \times_{U_i} V_{ij}$, and it follows that $\phi(Q) \subset f^{-1}(U_i) \times_{U_i} V_{ij}$, so ϕ factors uniquely through the open embedding $f^{-1}(U_i) \times_{U_i} V_{ij} \to X \times_Z V_{ij}$, and we have a unique morphism $Q \to f^{-1}(U_i) \times_{U_i} V_{ij}$. It follows that $f^{-1}(U_i) \times_{U_i} V_{ij} \cong f^{-1}(U_i) \times_{U_i} V_{ij}$ as desired. We thus have the following chain of isomorphisms:

$$\begin{aligned} \pi_Y^{-1}(V_{ij}) &\cong X \times_Z V_{ij} \\ &\cong f^{-1}(U_i) \otimes_{U_i} V_{ij} \\ &\cong \operatorname{Spec} A_i/I_i \otimes_{A_i} \operatorname{Spec} B_{ij} \\ &\cong \operatorname{Spec} A_i/I_i \otimes_{A_i} B_{ij} \end{aligned}$$

Let $\phi : A_i \to B_{ij}$ be the ring homomorphism making B_{ij} an A_i algebra, and set $J = \langle \phi(I_i) \rangle$, then we have that:

$$A_i/I_i \otimes_{A_i} B_{ij} \cong B_{ij}/J$$

hence:

$$\pi_Y^{-1}(V_i) \cong \operatorname{Spec} B_{ij}/J$$

so $\pi_Y : X \times_Z Y \to Y$ is a closed embedding by Corollary 3.1.2. □

These two lemmas each provide an example of classes of morphisms we are about to study, namely being *local on target* and *stable under base change*. More precisely, let $f : X \to Z$ be a morphism of schemes and P a property morphisms of schemes, then P is local on target if for any affine cover of $\{U_i\}$ of Z such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to Z$ satisfies P for all i we have that f satisfies P, and if $f : X \to Y$ satisfies P, then for all affine opens U, $f|_{f^{-1}(U)}$ satisfies P as well. In other words, a property of a morphism of schemes is called local on target if it can be checked affine locally. Let $g : Y \to Z$ be any other morphism of schemes, and let $f : X \to Z$ be a morphism satisfying P, then P is stable under base change if $X \times_Z Y \to Y$ also satisfies the property.

3.2 Reduced, Irreducible, and Integral Schemes

In the following sections, we will study some algebraic and topological properties schemes may have, and the interplay between them. We begin with the following definition:

Definition 3.2.1. Let X be a scheme, then X is **irreducible** if it is irreducible as a topological space as in Definition 1.4.3. We also have that X is **reduced** if $\mathscr{O}_X(U)$ has no nilpotents for all $U \subset X$, and is **integral** if $\mathscr{O}_X(U)$ is an integral domain for all $U \subset X$.

We first check that being reduced is an inherently local property.

Lemma 3.2.1. *Let X be a scheme, then the following are equivalent:*

a) X is reduced

b) There exists an affine open cover $\{U_i\}$ such that each U_i is reduced

c) Every stalk $(\mathscr{O}_X)_x$ is reduced

Proof. Clearly $a \Rightarrow b$, so we first show that $b \Rightarrow c$. Let $x \in U_i = \operatorname{Spec} A$, then $(\mathscr{O}_X)_x \cong A_{\mathfrak{p}}$ so it suffices to check that $A_{\mathfrak{p}}$ has no nilpotents. Let $a/g \in A_{\mathfrak{p}}$ where $g, h \notin \mathfrak{p}$. Then if $(a/g)^k = 0$ for some k there exists a $c \in \mathbb{A} - \mathfrak{p}$ such that:

$$c \cdot a^k = 0$$

We see that $c \neq 0$, so since A has not nilpotents we have that either $a = 0$ hence $a/g = 0$, implying the claim.

Now we show that $c \Rightarrow a$. Let U be an open set of X, and $s \in \mathscr{O}_X(U)$ such that $s^k = 0$ for some k. Then for every $x \in U$ we have that $(s^k)_x = s_x^k = 0$ implying that $s_x = 0$ for all s. However, the map:

$$\mathscr{O}_X(U) \longrightarrow \prod_{x \in U} (\mathscr{O}_X)_x$$

is injective so $s = 0$, hence $\mathscr{O}_X(U)$ has no nilpotents. □

Example 3.2.1. Let X be a scheme, and Y a closed subset of X, then Y equipped with the induced reduced closed subscheme structure is irreducible. Indeed, let $\{U_i = \operatorname{Spec} A\}$ be an affine open cover of X, then $U_i \cap Y \cong \operatorname{Spec} A_i/I_i$ determines an affine open cover of Y. Each I_i is radical, hence we have that if $[a] \in A_i/I_i$ satisfies $[a]^k = 0$ then $a^k \in I_i$, implying that $a \in I_i$. It follows that $[a] = 0$, so A_i/I_i is reduced. We thus have an affine open cover of Y such that each affine scheme is reduced, so by Lemma 3.2.1 we have that Y is reduced as well.

We now show some properties of X being irreducible. We need the following definition:

Definition 3.2.2. Let X be a topological space, then a **generic point** is a point $\eta \in X$ which is dense, i.e. $\overline{\{\eta\}} = X$.

Lemma 3.2.2. *Let X be an irreducible topological space, then every non empty open subset of X is irreducible when equipped with the subspace topology. Moreover, a topological space is irreducible if and only if the intersection of every two non empty open sets is non empty.*

Proof. Suppose that X is irreducible, then by Lemma 1.4.4 we have that X is connected and every open subset of X is dense. Let U be a non empty open subset of X, then we claim that U is irreducible when equipped with the subspace topology. Indeed, suppose that $U = Y_1 \cup Y_2$ for two proper closed subsets of U. Then $Y_1 = Z_1 \cap U$ and $Y_2 = Z_2 \cap U$, then we have that $U = (Z_1 \cup Z_2) \cap U$, implying that $U \subset Z_1 \cup Z_2$ hence U is contained in the closed subset $Z_1 \cup Z_2$. However, $\bar{U} = X$, so $X = Z_1 \cup Z_2$ implying that X is reducible. The claim follows from the contrapositive.

Now let U and V be two nonempty open subsets of X such that $U \cap V = \emptyset$. Then $U^c \cup V^c = X$, so X is reducible. By the contrapositive we have that if X is irreducible then $U \cap V \neq \emptyset$.

Suppose that $U \cap V \neq \emptyset$ for every open set, and let $Z_1, Z_2 \subset X$ be two proper closed subsets. We see that since $Z_1^c \cap Z_2^c \neq \emptyset$ that $Z_1 \cup Z_2 \neq X$, so X is irreducible □

Lemma 3.2.3. *Let X be a scheme, then X is irreducible if and only if X has unique generic point η.*

Proof. Suppose that X is reducible, then $X = Z_1 \cup Z_2$ for two closed proper subsets of X. It follows that every $x \in X$ lies in Z_1 or Z_2 so the closure of every point is contained in Z_1 or Z_2. We thus have that X has no generic points, let alone a unique one. By the contrapositive, we have that if X has a unique generic point, then X is irreducible.

Now let X be irreducible, by Lemma 3.2.2 we have that $U = \operatorname{Spec} A$ is a irreducible topological space as well. We claim that the nilradical:

$$I = \{a \in A : \exists k \in \mathbb{N}, a^k = 0\}$$

is prime. Let $f, g \in A$ then if $f, g \in I$ we have that $U_f = U_{f^k} = U_0 = \emptyset$ and similarly for g. Similarly, if $U_f = U_0$ then there is some k such that $f^k = 0$ so we have that a distinguished open is empty if and only if the element lies in I. Now suppose that $U_f \cap U_g$ is not empty, the fact that $U_f \cap U_g$ is not empty implies that $fg \notin I$. It follows by the contrapositive that if $fg \in I$ then either f or g are in I so I is prime. The closure of the singleton set $\{I\} \in \operatorname{Spec} A$ is given by $\mathbb{V}(I)$ and we claim that this is equal to $\operatorname{Spec} A$. We need only show that $I \subset \mathfrak{p}$ for any prime in A, however this is clear as $0 \in \mathfrak{p}$, and for any

$f \in I$ we have that $f^k = 0 \in \mathfrak{p}$, hence $f \in \mathfrak{p}$, so $I \subset \mathfrak{p}$. We show that I is unique, suppose that $\mathfrak{q}$ is prime, and satisfies $\mathfrak{q} \subset \mathfrak{p}$ for every prime. Then $\mathbb{V}(\mathfrak{q}) = \operatorname{Spec} A$, so $\mathfrak{q} = \sqrt{\mathfrak{q}} = \sqrt{I} = I$ implying uniqueness.

We now claim that the point $x \in X$ corresponding to $I \in \operatorname{Spec} A$ is actually a generic point of X. Indeed, suppose that $\overline{\{x\}} = V$ for some closed subset of X, then we have that:

$$V = \bigcap_{Z \ni x} Z$$

where $Z \subset X$ is closed. In the subspace topology, since x is a generic point, we have that:

$$U = \bigcap_{Y \ni x} Y$$

where $Y \subset U$ is closed. The subsets of U which are closed are of the form $Z \cap U$ where Z is closed in X, hence we have that:

$$U = \bigcap_{Z \ni x} Z \cap U = V \cap U$$

hence $U \subset V$. However, the only closed set of X which contains U is X itself, so $V = X$ and $\{x\}$ is generic.

To show uniqueness, note that x lies in every open set of $U \subset X$, as other wise, $x \in U^c$, which is closed and thus contradicts the fact that $\{x\}$ is dense. Now suppose that U is any open affine, and $y \in X$ is a generic point not equal to x. Then y is clearly a generic point of every open affine, so $y, x \in U$ are both generic points. But then $x = y$ as every irreducible affine scheme only has one generic point, implying the claim. $\square$

Note that if the nilradical of a ring A is prime then its vanishing locus is the whole of $\operatorname{Spec} A$, so $\operatorname{Spec} A$ contains a generic point, and is thus irreducible. In particular, $\operatorname{Spec} A$ is irreducible if and only if the nilradical is prime.

Lemma 3.2.4. *Let X be a scheme, which is not the empty scheme. Then the following are equivalent:*

a) *X is irreducible*
b) *There exists an affine open covering $\{U_i\}$ of X such that U_i is irreducible for all i, and $U_i \cap U_j \neq \emptyset$ for all i and j.*
c) *Every nonempty open affine $U \subset X$ is irreducible.*

Proof. Note that Lemma 3.2.2 implies that $a \Rightarrow b, c$. We show that $b \Rightarrow a$. Suppose that $X = Z_1 \cup Z_2$, then we have that since each U_i is irreducible $U_i \subset Z_1$ or Z_2. Indeed suppose otherwise, then $U_i \cap (Z_1 \cup Z_2) = (U_i \cap Z_1) \cup (U_i \cap Z_2)$ which are both closed in the subspace topology, thus $U_i \cap Z_j$ must equal Z_j for at least one j. Without loss of generality suppose that $U_i \subset Z_1$ and take any other U_j. Then $U_i \cap U_j$ is non empty and in dense in U_j. Since $U_i \subset Z_1$, we have that $U_i \cap U_j \subset Z_1 \cap U_j$, which is closed in U_j. It follows that the closure of $U_i \cap U_j$ is contained in Z_1, thus $U_j \subset Z_1$. We thus have that $\bigcup U_i = X \subset Z_1$, so $X = Z_1$, implying that X is irreducible.

For $c \Rightarrow a$, let $U \cap V$ be empty for some open affines, then $U \cup V$ is affine as it is trivially a disjoint union, and thus the coproduct in the category of schemes, and finite coproducts of affine schemes are affine by Example 2.1.3. However, irreducible spaces are connected, and $U \cup V$ is an affine open so is not irreducible contradicting c. It follows that the intersection of every open affine is non trivial, and since the open affines generate the topology on X we must have that the intersection of every open set is non empty, thus by Lemma 3.2.2 we have that X is irreducible. $\square$

Example 3.2.2. Note that any disconnected scheme is not irreducible, we now give an example of a connected but reducible scheme. We first note that an affine scheme $\operatorname{Spec} A$ is connected if and only if it only has no nontrivial idempotents. Indeed, suppose that A has a nontrivial idempotent a, then $a \cdot a = a$. Note that $\langle a \rangle + \langle 1 - a \rangle = A$, implying that

$$\mathbb{V}(a) \cap \mathbb{V}(1 - a) = \mathbb{V}(1) = \emptyset$$

Since $\langle a \rangle$ and $\langle 1 - a \rangle$ are coprime, we have that $\langle a \rangle \cap \langle 1 - a \rangle = \langle a \rangle \cdot \langle 1 - a \rangle = \langle 0 \rangle$. We thus have that:

$$\mathbb{V}(a) \cup \mathbb{V}(1 - a) = \mathbb{V}(0) = \operatorname{Spec} A$$

But this then implies that $\mathbb{V}(a)^c = \mathbb{V}(1-a)$, so we have that $\operatorname{Spec} A$ is the union of two open disjoint sets, and thus disconnected. It follows by the contrapositive that if $\operatorname{Spec} A$ is connected, then there are no nontrivial idempotents.

Now suppose $\operatorname{Spec} A$ is disconnected, then there exist open sets such that $U \cap V = \emptyset$, and $U \cup V = \operatorname{Spec} A$. It follows that U and V are both also closed so $U = \mathbb{V}(I)$ and $V = \mathbb{V}(J)$ for two radical ideals I and J. Now we have that $I + J = A$, and $I \cap J = \{0\}$ so by the Chinese remainder theorem there is an isomorphism:

$$A \to A/I \times A/J$$

It follows that A is a product of two rings A/I and A/J so $\operatorname{Spec} A$ is the disjoint union of two affine schemes. It follows that $(1,0)$ is a nontrivial idempotent of A, hence disconnected and affine implies the existence of an idempotent, and the claim follows from contradiction.

We thus wish to find a ring with no nontrivial idempotents and a nilradical which is not prime[48]. Consider $\mathbb{Z}[x]/\langle 2x\rangle$, then the nilradical contains $[0]$ but $[2]\cdot[x] = 0$ so the nilradical is not prime. It follows that $\operatorname{Spec} \mathbb{Z}[x]/\langle 2x\rangle$ is reducible, but there are no non trivial idempotents. Indeed, if $[p] \in \mathbb{Z}[x]/\langle 2x\rangle$ satisfies $[p]^2 = [p]$ then we have that $p^2 - p \in \langle 2x\rangle$, but the only way this can be true if $p^2 - p$ is divisible by $2x$ or is just actually equal to zero. This is only satisfied if $[p] = 0$ or if $p = 1$, hence $[p] = 1$. It follows that $\mathbb{Z}[x]/\langle 2x\rangle$ has no non trivial idempotents, and is thus connected but not irreducible.

We now turn to proving results regarding integral schemes. We have our first theorem of the section:

Theorem 3.2.1. *Let X be a scheme, then X is integral if and only if it is reduced and irreducible.*

Proof. Suppose that X is integral, then X is automatically reduced. Moreover, every open affine of X corresponds to $\operatorname{Spec} A$ where A is an integral domain, so every open affine is irreducible by Lemma 1.4.5. It follows by Lemma 3.2.4 that X is irreducible as well.

Now suppose that X is irreducible and reduced, then every open affine is irreducible and reduced, so we have that for each affine open $\operatorname{Spec} A$, A is an integral domain. Indeed, this implies that the generic point of A is the zero ideal, hence $\{0\}$ is a prime ideal implying that A is an integral domain. We now claim that the restriction map $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$ is injective whenever V is an open affine contained in U. Note that if $V \subset U$ is an open affine, then V is an open affine in X and is thus an integral scheme. Furthermore, for any affine scheme $\operatorname{Spec} A$ where A is an integral domain, the restriction maps $A \to \mathscr{O}_{\operatorname{Spec} A}(U)$ are injective, as for any cover of U by distinguished opens the localization maps $A \to A_g$ are injective. It follows that if $f \in A$ satisfies $f|_U = 0$, then $f|_{U_g} = 0$ so $f = 0$ as well. Now let W be an affine scheme such that $W \subset U$, then $f|_{W\cap V} = 0$ but this implies that $f|_W = 0$, as $f|_{W\cap V} = f|_W|_{W\cap V} = 0$. It follows that if $\{W_i, V\}$ is an open cover of U by affine schemes such that $f|_V = 0$ for then $f|_{W_i} = 0$ for all i as well. We thus have that $f = 0 \in \mathscr{O}_X(U)$ by sheaf axiom one. It follows that $\mathscr{O}_X(U)$ can be identified as a subring of the integral domain $\mathscr{O}_X(V)$, hence $\mathscr{O}_X(U)$ is an integral domain implying the claim. □

We now have the obvious corollary:

Corollary 3.2.1. *Let X be a scheme, then the following are equivalent:*

a) X is integral

b) There exists an affine cover $\{U_i\}$ of X such that $U_i \cap U_j \neq \emptyset$ and U_i is integral for all i.

c) Every open affine $U \subset X$ is integral

Proof. We have that $a \Rightarrow b$ as if X is integral then X is irreducible by Theorem 3.2.1, so there exists an affine open cover of X such that each U_i is irreducible and $U_i \cap U_j \neq \emptyset$. Since every affine open is reduced we have the claim by Theorem 3.2.1 as well.

For $a \Rightarrow c$, we see that every open set $\mathscr{O}_X(U)$ is an integral domain, so if $U = \operatorname{Spec} A$ is integral, we have that $\mathscr{O}_X(U) = A$ is an integral domain implying that $\operatorname{Spec} A$ irreducible by Lemma 1.4.5. Every affine open of U is reduced, so U is reduced, and irreducible implying that U is integral again by Theorem 3.2.1.

For $b \Rightarrow a$, note that each U_i is reduced and irreducible by Theorem 3.2.1, so by Lemma 3.2.4 we have that X irreducible, and by Lemma 3.2.1 we have that X is reduced. By Theorem 3.2.1, X is integral.

For $c \Rightarrow a$,the same argument holds. □

[48] As by the affine case in Lemma 3.2.3, if the nilradical is not prime then X is not irreducible.

Example 3.2.3. We claim that $\mathbb{P}^n_A$ is integral if A is integral domain. Indeed, we have an affine open cover by:

$$U_{x_i} = A[\{x_j/x_i\}_{j \neq i}]$$

such that $U_{x_i} \cap U_{x_j} = U_{x_i x_j} \neq \emptyset$. Each of these is integral, so we have that $\mathbb{P}^n_A$ is integral as well.

Proposition 3.2.1. *Let X and Y be integral schemes over an algebraically closed field k. If X is locally of finite type then $X \times_k Y$ is an integral scheme.*

Proof. It suffices to prove that for any affine opens $U = \operatorname{Spec} A \subset X$ and $V = \operatorname{Spec} B \subset Y$, that $A \otimes_k B$ is an integral domain. We first claim that the natural map:

$$A \longrightarrow \prod_{\mathfrak{m} \in |\operatorname{Spec} A|} A/\mathfrak{m}$$

is injective. Indeed, we can write $A \cong k[x_1, \dots, x_n]/I$ for some prime ideal I, and some $n \in \mathbb{N}$. The maximal ideals of A are then precisely the maximal ideals of $k[x_1, \dots, x_n]$ such that $I \subset \mathfrak{m}$. Suppose that $[f] \in A \mapsto (0_{\mathfrak{m}}) \in \prod_{\mathfrak{m} \in |\operatorname{Spec} A|} A/\mathfrak{m}$. Then we have that $f \in \mathfrak{m}$ for every $I \subset \mathfrak{m}$. By Hilbert's strong Nullstellensatz we have that there exists a k such that $f^k \in I$, but since I is prime we have that $f \in I$ hence $[f] = 0$.

Now note that for every $\mathfrak{m} \in |\operatorname{Spec} A|$, we have that $A/\mathfrak{m} \cong k$ as k is algebraically closed. For each $\mathfrak{m}$, let $\phi_{\mathfrak{m}}$ be the unique isomorphism $A \to k$ with kernel $\mathfrak{m}$, then we have the following chain of maps:

$$A \otimes_k B \longrightarrow A/\mathfrak{m} \otimes_k B \longrightarrow B$$

given on simple tensors by:

$$a \otimes b \longmapsto \phi_{\mathfrak{m}}([a]) \cdot b$$

Let:

$$x = \sum_i a_i \otimes b_i \qquad \text{and} \qquad y = \sum_i c_i \otimes d_i$$

be such that $x \cdot y = 0$. By the bilinearity of the tensor product, and the fact that A and B are both vector spaces, we can take $\{b_i\}$ and $\{d_i\}$ to be linearly independent sets over k. We see that for every $\mathfrak{m} \in |\operatorname{Spec} A|$:

$$x \cdot y \longmapsto (\phi_{\mathfrak{m}}([a_i])b_i) \cdot (\phi_{\mathfrak{m}}([c_i)]d_i) = 0$$

Since B is an integral domain, we have that it follows that either:

$$(\phi_{\mathfrak{m}}([a_i])b_i) = 0 \qquad \text{or} \qquad (\phi_{\mathfrak{m}}([c_i])d_i) = 0$$

Suppose the first summation is zero, then since $\{b_i\}$ is linearly independent, we have that $\phi_{\mathfrak{m}}([a_i]) = 0$ for all a_i. This implies that each $a_i \in \mathfrak{m}$ for all $\mathfrak{m}$. By the injectivity of the map $A_i \to \prod A_{\mathfrak{m}}$, it follows that each $a_i = 0 \in A$, hence:

$$x = \sum_i 0 \otimes b_i = 0$$

The same argument demonstrates that if the second sum is equal to zero, then $y = 0$, thus if $x \cdot y = 0$, we have that either $x = 0$ or $y = 0$ so $A \otimes_k B$ is an integral domain. ☐

3.3 Normal Schemes

Recall that if A is an integral domain, and $\eta = \langle 0 \rangle$ is the zero ideal, then $A_\eta = \operatorname{Frac}(A)$, that is the localization at the zero prime ideal is the field of fractions. This can be seen easily by noting that a), A_η is easily seen to be a field, and b), that the constructions of $\operatorname{Frac}(A)$ is identical to A_η. Further recall that if $A \subset B$, then B is an A algebra, and we say that $b \in B$ is *integral over* A, if there exists a monic polynomial $p \in A[x]$ such that $p(b) = 0$. We set the integral closure of A to be:

$$\bar{A} = \{b \in B : b \text{ is integral over } A\}$$

We now have the following definition:

Definition 3.3.1. Let A be an integral domain, then A is an **integrally closed domain** if $\bar{A} = A$, where A is being viewed as a subring of $\text{Frac}(A)$[49].

We have the following example:

Example 3.3.1. The integers are an integrally closed domain. Indeed, note that $\text{Frac}\,\mathbb{Z} = \mathbb{Q}$, clearly $\mathbb{Z} \subset \bar{\mathbb{Z}}$ as for any element in $a \in \mathbb{Z}$ we have that $x - a$ has a a root. Now let $a/b \in \bar{\mathbb{Z}}$, such that a and b have greatest common divisor equal to 1. Then their must exist some monic polynomial:

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with $a_i \in \mathbb{Z}$, such that $p(a/b) = 0$. It follows that:

$$a^n/b^n + a_{n-1}a^{n-1}/b^{n-1} + \cdots a_1 a/b + a_0 = 0$$

Multiplying throughout by b^n we obtain that:

$$a^n + a_{n-1}a^{n-1}b + \cdots a_1 ab^{n-1} + a_0 b^n = 0$$

however, since $a_{n-1}a^{n-1}b + \cdots a_1 ab^{n-1} + a_0 b^n$ is divisible by b, we must have that a^n is divisible by b. Since a and b both have unique factorizations into primes, it follows that a is divisible by b, a clear contradiction, implying the claim.

As a counter example, take $\mathbb{C}[x,y]/\langle x^2 - y^3 \rangle$. We first claim that this ring is isomorphic to $\mathbb{C}[t^2, t^3]$. Indeed, consider the ring homomorphism $\mathbb{C}[x,y] \to \mathbb{C}[t^2,t^3]$ given by $x \mapsto t^3$ and $y \mapsto t^2$, then we see that $x^2 - y^3 \mapsto t^6 - t^6 = 0$, so there is a unique ring homomorphism given by $[x] \mapsto t^3$ and $[y] \mapsto t^2$. We define an inverse by sending $t^3 \mapsto x$ and $t^2 \mapsto y$, and composing with the projection. This is easily seen to be an isomorphism, and $\mathbb{C}[t^2,t^3]$ is obviously an integral domain. It's field of fractions is the localization at the zero ideal, which contains $\mathbb{C}[t, t^{-1}]$, as $t^2 \cdot t^{-3} = t^{-1}$ and $t^3 \cdot t^{-2} = t$. However, t is integral over $\mathbb{C}[t^2,t^3]$ as it is the root of the polynomial $(\mathbb{C}[t^2,t^3])[\alpha]$ given by $\alpha^2 - t^2$.

We now develop a scheme theoretic analogue of the above construction:

Definition 3.3.2. Let X be a scheme, then X is **normal** if for all $x \in X$, the stalk $(\mathscr{O}_X)_x$ is an integrally closed domain.

We have the following (non)examples:

Example 3.3.2. We claim that $\mathbb{P}^n_{\mathbb{C}}$ is a normal scheme. Indeed, the $U_i = \text{Spec}(\mathbb{C}[x_0, \ldots, x_n]_{x_i})_0$ cover $\mathbb{P}^n_{\mathbb{C}}$, so suppose $x \in U_i$. Then x corresponds to a prime ideal $\mathfrak{p}$ of the ring $\mathbb{C}[\{x_j/x_i\}_{j \neq i}]$. Any polynomial ring is a unique factorization domain, and so is it's localization at $\mathfrak{p}$, so the argument that $\mathbb{Z}$ is an integrally closed domain holds pretty much verbatim for $\mathbb{C}[\{x_j/x_i\}_{j\neq i}]_{\mathfrak{p}}$, hence $\mathbb{P}^n_{\mathbb{C}}$ is normal.

As a counter example take $X = \text{Spec}\,\mathbb{C}[t^2,t^3]$, and consider the maximal ideal $\mathfrak{m} = \langle t^2, t^3 \rangle$. Then the stalk at $\mathfrak{m}$ does not invert t^2 or t^3, hence the same argument as in Example 3.3.1 demonstrates that X is not a normal scheme.

We now wish to describe a process in which we take an integral scheme X and normalize it. We first need the following definition:

Definition 3.3.3. Let $f : X \to Y$ be a morphism of schemes. Then f is **dominant** if $f(X)$ is a dense subset of Y.

We need the following lemma:

Lemma 3.3.1. *Let $f : X \to Y$ be a morphism of integral schemes, then the following are equivalent:*

a) f is dominant.

b) f takes the generic point of X to the generic point of Y.

c) For every open affines $U \subset X$, $V \subset Y$, such that $f(U) \subset V$, the ring homomorphism $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ is injective.

d) For all $x \in X$ the map of local rings $(\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ is injective.

Proof. Let $f : X \to Y$ be dominant and by Lemma 3.2.3 let $\eta_X \in X$ and $\eta_Y \in Y$ be the unique generic points. It follows that since f is dominant that $f(X) \subset Y$ is a dense subset. We first note that $f(X)$ is an irreducible subspace, as if $Z_1, Z_2 \subset f(X)$ are closed such that $Z_1 \cup Z_2 = f(X)$, then we can write

[49] Recall that the localization map for an integral domain is injective, so A is indeed a subring.

$Z_1 = W_1 \cap f(X)$, and $Z_2 = W_2 \cap f(X)$, hence $f(X) = (W_1 \cup W_2) \cap f(X)$, but then $f(X) \subset W_1 \cup W_2$, so $W_1 \cup W_2 = X$ as $f(X)$ is dense. Since Y is irreducible we must have that $W_1 = Y$ or $W_2 = Y$, either way it follows that $Z_1 = f(X)$ or $Z_2 = f(X)$. It follows that $f(X)$ must contain a unique generic point point η, and this point must also be a generic point for Y, so $\eta = \eta_Y$.

We now claim that $f(\eta_X) = \eta_Y$. Note that for any subset U we have that $f(\bar{U}) \subseteq \overline{f(U)}$. Indeed, if f is continuous, then $f^{-1}(\overline{f}(U))$ is closed, and since $f(U) \subset \overline{f(U)}$, we have that $f^{-1}(f(U)) \subset f^{-1}(\overline{f(U)})$, so $U \subset \overline{f}^{-1}(\overline{f(U)})$ implying that $\bar{U} \subset f^{-1}(\overline{f(U)})$, and finally that $f(\bar{U}) \subset \overline{f(U)}$. It follows that $f(X) = f(\bar{\eta_X}) \subset \overline{f(\eta_X)}$ which must be equal to Y as $f(X)$ is dense. It follows that $f(\eta_X)$ is dense, hence $f(\eta_X)$ must be η_Y. We thus have that $a \Rightarrow b$. Clearly if f takes the generic point of X to the generic point of Y then $f(X)$ is dense in Y so $b \Rightarrow a$ as well.

To see that $b \Rightarrow c$, let $U \subset X$, and $V \subset Y$ be affine opens such that $f(U) \subset V$. Then we have an induced morphism of affine schemes $f|_U : U \to V$. Since $\mathscr{O}_X(U)$ and $\mathscr{O}_Y(V)$ are integral domains, and $\eta_X \in U$ and $\eta_Y \in V$ both correspond to the zero ideal, we have that by b), $f|_U$ must come from a ring homomorphism ϕ satisfying $\phi^{-1}(\langle 0 \rangle) = \langle 0 \rangle$, hence $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ is injective. If this holds for all such open affine, then f must take the generic point to the generic point so $c \Rightarrow b$ as well.

For $c \Rightarrow d$, let $x \in U$ and $f(x) \in V$. Then writing $U = \operatorname{Spec} A$, and $V = \operatorname{Spec} B$, we let $x = \mathfrak{p}$, and $f(x) = \phi^{-1}(\mathfrak{p})$, where $\phi : B \to A$ is the ring homomorphism inducing $f|_U$. The map $(\mathscr{O}_Y)_{f(x)} \to (f_*\mathscr{O}_X)_{f(x)}$ is clearly injective, so it suffices to check that $(f_*\mathscr{O}_X)_{f(x)} \to (\mathscr{O}_X)_x$ is injective. Let $U_g \subset \operatorname{Spec} B$, and take $[U_g, s]_{\phi^{-1}(\mathfrak{p})} \in (f_*\mathscr{O}_X)_{f(x)} \cong ((f|_U)_*\mathscr{O}_{\operatorname{Spec} A})_{\phi^{-1}(p)}$, then we have that this maps to $[f|_U^{-1}(U_g), s]_{\mathfrak{p}} = [U_{\phi(g)}, s]_{\mathfrak{p}}$, where $\phi(g) \neq 0$. If this is zero, then there exists some distinguished open $U_h \subset U_{\phi(g)}$ such that $s|_{U_h} = 0$, but the restriction maps on an integral affine scheme are injective, so this implies $s = 0$, hence $[U_g, s] = 0$, hence $c \Rightarrow d$ as desired. To see that $d \Rightarrow c$, it suffices to reduce to the case of affine schemes, let $\phi : B \to A$ be the ring homomorphism inducing $\operatorname{Spec} A \to \operatorname{Spec} B$. The stalk map $(\mathscr{O}_{\operatorname{Spec} B})_{\phi^{-1}(\mathfrak{p})} \to (\mathscr{O}_{\operatorname{Spec} A})_{\mathfrak{p}}$ is then the localization of the map $B \to A_{\mathfrak{p}}$ at $\phi^{-1}(\mathfrak{p})$, which exists as $\phi(\phi^{-1}(\mathfrak{p})) \subset \mathfrak{p}$. We have the following commutative diagram:

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & A \arrow[d] \\
B_{\phi^{-1}(\mathfrak{p})} \arrow[r] & A_{\mathfrak{p}}
\end{tikzcd}$$

Since A and B are integral domains the vertical arrows are injective, and by hypothesis the bottom arrow is injective. It follows that if $\phi(b) = 0$, then $\phi(b)/1 \in A_{\mathfrak{p}}$ is zero, implying that $b/1 \in B_{\phi^{-1}(\mathfrak{p})}$ is zero hence $b \in B$ is zero. Therefore ϕ is injective as desired, so $c \Rightarrow d$. ☐

We have the following definition:

Definition 3.3.4. Let X be an integral scheme, then the **normalization of X** is the scheme $\tilde{X}$, equipped with a morphism $N : \tilde{X} \to X$, such that for every normal integral scheme Z, and every dominant $f : Z \to X$ the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "\exists! \tilde{f}"'] & X \\
\tilde{X} \arrow[ur, "N"'] &
\end{tikzcd}$$

where

As with every object defined this way we must show that such an object exists and is unique up to unique isomorphism. We do so now:

Theorem 3.3.1. *Let X be an integral scheme, then it's normalization, $\tilde{X}$ exists, and is unique up to unique isomorphism.*

Proof. If such an object exists it is obviously unique to up to unique isomorphism, as the morphism N we construct will be dominant so we need only check the universal property.

First consider the case where $X = \operatorname{Spec} A$ is affine, then we take $\tilde{X} = \operatorname{Spec} \tilde{A}$, where $\tilde{A}$ is the integral closure of A in $\operatorname{Frac}(A)$. This comes with a canonical injection map $A \to \tilde{A}$, so we get a dominant morphism $N : \operatorname{Spec} \tilde{A} \to \operatorname{Spec} A$. Now let Z be a normal integral scheme, and $f : Z \to \operatorname{Spec} A$ be a dominant morphism, then for every affine open $U \subset Z$, we have that $f|_U : U \to \operatorname{Spec} A$, is induced by an injective ring map. The homomorphism $A \to \mathscr{O}_Z(U)$ is given by the ring homomorphism $A \to \mathscr{O}_Z(Z)$ composed with restriction to $\mathscr{O}_Z(U)$. This second map is injective, and since the composition is injective, we must have that $A \to \mathscr{O}_Z(Z)$ is injective as well.

We want to show that the ring homomorphism $A \to \mathscr{O}_Z(Z)$ factors through the inclusion $A \to \tilde{A}$. We first show that $\mathscr{O}_Z(Z)$ is integrally closed. Let $a \in \operatorname{Frac}(\mathscr{O}_Z(Z))$ be integral over $\mathscr{O}_Z(Z)$, and let $\operatorname{Spec} B \subset Z$ be an affine open. Then since Z is integral, we have that $\mathscr{O}_Z(Z) \subset B$[50], so $\operatorname{Frac}(\mathscr{O}_Z(Z)) \subset \operatorname{Frac}(B)$. It follows that $a \in \operatorname{Frac}(B)$, and that a is integral over B. Let $I = \{b \in B : ab \in B\}$, if $I = B$, then $a \in B$ so we are done. If $I \neq B$, then $I \subset \mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset B$. We see that a is integral over $B_{\mathfrak{p}}$, and thus $a \in B_{\mathfrak{p}}$ as Z is normal. However, there then exists an $s \in B \setminus \mathfrak{p}$ such that $s \cdot a \in B$, implying that $s \in I$, contradicting the fact that $s \in B \setminus \mathfrak{p}$, so $I = B$. It follows that $a \in B = \mathscr{O}_Z(V)$. Cover Z with affine opens V_i, and the same argument shows that $b \in \mathscr{O}_Z(V_i)$ for all i. For all affine opens $V_{ijk} \subset V_i \cap V_j$, we have that we can identify $\mathscr{O}_Z(V_i)$ and $\mathscr{O}_Z(V_j)$ as subrings of $\mathscr{O}_Z(V_{ijk})$, so $b \in \mathscr{O}_Z(V_i)$ and $b \in \mathscr{O}_Z(V_j)$ both map to the same element in $\mathscr{O}_Z(V_{ijk})$[51]. Since the affine opens form a basis for the topology on Z, and thus determine a sheaf on a base, it follows that $b \in \mathscr{O}_Z(Z)$ so $\mathscr{O}_Z(Z)$ is indeed integrally closed.

It follows that since A injects into $\mathscr{O}_Z(Z)$, and $\mathscr{O}_Z(Z)$ is integrally closed, that $\tilde{A}$ injects into $\mathscr{O}_Z(Z)$ as well, thus we have a morphism $\tilde{A} \to Z$. Since A injects into $\tilde{A}$ we clearly have the following commutative diagram in the category of rings:

$$\begin{tikzcd}
\mathscr{O}_Z(Z) & A \arrow[l] \arrow[dl] \\
\tilde{A} \arrow[u] &
\end{tikzcd}$$

which yields the following commutative diagram in the category of schemes:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & \operatorname{Spec} A \\
\operatorname{Spec} \tilde{A} \arrow[ur] &
\end{tikzcd}$$

implying the result for affine integral schemes.

Now let X be an integral scheme, and $\{U_i = \operatorname{Spec} A_i\}$ be an open affine cover for X. Then we have isomorphisms $\beta_{ij} : U_{ij} \subset \operatorname{Spec} A_i \to \operatorname{Spec} A_j$ which agree on triple overlaps. For each i, set $\tilde{U}_i = \operatorname{Spec} \tilde{A}_i$, and let $N_i : \operatorname{Spec} \tilde{A}_i \to \operatorname{Spec} A_i$ be the normalization map. Finally set $\tilde{U}_{ij} \subset \operatorname{Spec} \tilde{A}_i$ to be $N_i^{-1}(U_{ij})$. We claim that $\tilde{U}_{ij}$ satisfies the universal property of the normalization of U_{ij}. Indeed, we have a morphism $N_i|_{\tilde{U}_{ij}} : \tilde{U}_{ij} \to U_{ij}$ which must be dominant as it sends the unique generic point of $\tilde{U}_{ij}$ to U_{ij}. Now let $f : Z \to U_{ij}$ be any dominant morphism from an integrally closed scheme Z, then the composition $\iota \circ f : Z \to \operatorname{Spec} A_i$ is dominant, and there is a unique morphism $g : Z \to \operatorname{Spec} \tilde{A}_i$ such that $N \circ g = \iota \circ f$. But this implies that $g(Z) \subset \tilde{U}_{ij}$, so g factors through the inclusion map$\tilde{U}_{ij} \to \operatorname{Spec} \tilde{A}_i$ implying that $\tilde{U}_{ij}$ is indeed the normalization of U_{ij}.

We want to show that there exist scheme isomorphisms $\phi_{ij} : \tilde{U}_{ij} \to \tilde{U}_{ji}$ which agree on triple overlaps. Fix the notation $N_i|_{\tilde{U}_{ij}} = N_{ij}$, and note that we have a dominant morphism $\beta_{ij} \circ N_{ij} : \tilde{U}_{ij} \to U_{ji}$. It follows that there is a unique morphism ϕ_{ij} such that the following diagram commutes:

$$\begin{tikzcd}
\tilde{U}_{ij} \arrow[r, "\beta_{ij} \circ N_{ij}"] \arrow[d, "\phi_{ij}"] & U_{ji} \\
\tilde{U}_{ji} \arrow[ur, "N_{ji}"] &
\end{tikzcd}$$

[50] Via the inclusion map.

[51] If one is unconvinced, then they can write out the restriction maps themselves, and find that this must be true by examining the induced injective maps $\operatorname{Frac}(\mathscr{O}_Z(Z)) \to \operatorname{Frac}(\mathscr{O}_Z(V_i))$.

Similarly, we have a morphism $\phi_{ji} : \tilde{U}_{ji} \to \tilde{U}_{ji}$ such that a similar diagram commutes. We thus claim that the following diagram commutes:

$$\begin{tikzcd} \tilde{U}_{ij} \arrow[r, "N_{ij}"] \arrow[d, "\phi_{ji}\circ\phi_{ij}"'] & U_{ij} \\ \tilde{U}_{ij} \arrow[ur, "N_{ij}"'] & \end{tikzcd}$$

Indeed, note that $N_{ij} \circ \phi_{ji} = \beta_{ji} \circ N_{ji}$, so:

$$\begin{aligned} N_{ij} \circ \phi_{ji} \circ \phi_{ij} =& \beta_{ji} \circ N_{ji} \circ \phi_{ij} \\ =& \beta_{ji} \circ \beta_{ij} \circ N_{ij} \\ =& N_{ij} \end{aligned}$$

so the diagram commutes. But the identity map also makes this diagram commutes so $\phi_{ji} \circ \phi_{ij} = \mathrm{Id}$, and similarly $\phi_{ij} \circ \phi_{ji}$ is the identity, implying that they are isomorphisms. It is easily seen by a similar argument that these morphisms agree on triple overlaps, as the β_{ij} agree on triple overlaps so the $\tilde{U}_i$ glue together to form an integral normal scheme $\tilde{X}$.

It follows that the N_i then also glue together to form a dominant morphism $N : \tilde{X} \to X$, such that $N|_{\tilde{U}_i} = N_i$. Given $f : Z \to X$ with f dominant and Z integral normal, we obtain an open cover of Z by $V_i = f^{-1}(U_i)$. Each of these schemes is normal integral, and the restriction is clearly dominant, so we obtain gives unique morphisms $V_i \to \tilde{U}_i$ which which clearly agree on $V_i \cap V_j$. These maps then glue to yield a unique dominant morphism $Z \to \tilde{X}$ such that the relevant diagram commutes, so $\tilde{X}$ is indeed the normalization of X. $\square$

3.4 Noetherian Schemes

We now turn to defining another important class of schemes, called Noetherian schemes, which again have an interesting interplay between the algebraic properties of their structure sheaf, and the topological properties of the total space. To begin, we review some commutative algebra:

Definition 3.4.1. Let A be a commutative ring, then A is **Noetherian** if every strictly increasing chain of ideals:

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

terminates. In other words, there exists some m such that $I_m = I_{m+k}$ for all $k \geq 0$.

Example 3.4.1. Any field is obviously Noetherian, any finite ring is also obviously Noetherian. Mildly more interestingly, $\mathbb{Z}$ is Noetherian. Indeed, every ideal of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$, so suppose we have the following infinite chain of ideals:

$$\langle n_1 \rangle \subset \langle n_2 \rangle \subset \cdots$$

We see that if $\langle n_1 \rangle \subset \langle n_2 \rangle$, then $n_1 \in \langle n_2 \rangle$, hence $n_1 = a \cdot n_2$ for some $a \in \mathbb{Z}$. It follows that n_2 divides n_1. If this is chain is infinite, then n_1 has infinitely many divisors, which is absurd implying the claim.

We have the following useful lemma which makes the example above a bit more immediate:

Lemma 3.4.1. *Let A be a ring, then A is Noetherian if and only if every ideal of A is finitely generated.*

Proof. Suppose that every ideal of A is finitely generated, and that:

$$I_1 \subset I_2 \subset \cdots$$

is a strictly increasing chain of ideals, and let:

$$I = \bigcup_i I_i$$

We claim that I is an ideal (no generating set needed!). Indeed, we see that if $0 \in I$, and that if $a, b \in I$ then $a \in I_i$ and $b \in I_j$ for some i and j. Without loss of generality suppose that $i \leq j$, then $a_i \in I_j$ so $a + b \in I_j$ hence $a + b \in I$. We see that I clearly contains all of it's inverses so I is a subgroup. Now let $a \in I$, and $b \in A$, then $a \in I_i$ for some i, and $a \cdot b \in I_i$ so $a \cdot b \in I$ as well implying that I is an ideal.

Since I is finitely generated, let $I = \langle a_1, \ldots, a_n \rangle$ for some $n \in \mathbb{Z}$. We have that each a_i lies in some I_{j_i} for some j_i, so let $j_k = \max(j_1, \cdots j_n)$, then since $I_{j_i} \subset I_{j_k}$ for all $i \in \{1, \ldots, n\}$ we must have that I_{j_k} contains each a_i. Let $j_k = m$, then it follows that $I \subset I_m$, so $I = I_m$ as $I_m \subset I$ by definition. For any $l \geq m$, we have that $I_m = I \subset I_l$ so the chain clearly terminates, and A is Noetherian.

Conversely, let $I \subset A$ be any ideal with minimal generating set $\{a_i\}_{i \in J}$ where J is a totally ordered set that is not finite. For any $j \in J$ we set $I_j = \{a_i\}_{i \leq j}$, and note that for any $j < k$, we have that $I_j \subset I_k$ and that this inclusion is strict. Indeed, if $I_j = I_k$ then for all $j < l \leq k$, we have that $a_l \in I_j$, implying that $a_l = \sum_{i \leq j} b_i a_i$ hence a_l is not a generating element of I, a contradiction, so $I_j \subset I_k$. We can label the initial segment of J with natural numbers regardless of it's cardinality, hence:

$$I_1 \subset I_2 \subset \cdots$$

is an infinite strictly increasing chain of ideals, so A is not Noetherian. The claim then follows by the contrapositive. □

We also have the following collection results:

Lemma 3.4.2. *Let A be a Noetherian ring then:*

a) If S is any multiplicatively closed subset then $S^{-1}A$ is Noetherian.

b) If $I \subset A$ is an ideal then A/I is Noetherian.

Proof. Let $I_S \subset S^{-1}A$ be an ideal, then we first claim that

$$I_S = S^{-1}I := \left\{ \frac{a}{s} : a \in I, s \in S \right\}$$

for some $I \subset A$. In particular, let $I = \pi^{-1}(I_S)$ where $\pi : A \to S^{-1}A$ is the localization map. Indeed, we have that:

$$S^{-1}\pi^{-1}(I_S) = \left\{ \frac{a}{s} : a \in \pi^{-1}(I_S), s \in S \right\}$$

Suppose that $a/s \in I_S$, then we have that $a/1 \in I_S$, so $a \in \pi^{-1}(I_S)$. It follows that $a/s \in S^{-1}\pi^{-1}$ giving us one inclusion. Now suppose that $a/s \in S^{-1}\pi^{-1}$, then $a \in \pi^{-1}$, so $a/1 \in I_S$ by definition. It follows that $a/s \in I$, by $a/1 \cdot 1/s = a/s$, hence $I = S^{-1}\pi^{-1}(I_S)$ implying the claim.

Since A is Noetherian, it follows that $\pi^{-1}(I_S)$ is finitely generated. In particular, since any of ideal of $S^{-1}A$ is generated by elements of the form $a/1$ as $1/s$ is invertible, we clearly see that $S^{-1}I$ is finitely generated as well. By the above paragraph, it follows that I_S is finitely generated, hence $S^{-1}A$ is Noetherian by Lemma 3.4.1 implying b).

Now let $I \subset A$ be an ideal. We see that if J is an ideal of A/I, then J is of the form $\pi(\pi^{-1}(J))$ as the quotient map $\pi : A \to A/J$ is surjective. We see that $\pi^{-1}(J)$ is finitely generated as A is Noetherian, so J itself must be finitely generated as well. Indeed suppose that $\{a_1, \ldots, a_n\}$ are generating elements of $\pi^{-1}(J)$, and let $[j] \in J$. We see that $j \in \pi^{-1}(J)$ can be written as $\sum_i b_i a_i$, hence $[j] = \sum_i [b_i][a_i]$, so $\{[a_1], \ldots, [a_n]\}$ generates J. It follows that every ideal of A/I is finitely generated, hence A/I is Noetherian by Lemma 3.4.1 implying b). □

The following results are some of the most famous results in commutative algebra, the first of which is known as the Hilbert Basis theorem.

Theorem 3.4.1. *Let A be a ring, then $A[x_1, \ldots, x_n]$ is Noetherian if and only if A is Noetherian.*

Proof. We see that if $A[x_1, \ldots, x_n]$ is Noetherian, then $A[x_1, \ldots, x_n]/\langle x_1, \ldots, x_n \rangle \cong A$ must be Noetherian by Lemma 3.4.3.

Now suppose that A is Noetherian, since we trivially have that $A[x, y] \cong (A[x])[y]$, it suffices by an induction argument to show that $A[x]$ is Noetherian. Let $I \subset A[x]$ be an ideal, we will show that I is finitely generated. We have a partial order on I, by writing:

$$f = a_n x^n + \cdots + a_1 x^1 + a_0 \qquad g = b_k x^k + \cdots + b_1 x^1 + b_0$$

and saying that $f \leq g$ if and only if $n \leq k$, we call n and k the degree of f and g respectively, and write it as $\deg f$. Choose an element of least degree $f_0 \in I$, i.e. an element f_0 such that there is no g in I satisfying $\deg g < \deg f$. If $\langle f_0 \rangle = I$ we are done, if not, then we choose an element f_2 in $I \setminus \langle f_0 \rangle$ of least degree. We perform this recursively obtaining a sequence[52] $\langle f_0, f_1, \dots \rangle \subset I$. For each f_i, let $a_{\deg f_i}$ be the leading coefficient of f_i, and consider the ideal $J = \langle a_{\deg f_0}, \dots \rangle \subset A$. Then, since A is Noetherian, we know that the sequence:

$$\langle a_{\deg f_0} \rangle \subset \langle a_{\deg f_0}, a_{\deg f_1} \rangle \subset \cdots$$

terminates, so for some $m \geq 0$, we have that this chain must terminate with $\langle a_{\deg f_0}, \dots, a_{\deg f_m} \rangle$ implying that $J = \langle a_{\deg f_0}, \dots, a_{\deg f_m} \rangle$. We claim that $I = \langle f_0, \dots, f_m \rangle$. Suppose otherwise, then by construction $f_{m+1} \notin \langle f_0, \dots, f_m \rangle$, but $a_{\deg f_{m+1}} \in J$, so we can write:

$$a_{\deg f_{m+1}} = \sum_{i=0}^{m} a_{\deg f_i} b_i$$

for some $b_i \in A$. Define g by:

$$g = \sum_i b_i f_i x^{\deg f_{m+1} - \deg f_i}$$

Note that this clearly lies in $\langle f_0, \dots, f_m \rangle$, but this element as the same degree as f_{m+1} with $a_{\deg g} = a_{\deg f_{m+1}}$. We thus see that $f_{m+1} - g$ has degree strictly less than f_{m+1}, and that $f_{m+1} - g \notin \langle f_0, \dots, f_m \rangle$, so $f_{m+1} - g$ is the minimal element of $I \setminus \langle f_0, \dots, f_m \rangle$, a contradiction. It follows that $I = \langle f_0, \dots, f_m \rangle$, so every ideal of $A[x]$ is finitely generated and thus by Lemma 3.4.1 we have that $A[x]$ is Noetherian. □

We now have the following obvious corollary:

Corollary 3.4.1. *Let A be a Noetherian and B be any finitely generated A algebra, then B is Noetherian.*

To prove our second famous result, we need to extend the idea of a Noetherian ring to modules.

Definition 3.4.2. Let M be an A module, then M is Noetherian if for every strictly increasing chain of submodules:

$$N_1 \subset N_2 \subset \cdots$$

terminates.

We prove the following analogue Lemma 3.4.1

Lemma 3.4.3. *Let M be an A module, then the following hold:*

a) M is Noetherian if and only if every submodule is finitely generated.

b) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence, then M_2 is Noetherian if and only if M_1 and M_3 are.

c) If A is Noetherian, and M is finitely generated then M is Noetherian.

Proof. We begin with a). Suppose that every submodule of M finitely generated, and consider the following sequence of submodules:

$$N_1 \subset N_2 \subset \cdots$$

Let:

$$N' = \bigcup_i N_i$$

Then this has finitely many generators $(m_1, \dots, m_n)$ for some n, and each must lie in N_i for some i, so choose the largest such i, and call it k. We have that $(m_1, \dots, m_n) \subset N_k$ essentially by construction, $N' = N_k$. It follows that for any $l \geq k$, we have that $N_l \subset N' = N_k$, so for all $l \geq k$ we have that $N_l = N_k$, so M is Noetherian.

[52] This is equivalent to using the axiom of dependent choice.

Now suppose that M is Noetherian, and let N be a submodule which is not finitely generated. Let $\{m_i\}_{i\in I}$ be the minimal generating set of N where I is a totally ordered set of any cardinality. For any $j \in I$ let N_j be the submodule generated by the elements $\{m_i\}_{i\leq j}$, then for any $k < j$, we have that $N_k \subseteq N_j$. If $N_k = N_j$ then for each $k < l \leq j$, we have that m_l can be written as a linear combination of $\{m_i\}_{i\leq k}$, hence m_l is not a generating element. It follows that N_k is a strict subset of N_j for each $k < j$. Since we can write the initial segment of any totally ordered set as the natural numbers, we have that:

$$N_1 \subset N_2 \subset N_3 \subset \cdots$$

is a strictly increasing chain of ideals which does not terminate, hence M is not Noetherian, a contradiction. It follows that every submodule of M (including M itself) must be finitely generated, thus we have $a)$.

Now suppose that we have an exact sequence:

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

is an exact sequence of A modules. If M_2 is Noetherian, then we have that $M_3 \cong M_2/\ker g$ so M_3 is Noetherian, as every submodule of M_3 must be finitely generated. Moreover, every submodule of M_1 is a submodule of M_2, so we have that every submodule of M_1 is finitely generated hence M_1 is also Noetherian.

Let M_1 and M_3 be Noetherian modules, and consider the following chain of strictly increasing submodules of M_2:

$$N_1 \subset N_2 \subset \cdots$$

Then we have that:

$$f^{-1}(N_1) \subset f^1(N_2) \cdots \qquad \text{and} \qquad g(N_1) \subset g_2(N_2) \subset \cdots$$

are are strictly increasing chains of ideals in M_1 and M_3 respectively. Since M_1 and M_3 it follows that there exist n_1 and n_3 such that $f^{-1}(N_{n_1})$ and $g(N_{n_3})$ make the above chains terminate. Without loss of generality let $n_3 > n_2$, and denote n_3 by n. Then we claim that for all $k > n$, $N_k = N_n$. Indeed consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f^{-1}(N_n) & \xrightarrow{f} & N_n & \xrightarrow{g} & g(N_n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & f^{-1}(N_k) & \xrightarrow{f} & N_k & \xrightarrow{g} & g(N_k) & \longrightarrow & 0
\end{array}$$

The vertical arrows are inclusion maps, so the leftmost and rightmost arrows are the identities. We want to show that the middle arrow is the identity as well, and it suffices to show that $N_k \subset N_n$. Let $m \in N_k$, and consider $g(m) \in g(N_k)$. Since the right most arrow is the identity, we have that $g(m) \in g(N_n)$, since g is surjective there exists an element $l \in N_n$ which maps to $g(m)$. Let $\iota : N_n \to N_k$ denote the inclusion map, then since:

$$g(\iota(l)) = g(m)$$

It follows that $\iota(l) - m \in \ker g$, but the kernel of g is the image of f, so we have that there exists an $\eta \in f^{-1}(N_k)$ such that $f(\eta) = \iota(l) - m$. Since the left most arrow is the identity, we have that $\eta \in f^{-1}(N_k)$, so $f(\eta) \in N_n$. It follows that $\iota(l) - m \in N_n \subset N_l$ hence $m \in N_n$ as well so $N_k = N_n$, and the middle arrow is the identity. We thus have that N_n makes the chain terminate implying $b)$.

To prove $c)$, let A be a Noetherian ring, and suppose that M is finitely generated. Then M is a quotient of the free module A^n for some n, and so it suffices to check that A^n is a Noetherian A-module. Note that every submodule of A is by definition of an ideal, as it is a subgroup and swallows multiplication, so A is Noetherian as a module over itself as well. We proceed by induction, suppose that A^n is Noetherian, then we have the following short exact sequence:

$$0 \longrightarrow A \xrightarrow{f} A^{n+1} \xrightarrow{g} A^n \longrightarrow 0$$

Since A^n is Noetherian and A are Noetherian, it follows by $b)$ that A^{n+1} is Noetherian implying $c)$ as desired.

$\square$

We are now in position to prove the following result, known as the Artin-Tate lemma:

Theorem 3.4.2. *Let $A \subset B \subset C$ be rings where A is Noetherian, C is finitely generated over A, and C is a finite B module. Then B is finitely generated as an A algebra.*

Proof. Let $\{x_1, \ldots, x_n\}$ be the generators of C as an A algebra, and let $\{y_1, \ldots, y_m\}$ be the generators of C as a B module. Then we have that for some $b_{ij}, b_{ijk} \in B$ that:

$$x_i = \sum_j b_{ij} y_j \qquad \text{and} \qquad y_i y_j = \sum_k b_{ijk} y_k \tag{3.4.1}$$

Let B_0 be the A algebra generated by $\{b_{ij}, b_{ijk}\}$. By Corollary 3.4.1, we have that B_0 is Noetherian, and we have that $A \subseteq B_0 \subseteq B$.

It is clear that C is a B_0-algebra, so we claim that C is finite over B_0, i.e. is a finitely generated B_0 module. Every element $c \in C$ can be written as:

$$c = \sum_{i_1 \cdots i_n} a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

By making repeated use of the equations in (2.4.1) we can rewrite this in terms of a linear combination of y_k and elements of B_0, hence C is a finitely generated B_0 module. It follows from Lemma 3.4.2 part c) that C is a Noetherian B_0 module, so every submodule of C is finitely generated. We thus have that B is a finitely generated B_0 module, and thus a finitely generated A algebra as desired. □

After our brief detour into commutative algebra, we are now ready to dive back into scheme theory. It should be no surprise that the class of schemes we are about to study are intimately related to Noetherian rings. We begin with the following definition:

Definition 3.4.3. Let X be a topological space, then X is Noetherian if every decreasing sequence of closed subsets:

$$Y_1 \supset Y_2 \supset \cdots$$

terminates. In other words there exits an integer m such that for all $k \geq m$ we have $Y_m = Y_k$.

Example 3.4.2. Let A be a Noetherian ring, then $\operatorname{Spec} A$ is a Noetherian topological space. Indeed, any descending sequence of closed subsets can be written uniquely as a sequence of the vanishing locus of radical ideals I_k:

$$\mathbb{V}(I_1) \supset \mathbb{V}(I_2) \supset \cdots$$

This then corresponds to an increasing sequence of ideals:

$$I_1 \subset I_2 \subset \cdots$$

which must terminate as A is Noetherian. It follows that the chain $\mathbb{V}(I_1) \supset \mathbb{V}(I_2) \supset \cdots$ must terminate as well.

Note that not every affine scheme which is a Noetherian topological space comes from a Noetherian ring. Indeed consider the infinite polynomial ring $A = k[x_1, x_2, \ldots]/\left\langle x_1^2, x_2^2, \ldots \right\rangle$ over a field k. Every prime ideal must contain the nilpotents $[x_i]$ for all i, so the only prime is given by $\mathfrak{p} = \langle [x_1], [x_2], \ldots \rangle$ implying that $\operatorname{Spec} A$ is a single point and thus Noetherian. It clear that A is not Noetherian as $\mathfrak{p}$ is not finitely generated.

Lemma 3.4.4. *Let X be a Noetherian topological space, then every non empty closed subset $Z \subset X$ can be expressed uniquely as $Z = Z_1 \cup \cdots \cup Z_n$ where each Z_n is an irreducible closed subspace, and for all i, j we have that $Z_i \not\subset Z_j$.*

Proof. Suppose there exists a closed subset Y_1 that cannot be expressed as a finite union of irreducible closed subspaces. If Y_1 contains another such closed subset Y_2, then we have that $Y_1 \supset Y_2$. We can repeat this process ad infinitum, but since X is Noetherian, we must have that this chain eventually terminates for some Y_r. Now since this chain terminates, it follows that every proper closed subset of Y_r can be written as the finite union of irreducible closed subspaces. We see that Y_r is not irreducible as other wise it is trivially a finite of union of irreducible closed subspaces, hence $Y_r = W_1 \cup W_2$ for proper closed subsets of Y_r. However, W_1 and W_2 can be written as a finite union of irreducible closed subsets, a contradiction. It follows that every closed subset of X can thus be written as a finite union of irreducible closed subsets, and by discarding those that satisfy $W_i \subset W_j$, we have that every closed subset of X can be written as a finite union of irreducible closed subspaces none of which fully contain each other.

To show uniqueness, suppose that:

$$Z = Z_1 \cup \cdots \cup Z_n = Y_1 \cup \cdots \cup Y_m$$

where Z_i and Y_j are irreducible closed subspaces none of which contain the other. It follows that for any $Z_1 \subset Y_1 \cup \cdots \cup Y_m$, so $Z_1 = (Y_1 \cap Z_1) \cup \cdots \cup (Y_m \cap Z_1)$, but then for some i we have that $Z_1 = Y_i \cap Z_1$ as Z_1 is irreducible. It follows that $Z_1 \subset Y_i$, and similarly for some j we have that $Y_i \subset Z_j$, but then $j = 1$ as we have that $Z_1 \subset Y_i \subset Z_j$ and Z_1 is only contained in Z_1. It follows that $Y_i = Z_1$, so repeating this process for all $1 \leq i \leq n$ we have that the lists are the same, implying the claim. □

Definition 3.4.4. Let X be a scheme, then X is **locally Noetherian** if there exists a cover $\{U_i\}$ of X by affine schemes such that each U_i is the spectrum of a Noetherian ring. Moreover, X is **Noetherian** if it can be covered by finitely many such affine schemes.

Example 3.4.3. Lemma 3.4.3 demonstrates that every affine scheme $\operatorname{Spec} A$ where A is Noetherian is Noetherian.

Lemma 3.4.5. *A topological space X is Noetherian if and only if every subspace of X is quasi-compact. In particular, X is quasi-compact, and every subspace of X is Noetherian.*

Proof. Let $Y \subset X$ be a subset equipped with subspace topology, and $\{U_i \cap Y\}_{i \in I}$ be an open cover of Y. Consider the set:

$$\mathscr{U} = \{\text{finite unions of elements in } \{U_i\}\}$$

and equip this set with the partial order given by $V \leq W$ if and only if $V \subset W$. Consider an ascending chain of elements in $\mathscr{U}$:

$$V_1 \subset V_2 \subset \cdots$$

then we obtain the descending chain of closed subsets of X:

$$V_1^c \supset V_2^c \supset \cdots$$

which must terminate for some m as X is Noetherian. By Zorn's lemma, there must then be a maximal element of $\mathscr{U}$, call it W. Then we have that for some $\{i_1, \ldots, i_n\}$

$$W = U_{i_1} \cup \cdots \cup U_{i_n}$$

and moreover that:

$$W \cap Y = (U_{i_1} \cap Y) \cup \cdots \cup (U_{i_n} \cap Y)$$

Suppose that $Y \not\subset W$, then there is a $y \in Y$ such that $y \notin W$. However, $\{U_i \cap\}_{i \in I}$ covers Y, so for some $k \in I$, we have that $y \in U_k$. It follows that $W \subset W \cup U_k$, contradicting the fact that W is maximal, hence $Y \subset W$. Therefore, $Y = W \cap Y$, and the set $\{U_{i_j} \cap Y\}_{j=1}^n$ is a finite subcover, so Y is quasi-compact. In particular, we have that X is quasi-compact.

Now suppose that every subspace of X is compact, and let:

$$V_1 \supset V_2 \supset \cdots$$

be a descending chain of closed subsets of X. Then we obtain an ascending chain of open sets:

$$U_1 \subset U_2 \subset \cdots$$

by letting $U_i = V_1^c$. Consider the open subspace:

$$U = \bigcup_{i=1}^{\infty} U_i$$

which has an open cover given by $\{U_i\}_{i\in\mathbb{N}}$. Since U is quasi-compact, this subspace has an open cover given $U_{i_1} \cup \cdots U_{i_n}$ for some $\{i_1, \cdots i_n\} \subset \mathbb{N}$. Via reordering we can assume that $U_{i_1} \subset \cdots \subset U_{i_n}$, so $U = U_{i_n}$. We claim that the ascending chain stabilize with U_{i_n}. Indeed suppose that $m > i_n$, then $U_{i_n} \subset U_m$, however, by construction, $U_m \in U$, so $U_m = U_{i_n}$. By taking compliments again we obtain that the descending chain of closed subsets:

$$V_1 \supset V_2 \supset \cdots$$

stabilizes so X is Noetherian.

Now finally let Y be a subspace of a Noetherian topological space X. Let $W \subset Y$, then the subspace topology on W induced from Y is the same as the one induced from X. That is, $U \subset W$ is open in the subspace topology if and only $U = Y \cap V$ for some open set $V \subset Y$. However, V is open in Y if and only if $V = X \cap Z$ for some Z open in X. It follows that U is open in W if and only if $U = Y \cap V = X \cap Y \cap Z = X \cap Z$, hence the topologies are equivalent. Since W is quasi-compact as a subspace of X, it follows that W is quasi-compact as subspace Y, hence Y must be Noetherian by argument above. □

Proposition 3.4.1. *Let X be a Noetherian scheme, then X is a Noetherian topological space*

Proof. By Example 3.4.2 we have that X is the union of finitely many Noetherian topological spaces, so it suffices to prove that any such topological space is Noetherian. Let $\{U_i\}_{i\in I}$ be the finite cover of X by Noetherian affine schemes, and suppose that:

$$Y_1 \supset Y_2 \supset \cdots$$

is a descending chain of closed subsets. Then for each i we have that there exists an m_i such that the following chain terminates at m_i:

$$Y_1 \cap U_i \supset Y_2 \cap U_i \supset \cdots \supset Y_{m_i} \cap U_i$$

Take $\max\{m_i\}_{i\in I}$ which exists as I is a finite set, and let m be the maximum number. Then we claim that for any $k \geq m$ we have $Y_m = Y_k$. Indeed, we can write:

$$Y_m = \bigcup_i Y_m \cap U_i = \bigcup_i Y_k \cap U_i = Y_k \tag{3.4.2}$$

as the $\{U_i\}$ cover X, implying the claim. □

As Example 3.4.2 shows, the converse does not hold. We continue to prove topological properties of Noetherian schemes:

Lemma 3.4.6. *Let X be a Noetherian scheme, then X has finitely many irreducible components. In particular, X a finite number of connected components, each of which is the finite union of irreducible components.*

Proof. Note that any irreducible component is closed. Indeed, $Z \subset X$ is irreducible then clearly so is $\bar{Z}$, so it follows that Z is maximal that $Z = \bar{Z}$ as Z is by definition a subset of $\bar{Z}$. Since X is a Noetherian topological space by Proposition 3.4.1, and by Lemma 3.4.4 we have that every closed subset of X can be written as the finite union of irreducible closed subsets it follows that:

$$X = Z_1 \cup \cdots \cup Z_n$$

where each Z_i irreducible. Let $\{Y_i\}$ be the set of irreducible components, then since each Z_i must be contained in one of these irreducible components, it follows that:

$$X = \bigcup_i Y_i$$

However, this is a decomposition of X into irreducible closed subspaces, each of which is not contained in the other as they are all maximal. It follows that each Y_i must be equal to some Z_j for some i and j by the uniqueness part of Lemma 3.4.4, hence there can only be finitely many irreducible components.

Let $\{X_i\}$ be the set of connected components, then since each X_i is closed we have that by Lemma 3.4.4:

$$X_i = Z_{1_i} \cup \cdots \cup Z_{n_i}$$

for irreducible closed subsets of X_i. It follows that:

$$X = \bigcup_i Z_{1_i} \cup \cdots \cup Z_{n_i}$$

which must be a finite union as X is Noetherian, implying there are only finitely many X_i. It follows that each X_i must be a finite union of irreducible components Y_i of X by uniqueness of the decomposition of X into irreducible components, again by Lemma 3.4.4, implying the claim. □

It turns out we can check the locally Noetherian condition affine locally (hence the name):

Proposition 3.4.2. *Let X be a scheme, then X is locally Noetherian if and only if every open affine is Noetherian.*

Proof. If every open affine is Noetherian, then clearly X is locally Noetherian.

Suppose that $\{U_i = \operatorname{Spec} A_i\}$ is an affine open cover of X with A_i Noetherian for all i, and $V = \operatorname{Spec} B \subset X$ be any open affine. Then we obtain an open cover of V by $\{V \cap U_i\}$, and for each of these there is an open cover by distinguished opens $U_f \subset \operatorname{Spec} B$ and $U_g \subset \operatorname{Spec} A_i$. Since the schemes $U_i \cap V \subset \operatorname{Spec} A_i$ and $U_i \cap V \subset \operatorname{Spec} B$ are clearly isomorphic (just take the identity), it follows that V admits a cover of distinguished opens all of which are Noetherian schemes. In particular, we have that there exists a finite set of elements $\{f_i\}$ of B which generate the unit ideal $\langle 1 \rangle$ such that for all i B_{f_i} is a Noetherian ring.

Now let $I \subset B$ be an ideal, and let $\pi_i : A \to A_{f_i}$ be the localization map. If I_{f_i} is the localized ideal in B_{f_i} then we claim that:

$$I = \bigcap_i \pi_i^{-1}(I_{f_i})$$

For each i we have that $I \subset \pi_i^{-1}(\pi_i(I))$ so it follows that $I \subset \bigcap_i \pi_i^{-1}(I_{f_i})$. Now let $b \in \bigcap_i \pi_i^{-1}(I_{f_i})$, then for each i we have that $\pi_i(b) \in I_{f_i}$, so we have that for some $a_i \in I$, and some integer m_i:

$$\frac{b}{1} = \frac{a_i}{f_i^{m_i}}$$

It follows that there exists an M_i such that $f_i^{M_i} b \in I$. Let M be the maximum of all such M, then since $\langle \{f_i\} \rangle = \langle 1 \rangle$, we have that $\langle \{f_i^M\} \rangle = \langle 1 \rangle$ so we there exist c_i in B such that:

$$1 = \sum_i c_i f_i^M$$

hence:

$$b = \sum_i c_i f_i^M b \in I$$

Now suppose that:

$$I_1 \subset I_2 \subset \cdots$$

is an increasing chain of ideals, then for each i we have that:

$$I_{1_{f_i}} \subset I_{2_{f_i}} \subset \cdots$$

terminates for some m_{f_i}. Let m be the maximum of all such m_{f_i}, then for all $k > m$ and all i, we have that $I_{k_{f_i}} = I_{m_{f_i}}$. It follows that for all $k > m$ we have that:

$$I_k = \bigcap_i \pi_i^{-1}(I_{k_{f_i}}) = \bigcap_i \pi_i^{-1}(I_{m_{k_{f_i}}}) = I_m$$

so the chain in B terminates with I_m, implying that B is Noetherian, and that V is Noetherian. □

We have the following corollary:

Corollary 3.4.2. *Let X be a scheme, then X is Noetherian if and only if it is quasi-compact, and for every affine open $\mathscr{O}_X(U)$ is a Noetherian ring.*

The condition that X is Noetherian is in a sense a finiteness condition that allows us to prove some striking results. Often times we will restrict to the case where we deal with Noetherian or locally Noetherian schemes, as they are easier to work with, and the condition is actually quite a reasonable one. As an example, note that we showed that X is a reduced scheme if and only if all of it's stalks have no nontrivial nilpotents. The astute reader will recognize that we did not have a similar equivalent condition for a scheme to be integral. As the following theorem shows, we can deduce such a result if we work with sufficiently nice schemes:

Theorem 3.4.3. *Let X be a connected and Noetherian scheme, then X is integral if and only if the stalk $(\mathscr{O}_X)_x$ is an integral domain.*

Proof. Note that if X is integral then stalks are integral domains.

Conversely, suppose that X is a connected Noetherian scheme, such that all the stalks are integral domains. Then all the stalks also contain no nontrivial nilpotents hence X is reduced by Lemma 3.2.1.

By Theorem 3.2.1 need only show that X is irreducible. As X is connected we have only one connected component, and by Lemma 3.4.6 we have that X has finitely many irreducible components. Let X have a decomposition into:

$$Z_1 \cup \cdots \cup Z_n$$

where each Z_i is an irreducible component. We see that if $Z_1 \cap Z_j = \emptyset$ for all j then Z_1 is open as its compliment is the finite union of closed subset. Since Z_1 is irreducible and thus connected, it follows that either $n = 1$ and $Z_1 = X$ so we are done, or that Z_1 and $Z_2 \cup \cdots \cup Z_n$ are disjoint open sets that cover X so X is disconnected. It follows that if $n \neq 1$, every irreducible component of X must intersect with at least one other irreducible component.

Suppose that $n \neq 1$, then there exist irreducible components Z and Y such that $Z \cap Y \neq \emptyset$. Let $x \in Z \cap Y$ and let $U = \operatorname{Spec} A$ be a affine open containing x. Note that if for all $x \in Z \cap Y$ and all $U = \operatorname{Spec} A$ we have that $Z \cap \operatorname{Spec} A = Y \cap \operatorname{Spec} A$, we can conclude that $Y = Z$, so without loss of generality assume that $Z \cap \operatorname{Spec} A \neq Y \cap \operatorname{Spec} A$. By Lemma 3.2.2, we have that $Z \cap \operatorname{Spec} A$ and $Y \cap \operatorname{Spec} A$ are irreducible closed subsets of $\operatorname{Spec} A$. We claim that they are irreducible components, indeed, suppose that there was an irreducible closed subset $S \subset \operatorname{Spec} A$ such that $Z \cap \operatorname{Spec} A \subset S$, then the closure of S in X is an irreducible closed subset of X containing the closure of $Z \cap \operatorname{Spec} A$, however this is equal to $\bar{Z} = Z$ contradicting the fact that Z is irreducible. It follows that $Z \cap \operatorname{Spec} A$ and $Y \cap \operatorname{Spec} A$ are irreducible components of $\operatorname{Spec} A$.

Now let x correspond to the prime ideal $\mathfrak{p} \subset A$, $Z \cap \operatorname{Spec} A = \mathbb{V}(I)$, and $Y \cap \operatorname{Spec} A = \mathbb{V}(J)$ for radical ideals $I \neq J \subset A$. We claim that I and J are minimal prime ideals over $\langle 0 \rangle$, in the sense that a) they are prime ideals, and b) for every prime ideal we have that if $\mathfrak{q} \subset I$ then $I = \mathfrak{q}$. Let $a, b \in A$ such that $a \cdot b \in I$, then we have that:

$$U_{ab} \cap \mathbb{V}(I) = (U_a \cap \mathbb{V}(I)) \cap (U_b \cap \mathbb{V}(I)) = \emptyset$$

Since $\mathbb{V}(I)$ is irreducible, it follows that either $U_a \cap \mathbb{V}(I)$ or $U_b \cap \mathbb{V}(I)$ are empty, hence either $a \in I$ or $b \in I$ so I and J are both prime. To see that they are minimal, suppose that there exists a prime ideal $\mathfrak{q} \subset I$, then $\mathbb{V}(I) \subset \mathbb{V}(\mathfrak{q})$, but by reversing the argument above we have that $\mathbb{V}(\mathfrak{q})$ is an irreducible closed subset so it follows that $\mathbb{V}(I) = \mathbb{V}(\mathfrak{q})$ as $\mathbb{V}(I)$ is maximal. We thus have that $I = \sqrt{I} = \sqrt{\mathfrak{q}} = \mathfrak{q}$ so I and J are both minimal prime ideals over $\langle 0 \rangle$.

We see that A is not an integral domain. Indeed, if A were an integral domain, then $\langle 0 \rangle$ is the unique minimal prime ideal over $\langle 0 \rangle$. In particular, there is a bijection between prime ideals which are contained in $\mathfrak{p}$ and prime ideals of $A_{\mathfrak{p}}$, hence we must have that there $I_{\mathfrak{p}}$ and $I_{\mathfrak{q}}$ are minimal primes of $A_{\mathfrak{p}}$. It follows that $A_{\mathfrak{p}} \cong (\mathscr{O}_X)_x$ is not an integral domain, a contradiction, hence we must have that $n = 1$, implying that X is irreducible, so X is reduced and irreducible and thus by Theorem 3.2.1 an integral scheme as desired. □

3.5 Morphisms of Finite Type

Recall that in Definition 2.3.4 we defined what it meant for a k-scheme to be locally of finite type. We now extend this definition to arbitrary schemes as follows:

Definition 3.5.1. Let $f : X \to Y$ be a morphism of schemes. Then f is **locally of finite type** if there exists an affine open cover $\{V_i = \operatorname{Spec} B_i\}$ of Y, such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$ where A_{ij} is a finitely generated B_i algebra. The morphism is of **finite type** if the cover of $f^{-1}(V_i)$ is finite.

We have the following obvious examples:

Example 3.5.1. Let A be a finitely generated B algebra, then $\operatorname{Spec} A \to \operatorname{Spec} B$ is obviously of finite type. Let X be a k-scheme of locally finite type, and $f : X \to \operatorname{Spec} k$ the morphism making X a k-scheme, then f is also trivially locally of finite type. If we can take X to be Noetherian k-scheme of locally finite type, then we also have that f is of finite type.

We now show that being locally of finite type is local on target:

Proposition 3.5.1. *Let $f : X \to Y$ be a morphism of schemes, then f is locally of finite type if and only if for every affine open $V \subset Y$ we have that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is of locally finite type.*

Proof. Clearly we have that if for every affine open $V \subset Y$ the morphism $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is locally of finite type then f is.

Now suppose that $f : X \to Y$ is a morphism of locally finite type. Let $\{V_i = \operatorname{Spec} B_i\}$ be an open cover for Y, and for each i, let $\{U_{ij} = \operatorname{Spec} A_{ij}\}$ be an affine open cover for $f^{-1}(V_i)$. Let $V = \operatorname{Spec} B$ be any affine open, then we can write:

$$V = \bigcup_i V_i \cap V$$

hence:

$$\begin{aligned} f^{-1}(V) =& \bigcup_i f^{-1}(V_i) \cap f^{-1}(V) \\ =& \bigcup_{i,j} U_{ij} \cap f^{-1}(V) \end{aligned}$$

Now note that $U_{ij} \cap f^{-1}(V) \subset U_{ij} \cong \operatorname{Spec} A_{ij}$, thus there exist elements $f_{ijk} \in A_{ij}$ such that:

$$U_{ij} \cap f^{-1}(V) = \bigcup_k U_{f_{ijk}}$$

We note that $U_{f_{ijk}} \cong \operatorname{Spec}(A_{ij})_{f_{ijk}}$, hence doing this for all i and j we have obtained an affine open cover:

$$f^{-1}(V) = \bigcup_{i,j,k} U_{f_{ijk}} = \bigcup_{i,j,k} \operatorname{Spec}(A_{ij})_{f_{ijk}}$$

It thus suffices to show that if A is a finitely generated B algebra, then A_f is also a finitely generated B algebra for all $f \in A$. Let $\pi : A \to A_f$ be the localization map, and $\phi : B \to A$ be the map making A a finitely generated B algebra. The map $\pi \circ \phi$ which takes $b \mapsto \phi(b)/1$ is then the map making A_f a B algebra. Let $\{a_1, \dots, a_n\}$ be the generators of A as a B algebra, then any element $a \in A$ can be written as:

$$a = \sum_{i_1 \cdots i_n} \phi(b_{i_1 \cdots i_k}) a_1^{i_1} \cdots a_n^{i_n}$$

We claim that $\{a_1/1, \dots, a_n/1, 1/f\}$ is a generating set for A_f. Indeed, we see that any element in A_f, can be written as a/f^k, hence:

$$\begin{aligned} a/f^k =& (1/f^k) \cdot a/1 \\ =& (1/f^k) \cdot \frac{\sum_{i_1 \cdots i_n} \phi(b_{i_1 \cdots i_k}) a_1^{i_1} \cdots a_n^{i_n}}{1} \\ =& \sum_{i_1 \cdots i_n} \frac{1}{f^k} \cdot \frac{\phi(b_{i_1 \cdots i_n})}{1} \cdot \frac{a_1^{i_1}}{1} \cdots \frac{a_n^{i_n}}{1} \end{aligned}$$

implying the claim. □

We also have that morphisms of locally finite type are stable under base change:

Proposition 3.5.2. *Let $f : X \to Z$ be a morphism of locally finite type, and $g : Y \to Z$ be ay other morphism. Then $\pi_Y : X \times_Z Y \to Y$ is a morphism of locally finite type.*

Proof. Let $\{V_i = \operatorname{Spec} B_i\}$ be a cover of Z by affine opens, and $\{U_{ij} = \operatorname{Spec} A_{ij}\}$ a cover of X by affine opens such that $f(U_{ij}) \subset V_i$. Moreover, let $\{W_{ij} = \operatorname{Spec} C_{ij}\}$ be a cover of Y of affine opens such that $g(W_{ij}) \subset V_i$. It follows that $\pi_Y^{-1}(W_{ij}) \cong X \times_{V_i} W_{ij} \cong f^{-1}(V_i) \times_{V_i} W_{ij}$. Now $f^{-1}(V_i) \times_{V_i} W_{ij}$ admits an open affine cover of the form $U_{ik} \times_{V_i} W_{ij} = \operatorname{Spec}(A_{ik} \otimes_{B_i} C_{ij})$. We then need only show that $A_{ik} \otimes_{B_i} C_{ij}$ is a finitely generated C_{ij} algebra. However, this is then clear, as if $\{a_1, \dots, a_n\}$ are the generators of A_{ik} as a B_i algebra, then $\{a_1 \otimes 1, \dots, a_n \otimes 1\}$ are generators of $A_{ik} \otimes_{B_i} C_{ij}$ as C_{ij} algebra. Indeed, we can write any element ω in $A_{ik} \otimes B_i C_{ij}$ as a sum of trivial tensors:

$$\omega = \sum_i \alpha_i \otimes c_i = \sum_i (\alpha_i \otimes 1) \cdot (1 \otimes c_i)$$

Each α_i can be written as the finite sum:

$$\alpha_i = \sum_{j_1 \cdots j_n} b_{ij_1 \cdots j_n} a_1^{j_1} \cdots a_n^{j_n}$$

hence:

$$\begin{aligned} \omega =& \sum_i \sum_{j_1 \cdots j_n} (b_i j_1 \cdots j_n a_1^{j_1} \cdots a_n^{j_n} \otimes 1) \cdot (1 \otimes c_i) \\ =& \sum_i \sum_{j_1 \cdots j_n} (a_1 \otimes 1)^{j_1} \cdots (a_n \otimes 1)^{j_n} \cdot (1 \otimes b_{ij_1 \cdots j_n} c_i) \end{aligned}$$

By collecting terms, and relabeling we obtain that:

$$\omega = \sum_{i_1 \cdots i_n} (a_1 \otimes 1)^{i_1} \cdots (a_n \otimes 1)^{i_n} \cdot (1 \otimes c_{i_1 \cdots i_n})$$

implying that $A_{ik} \otimes_{B_i} C_{ij}$ is indeed a finitely generated C_{ij} algebra. $\square$

Proposition 3.5.3. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of (locally) finite type. Then $g \circ f$ is (locally) of finite type.*

Proof. Let $\{W_i = \operatorname{Spec} C_i\}$ be an open affine cover for Z. Since g is (locally) of finite type, there exists an open affine cover $g^{-1}(W_i)$, $\{V_{ij} = \operatorname{Spec} B_{ij}\}_j$, such that each B_{ij} is a a finitely generated C_i algebra. By the same logic, there exists an affine open cover of each $f^{-1}(V_{ij})$, $\{U_{ijk} = \operatorname{Spec} A_{ijk}\}_k$, such that each A_{ijk} is a finitely generated B_{ij} algebra. Now note that for each i:

$$\begin{aligned} \bigcup_{jk} U_{ijk} =& \bigcup_{ij} \left(\bigcup_k U_{ijk} \right) \\ =& \bigcup_j f^{-1}(V_{ij}) \\ =& f^{-1} \left(\bigcup_j V_{ij} \right) \\ =& f^{-1}(g^{-1}(W_i)) \end{aligned}$$

hence for each i, the $\{U_{ijk}\}_{jk}$ form an affine open cover of $(g \circ f)^{-1}(W_i)$. It now suffices to show that each A_{ijk} is a finitely generated C_i algebra. Each A_{ijk} is a finitely generated B_{ij} algebra, so let $\{a_1, \dots, a_n\}$ generate A_{ijk} as a B_{ijk} algebra. Moreover, we have that each B_{ij} is a finitely generated C_i algebra so let $\{b_1, \dots, b_m\}$ generate B_{ij} as a C_i algebra. We claim that $\{a_1, \dots, a_n, b_1, \dots, b_m\}$[53] generates A_{ijk} as C_i algebra. Indeed, let $a \in A$, then:

$$a = \sum_{l_1 \cdots l_n} b_{l_1 \cdots l_n} a_1^{l_1} \cdots a_n^{l_n}$$

[53]Here it understood that by b_l we mean the image of b_l in A_{ijk} under the homomorphism making A_{ijk} a B_{ij} algebra.

We can write:

$$b_{l_1\cdots l_n} = \sum_{\lambda_1\cdots\lambda_m} c_{l_1\cdots l_n\lambda_1\cdots\lambda_m} b_1^{\lambda_1}\cdots b_m^{\lambda_m}$$

hence:

$$a = \sum_{l_1\cdots l_n\lambda_1\cdots\lambda_m} c_{l_1\cdots l_n\lambda_1\cdots\lambda_m} b_1^{\lambda_1}\cdots b_m^{\lambda_m} a_1^{l_1}\cdots a_n^{l_n}$$

implying the claim.

If g and f are of finite type, then every cover can be taken to be finite, hence $\{U_{ijk}\}_{jk}$ is a finite cover of $(g\circ f)^{-1}(W_i)$, so $g\circ f$ is of finite type as well. □

Example 3.5.2. Let $f : X \to Y$ be a closed embedding, then f is of finite type. Indeed, for every affine open $U = \operatorname{Spec} A \subset Y$, we have that $f^{-1}(U) = \operatorname{Spec} A/I$, so admits a finite cover of affine opens of X. It remains to show that A/I is a finitely generated A algebra, however this clear as any $[a] \in A$ can be written as $a \cdot [1] = [a \cdot 1] = [a]$, hence A/I is finitely generated over A by $[1]$.

Let $\iota : U \to X$ be an open embedding, then ι is locally of finite type. Indeed, let $\{V_i = \operatorname{Spec} B_i\}$ be an affine open cover of X, then $\iota^{-1}(V_i) = U \cap V_i$ and $\iota|_{U\cap V_i} : U \cap V_i \to V_i$ is an open embedding into an affine scheme. We can cover each $U \cap V_i$ with $U_{f_{ij}} \subset \operatorname{Spec} B_i$ for some $f_{ij} \in B$. It follows that $\{U_{f_{ij}}\}_j$ is a cover for $\iota^{-1}(V_i)$, and that $\iota|_{U_{f_{ij}}}$ is the given by the localization map $\pi_{ij} : B_i \to (B_i)_{f_{ij}}$. Consider the morphism:

$$\begin{aligned} \phi : B_i[x] &\longrightarrow (B_i)_{f_{ij}} \\ x &\longmapsto 1/f_{ij} \end{aligned}$$

Let $b/f_{ij}^n \in (B_i)_{f_{ij}}$, then $bx^n \mapsto b/f_{ij}^n$ so $(B_i)_{f_{ij}}$ is finitely generated by $\{1, 1/f_{ij}\}$ as a B_i algebra. If X is Noetherian, then we can take ι to be of finite type.

3.6 Separated Z-Schemes

In the category of topological spaces, direct products exist, and a space is Hausdorff if and only if the map $\Delta : X \to X \times X$ has closed image. In the category of schemes, the topological spaces we are dealing with are almost never dealing with Hausdorff spaces and we do not have product. Indeed, consider the affine plane $\mathbb{A}^n_{\mathbb{C}}$, then this space is modeled off of $\mathbb{C}^n$, but is certainly not Hausdorff, as the unique generic point is contained in every open set. Moreover, we have that $\mathbb{A}^n_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^m_{\mathbb{C}} \cong \mathbb{A}^{n+m}_{\mathbb{C}}$, so fibre products mildly behave like direct products, but $\mathbb{A}^{n+m}_{\mathbb{C}}$ has many more points than the naive cartesian product[54].

However, if we restrict ourself to the category of Z-schemes, then fibre product, $X \times_Z Y$, does satisfy the universal property of the direct product. Indeed, this is true essentially by constriction, if $f_X : X \to Z$ and $f_Y : Y \to Z$ are Z-schemes, then their fibre product is a Z-scheme. If $f_Q : Q \to Z$ is a Z-scheme, and $p_X : Q \to X$ and $p_Y : Q \to Y$ are morphisms of Z-schemes, then we automatically have $f_X \circ p_X = Q_X$ and $f_Y \circ p_Y = Q_Y$, so there exists a unique morphism $Q \to X \times_Z Y$ of Z-schemes which satisfies the direct product diagram. With this in mind, we wish to develope an analogue to a scheme being Hausdorff, which mimics the definition of Hausdorff in the category of topological spaces, leading us to the next definition:

Definition 3.6.1. Let X be a Z-scheme, then X is **separated over Z**, or alternatively a separated **Z**-scheme, if the diagonal map $\Delta : X \to X \times_Z X$ has closed image. A morphism $f : X \to Z$ is **separated** if $\Delta : X \to X \times_Z X$ is a closed embedding.

The notion of separatedness is our analogue of Hausdorff in the category of schemes, and we will spend the next few pages discussing the implications of such a result.

Example 3.6.1. Let $X = \mathbb{A}^n_{\mathbb{C}}$, then we claim that $\mathbb{A}^n_{\mathbb{C}}$ is separated over $\mathbb{C}$. Indeed, we have that $X \times_{\mathbb{C}} X \cong \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}[x_1, \ldots, x_n]$, and that the diagonal morphism is induced by the ring homomorphism given on simple tensors by $\phi : f \otimes g \mapsto fg$. This is a surjective ring homomorphism, so if $I = \ker \phi$, we have that $\Delta(X) = \mathbb{V}(I) \subset X \times_{\mathbb{C}} X$. It follows that X is separated over $\mathbb{C}$.

[54] Which is not even a scheme!

We would actually like to show that the notion of being separated over a scheme Z is the same as the morphism $f : X \to Z$ being a separated morphism. To do so we will need to show that the diagonal map is a closed embedding if it has closed image[55] We need the following definition:

Definition 3.6.2. A morphism $f : X \to Y$ is a **locally closed immersion**[56] if f factors as a closed embedding followed by an open embedding. In other words we have the following commutative diagram for some open subset $U \subset Y$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & U \arrow[ur, "\iota"'] & \end{tikzcd}$$

where g is a closed embedding, and ι is the inclusion.

We want to show every diagonal map is a locally closed immersion.

Lemma 3.6.1. *Let $f : X \to Z$ be a morphism, then $\Delta : X \to X \times_Z X$ is a locally closed immersion.*

Proof. Let $\{V_i\}$ be an affine open cover for Z, and for each i let: $\{U_{ij}\}$ be an affine open cover for $f^{-1}(V_i)$. We have that $\{U_{ij} \times_{V_i} U_{ik}\}_{i,j,k}$ is an open affine cover for $X \times_Z X$, and claim that:

$$U = \bigcup_{ij} U_{ij} \times_{V_i} U_{ij}$$

contains the image of Δ. However, this clear because $\Delta|_{U_{ij}}$ has image in $U_{ij} \times_{V_i} U_{ij}$, so Δ has image in U, and we have that Δ factors as:

$$X \longrightarrow U \longrightarrow X \times_Z X$$

The second morphism is clearly an open embedding, so we need only show that the morphism with restricted image, which we denote by g, is a closed embedding. This is also clear, as if $U_{ij} = \operatorname{Spec} A_{ij}$, and $V_i = \operatorname{Spec} B$, then $g|_{U_{ij}}$ is induced by the ring homomorphism $A_{ij} \otimes_{B_i} A_{ij} \to A_{ij}$ which is surjective, and is thus a quotient map. By Corollary 3.1.2 we have that g is a closed embedding, implying the claim. $\square$

We now prove the following more general statement:

Proposition 3.6.1. *Let $f : X \to Y$ be a locally closed embedding, then f is a closed embedding if and only if $f(X)$ has closed image in Y.*

Proof. Suppose that f is a closed embedding, then f trivially has closed image. Moreover, every closed embedding is a locally closed immersion as Y is trivially an open subscheme of Y.

Now let f be a locally closed immersion, and factor as $\iota \circ g$ where $g : X \to U$ is a closed embedding, and ι is the inclusion map into Y. Suppose $f(X)$ has closed image in Y, then by Corollary 3.1.2 we need to find an open cover of Y such that f restricts to a closed embedding. Note that:

$$Y = U \cup f(X)^c$$

as $f(X) \subset U$. We have that $f|_{f^{-1}U} : f^{-1}(U) = X \to U$ is a closed embedding as it is equal to g, and that $f^{-1}(f(X)^c)$ is the empty scheme $\emptyset$, so $f|_\emptyset$ is the empty map which is also trivially a closed embedding, implying the claim. $\square$

We now have the following corollary:

Corollary 3.6.1. *Let $f : X \to Z$ be a morphism of schemes, then f is separated if and only if X is separated over Z.*

We now list some examples (and non-examples) of separated schemes and morphisms:

Example 3.6.2. Every morphism of affine schemes is separated. Indeed, let $\operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine scheme, then $\operatorname{Spec} A \times_B \operatorname{Spec} A = \operatorname{Spec} A \otimes_B A$, and the diagonal morphism is given by $a_1 \otimes a_2 \mapsto a_1 \dot{a}_2$ which is surjective, so Δ is a closed embedding. In particular, $\operatorname{Spec} A$ is separated over $\operatorname{Spec} B$.

[55] The other direction is immediate.

[56] In the literature this is sometimes called a locally closed embedding, or simply an immersion.

Example 3.6.3. We claim that $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$ is separated over A. We construct the map $\mathbb{P}^n_A \to \operatorname{Spec} A$ given on the open cover $\{U_{x_i} = \operatorname{Spec}(A[x_0, \dots, x_n]_{x_i})_0\}$ by the morphism of affine schemes induced by the ring homomorphisms $A \hookrightarrow (A[x_0, \dots, x_n]_{x_i})_0$. We have an open cover of $\mathbb{P}^n_A \times_A \mathbb{P}^n_A$ by $\{U_{x_i} \times_A U_{x_j}\}_{i,j}$. Now note that $\Delta^{-1}(U_{x_i} \times_A U_{x_j})$ is equal to the intersection:

$$U_{x_i} \cap U_{x_j} = (\operatorname{Spec}(A[x_0, \dots, x_n]_{x_i})_0)_{x_j/x_i} \cong \operatorname{Spec} A[\{x_k/x_i\}_{k\neq i}, x_i/x_j]$$

We have that:

$$U_{x_i} \times_A U_{x_j} = \operatorname{Spec} A[\{z_k/z_i\}_{k\neq i}, \{y_k/y_j\}_{k\neq j}]$$

We have that $\Delta|_{U_{x_i} \cap U_{x_j}}$ is induced by the ring homomorphism which makes the following diagram of rings commute:

$$\begin{tikzcd}
A[\{x_k/x_i\}_{k\neq i}, x_i/x_j] & & \\
& A[\{z_k/z_i\}_{k\neq i}, \{y_k/y_j\}_{k\neq j}] \arrow[ul, "\Delta^\sharp"'] & A[\{x_k/x_j\}_{k\neq j}] \arrow[l, "\iota_j"'] \arrow[ull, bend right, "\phi"'] \\
& A[\{x_k/x_i\}_{k\neq i}] \arrow[u, "\iota_i"] \arrow[uul, bend left, "\psi"] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

where ψ is the inclusion, ϕ is the morphism $x_k/x_j \mapsto x_k/x_i \cdot x_i/x_j$ for $j \neq i$, and $x_i/x_j \mapsto x_i/x_j$. The maps ι_i and ι_j take x_k/x_i and x_k/x_j to z_k/z_i and y_k/y_j respectively. It follows that $\Delta^\sharp(z_k/z_i) = x_k/x_i$, and that $\Delta^\sharp(y_i/y_j) = x_i/x_j$ so $\Delta^\sharp$ is indeed surjective. Therefore, on the open cover $U_{x_i} \times U_{x_j}$ we have that $\Delta|_{U_{x_i} \cap U_{x_j}}$ is a closed embedding, so Δ is a closed embedding thus $\mathbb{P}^n_A$ is separated over $\operatorname{Spec} A$.

We have the following non example:

Example 3.6.4. Let Z be the scheme obtained by gluing $X = \operatorname{Spec} \mathbb{C}[x]$ and $Y = \operatorname{Spec} \mathbb{C}[y]$ along the affine open U_x and U_y via the isomorphism induced by $x \mapsto y$. We claim that Z is not separated over $\operatorname{Spec} \mathbb{C}$. If ψ_X and ψ_Y are the open embeddings $X \to Z$ and $Y \to Z$ respectively, we have that Z has an open cover given by $\psi_X(X)$ and $\psi_Y(Y)$. It follows that $Z \times_{\mathbb{C}} Z$ has an open cover given by:

$$\{\psi_X(X) \times_{\mathbb{C}} \psi_X(X), \psi_Y(Y) \times_{\mathbb{C}} \psi_Y(Y), \psi_X(X) \times_{\mathbb{C}} \psi_Y(Y), \psi_Y(Y) \times_{\mathbb{C}} \psi_X(X)\}$$

Each of these is isomorphic to the affine plane $\mathbb{A}^2_{\mathbb{C}}$, so we need to determine how these schemes glue together. We label these schemes by X_1, X_2, X_3 and X_4 in the order which they appear, and set:

$$X_i = \operatorname{Spec} \mathbb{C}[x_i] \times_{\mathbb{C}} \operatorname{Spec} \mathbb{C}[y_i] = \operatorname{Spec} \mathbb{C}[x_i, y_i]$$

Then X_1 and X_2 are glued on $U_{x_1} \cap U_{y_1}$ and $U_{x_2} \cap U_{y_2}$, X_1 and X_3 are glued along U_{y_1} and U_{y_3}, X_1 and X_4 are glued along U_{x_1} and U_{x_4}, X_2 and X_3 are glued along U_{x_2} and U_{x_3}, X_2 and X_4 are glued on U_{y_2} and U_{y_4} and X_3 and X_4 are glued along $U_{x_3} \cap U_{y_3}$ and $U_{x_4} \cap U_{y_4}$. All of these morphisms are induced by the by isomorphism $x_i, y_i \mapsto x_j, y_j$.

It follows that $Z \times_{\mathbb{C}} Z$ is the affine plane with four origins, and doubled axis. The diagonal $\Delta(Z)$ is equal to $\Delta(\psi_X(X)) \cup \Delta(\psi_Y(Y))$, which via the above identification is contained in $X_1 \cup X_2$[57]. In particular, geometrically $\Delta(Z) \cap X_1$ and $\Delta(Z) \cap X_2$ is the diagonal in $\mathbb{A}^2_{\mathbb{C}}$, while $\Delta(Z) \cap X_3$ and $\Delta(Z) \cap X_4$ is the diagonal of $\mathbb{A}^2_{\mathbb{C}}$ minus the origin. Therefore, $\Delta(Z) \cap X_i$ is not closed for all i, hence by definition of the topology on $Z \times_{\mathbb{C}} Z$, we have that Z is not separated.

Note that this also shows that Z is not an affine scheme by Example 3.6.2.

We now show that every open and closed embedding is also a separated morphism:

Proposition 3.6.2. *Let $f : X \to Z$ be a closed or open embedding, then X is separated over Z.*

[57] Abuse of notation alert! Technically, each X_i is a copy of $\mathbb{A}^2_{\mathbb{C}}$ which we glue together to get $Z \times_{\mathbb{C}} Z$, so only their images under the canonical open embeddings are contained in $Z \times_{\mathbb{C}} Z$. We employ this abuse so as to not clutter the paige with notation.

Proof. First suppose that $f : X \to Z$ is a closed embedding, then there exist an open affine cover $\{V_i = \operatorname{Spec} B_i\}$ of Z such that $U_i = f^{-1}(V_i) = \operatorname{Spec} B_i/I_i$ for some ideal I. It follows that $X \times_Z X$ admits an open affine cover of the form:

$$\{U_i \times_{V_i} U_i = \operatorname{Spec}(B_i/I_i \otimes_{B_i} B_i/I_i)\}$$

Since $B_i/I_i \otimes_{B_i} B_i/I_i \cong B_i/I_i$ we have that $U_i \times_{V_i} U_i \cong U_i$ so $X \times_Z X \cong X$ and the diagonal map is just the identity. In particular, one can also see this by noting the $f(X) \times_Z f(X) \cong f(X) \cap f(X) \cong f(X) \cong X$.

Now suppose that $f : X \to Z$ is an open embedding, then $X \cong U$ for some open subscheme of Z. We have that $X \times_Z X \cong U \times_Z U \cong U \times_U U = U$, so again the diagonal map is just the identity, implying the claim. □

Recall that morphisms/topological properties of schemes are generally considered 'nice' if they are either local on target or stable under base change. We want to see that separated morphisms fall into this category as well:

Proposition 3.6.3. *Let $f : X \to Z$ be a morphism of schemes, then the following hold:*

a) f is separated if and only if there exists an affine open cover $\{V_i\}$ of Z such that $f|_{f^{-1}(V_i)}$ is separated.

b) If f is separated, and $Y \to Z$ is another morphism, then $X \times_Z Y$ is separated over Y.

Proof. To show *a*), we first assume that f is separated. It follows that $\Delta : X \to X \times_Z X$ is a closed embedding, so $\Delta|_{f^{-1}(V_i)} : f^{-1}(V_i) \to f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$ is also a closed embedding. It follows that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is a separated morphism as well.

Now suppose that we have affine open cover $\{V_i = \operatorname{Spec} B_i\}$ such that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is separated. This then implies that $\Delta|_{f^{-1}(V_i)} : f^{-1}(V_i) \to f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$ is a closed embedding. Let $\{U_{ij} = \operatorname{Spec} A_{ij}\}$ be an affine open cover for $f^{-1}(V_i)$, then we have that $\{U_{ij} \times_{V_i} U_{ik}\}$ is an affine open cover for $f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$, and that $\Delta|_{f^{-1}(V_i)}^{-1}(U_{ij} \times_{V_i} U_{ik}) = U_{ij} \cap U_{ik}$. Since this is a closed embedding, we thus have that $U_{ij} \cap U_{ik}$ is affine and of the form $\operatorname{Spec} A_{ij} \otimes_{B_i} A_{ik}/I$ for some ideal I. Doing this for all i, we obtain an open affine cover of $X \times_Z X$ such Δ restricts to a closed embedding on $\Delta^{-1}(U_{ij} \times_{V_i} U_{ik})$ so it follows that Δ itself is a closed embedding.

To show *b*), suppose that $f : X \to Z$ is separated, and let $g : Y \to Z$ be any morphism. We want to show that $X \times_Z Y$ is separated over Y. We have that:

$$\begin{aligned}
(X \times_Z Y) \times_Y (X \times_Z Y) &\cong (X \times_Z Y) \times_Y (Y \times_Z X) \\
&\cong ((X \times_Z Y) \times_Y Y) \times_Z X \\
&\cong (X \times_Z Y) \times_Z X \\
&\cong (X \times_Z X) \times_Z Y
\end{aligned}$$

The diagonal map $\Delta : X \times_Z Y \to (X \times_Z Y) \times_Y (X \times_Z Y)$ is then the map induced by $\Delta_X : X \to X \times_Z X$ and the identity on Y, composed with the above chain of isomorphisms. In other words we have the following diagram:

$$\begin{tikzcd}
X \times_Z Y \arrow[dr, "\Delta"] \arrow[drr, bend left, "\pi_Y"] \arrow[ddr, bend right, "\Delta_X \circ \pi_X"'] & & \\
& (X \times_Z X) \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_{X \times_Z X}"] & Y \arrow[d, "g"] \\
& X \times_Z X \arrow[r, "f \circ \pi_X"] & Z
\end{tikzcd}$$

The claim then follows from Lemma 3.6.2 which we prove below. □

Lemma 3.6.2. *Let $f : X \to Z$, and $g : Y \to Z$ be Z-schemes. Suppose that $f' : X' \to X$ and $g' : Y' \to Y$ are closed embeddings, then the induced map $f' \times g' : X' \times_Z Y' \to X \times_Z Y$ is a closed embedding.*

Proof. Note that $X' \times_Z Y'$ comes from the following Cartesian square:

$$\begin{tikzcd}
X' \times_Z Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "\pi_{X'}"'] & Y' \arrow[d, "g \circ g'"] \\
X' \arrow[r, "f \circ f'"] & Z
\end{tikzcd}$$

The map $f' \times g'$ is the then the unique map making the following diagram commute:

$$\begin{tikzcd}
X' \times_Z Y' \arrow[dr, "f' \times g'"] \arrow[drr, bend left, "g' \circ \pi_{Y'}"] \arrow[ddr, bend right, "f' \circ \pi_X"'] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

Let $\{W_i = \operatorname{Spec} C_i\}$ be an open affine cover for Z, and $\{U_{ij} = \operatorname{Spec} A_{ij}\}$, $\{V_{ik} = \operatorname{Spec} B_{ik}\}$ be an open affine cover for X and Y such that U_{ij} and V_{ik} map into W_i. It follows that $\{U_{ij} \times_{W_i} V_{ik}\}$ is an open affine cover for $X \times_Z Y$. By the commutativity of the diagram, and the universal property of the fibre product, we have that $(f' \times g')^{-1}(U_{ij} \times_{W_i} V_{ik})$ is isomorphic to $f'^{-1}(U_{ij}) \times_{W_i} g'^{-1}(V_{ik})$. Since f' and g' are closed embeddings, we have the following chain of isomorphisms for some ideals I and J:

$$\begin{aligned}
f'^{-1}(U_{ij}) \times_{W_i} g'^{-1}(V_{ik}) &\cong \operatorname{Spec} A_{ij}/I \times_{C_i} \operatorname{Spec} B_{ik}/J \\
&\cong \operatorname{Spec} A_{ij}/I \otimes_{C_i} B_{ik}/J \\
&\cong \operatorname{Spec}(A_{ij} \otimes C_i B_{ik}) / \langle I \otimes B_{ik}, A_{ij} \otimes J \rangle
\end{aligned}$$

hence $f' \times g'$ is a closed embedding as desired. □

Note that there is an alternative proof of Proposition 3.6.3 part *b*) that relies entirely on abstract nonsense. Indeed, set $X' = X \times_Z Y$, then we wish to show that $\Delta : X' \to X' \times_Y X'$ is a closed embedding. Well, we have the following commutative diagram:

$$\begin{tikzcd}
X' \times_Y X' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_{X'}"'] & X' \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \arrow[d, "f"] \\
X' \arrow[r, "\pi_Y"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

The left square, and the right square are Cartesian diagrams, so it follows by Lemma 2.3.4 that the outer square is cartesian as well. We then have the following commutative diagram:

$$\begin{tikzcd}
X' \times_Y X' \arrow[r, "\Delta_X \circ \pi_X \circ \pi_{X'}"] \arrow[d, "\pi_{X'}"'] & X \times_Z X \arrow[r, "\pi_X"] \arrow[d, "\pi_X"] & X \arrow[d, "f"] \\
X' \arrow[r, "\pi_X"] & X \arrow[r, "f"] & Z
\end{tikzcd}$$

We see that $\pi_X \circ \Delta_X \circ \pi_X \circ \pi_{X'} = \pi_X \circ \pi_{X'}$, and that $f \circ \pi_X = g \circ \pi_Y$, so the outer rectangle is Cartesian, and the right rectangle is Cartesian. It follows again by Lemma 2.3.4 that the left rectangle is then Cartesian as well. We thus finally have the following commutative diagram:

$$\begin{tikzcd}
X' \arrow[r, "\Delta"] \arrow[d, "\pi_X"'] & X' \times_Y X' \arrow[r, "\pi_{X'}"] \arrow[d, "\Delta_X \circ \pi_X \circ \pi_{X'}"] & X' \arrow[d, "\pi_X"] \\
X \arrow[r, "\Delta_X"] & X \times_Z X \arrow[r, "\pi_X"] & X
\end{tikzcd}$$

The right square is Cartesian by our previous argument, and the outer square is Cartesian as $\pi_{X'} \circ \Delta = \mathrm{Id}_{X'}$, while $\pi_X \circ \Delta_X \circ \pi_X = \pi_X$, so we obtain the diagram form $X' \times_X X \cong X'$. It follows that the left square is Cartesian, and that Δ is the base change of Δ_X. Since Δ_X is a closed embedding, and closed embeddings are stable under base change we must have that Δ is a closed embedding as well, implying the claim.

We now note that the two other classes of morphisms we have mentioned, those of finite type, and closed embeddings, are both preserved under composition. Indeed, if $f : X \to Y$, and $g : Y \to Z$ are both locally of finite type, then we have that an affine open cover $\{V_i = \operatorname{Spec} B_i\}$ of Z, $g|_{g^{-1}(V_i)} : g^{-1}(V_i) \to V_i$ is locally of finite type. Moreover, we have that if U_{ij} is open cover of Y such that $g(U_{ij}) \subset V_i$, then $f|_{f^{-1}(U_{ij})} : f^{-1}(U_{ij}) \to U_{ij}$ is locally of finite type. Each $f^{-1}(U_{ij})$ has a cover of affine opens W_{ijk} such that $f(W_{ijk}) \subset U_{ij}$. It follows that varying over j and k, we get that $g(f(W_{ijk})) \subset V_i$, and that W_{ijk} cover $f^{-1}(g^{-1}(V_i))$. It is then clear that with $W_{ijk} = \operatorname{Spec} A_{ijk}$, A_{ijk} is a finitely generated B_i algebra so $g \circ f$ is locally of finite type.

If f and g are closed embeddings, then for every affine open $V = \operatorname{Spec} B$, we have that $g^{-1}(V) = \operatorname{Spec} A/I$. Since f is a closed embedding, we have that $f^{-1}(g^{-1}(V)) = \operatorname{Spec}(A/I)/J$, so the composition $g \circ f$ is also a closed embedding.

These facts are essentially obvious from our results characterizing closed embeddings, and morphisms of locally finite type. We wish to show the same result holds for separated morphisms.

Proposition 3.6.4. *Let $f : X \to Y$ and $g : Y \to Z$ be separated morphisms, then $g \circ f : X \to Z$ is also separated.*

Proof. We write $\Delta : X \to X \times_Z X$ for the diagonal map we wish to prove is a closed embedding, and $\Delta_X : X \to X \times_Y X$, $\Delta_Y : Y \to Y \times_Z Y$ for the diagonal maps we know to be closed embeddings. From Theorem 2.3.1 we have the following Cartesian square:

$$\begin{tikzcd} X \times_Y X \arrow[r, "\psi"] \arrow[d, "f \circ \pi_X"'] & X \times_Z X \arrow[d, "f \times f"] \\ Y \arrow[r, "\Delta_Y"] & Y \times_Z Y \end{tikzcd}$$

where ψ is the map coming from the following diagram[58]:

$$\begin{tikzcd} X \times_Y X \arrow[dr, "\psi"] \arrow[drr, bend left, "\pi_X"] \arrow[ddr, bend right, "\pi_X"'] & & \\ & X \times_Z X \arrow[r, "\pi_X"] \arrow[d, "\pi_X"] & X \arrow[d, "g \circ f"] \\ & X \arrow[r, "g \circ f"] & Z \end{tikzcd}$$

It follows that ψ is closed embedding as it is the base change of the closed embedding Δ_Y. We claim that $\Delta = \psi \circ \Delta_X$. Indeed, Δ comes from the following diagram:

$$\begin{tikzcd} X \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{Id}_X"] \arrow[ddr, bend right, "\mathrm{Id}_X"'] & & \\ & X \times_Z X \arrow[r, "\pi_X"] \arrow[d, "\pi_X"] & X \arrow[d, "g \circ f"] \\ & X \arrow[r, "g \circ f"] & Z \end{tikzcd}$$

[58] Abuse of notation alert! We are once again using the notation π_X to refer to multiple maps.

Now $\pi_X \circ \psi \circ \Delta_X = \pi_X \circ \Delta_X = \mathrm{Id}_X$, so $\psi \circ \Delta_X$ makes the above diagram commute. It follows that Δ is the composition of a closed embeddings, and thus a closed embedding, hence $g \circ f$ is separated. □

Note that the intersection of two affine opens need not be affine. Indeed, let X be the affine plane over $\mathbb{C}$ with doubled origin, then their are two copies of $\mathbb{A}^2_{\mathbb{C}}$ contained in X, but their intersection is two copies of the zero ideal $\langle 0\rangle$ which is manifestly not affine, i.e. no ring has two copies of the zero ideal as a prime spectrum. We now show that separated morphisms provide a solution to this problem:

Proposition 3.6.5. *Let $f : X \to Z$ be a separated morphism, and let $V = \operatorname{Spec} B \subset Z$ be an open affine. Then for every open affine $U_i = \operatorname{Spec} A_i \subset X$ which maps into V, we have that $U_i \cap U_j$ is an open affine.*

Proof. Let $\Delta : X \to X \times_Z X$, and then note that $\Delta(f^{-1}(V))$ is contained in $f^{-1}(V) \times_V f^{-1}(V)$. We see that if U_i and U_j are as above, we have that $\Delta^{-1}(U_i \times_V U_j) = U_i \cap U_j$, but Δ is a closed embedding so $U_i \cap U_j$ is of the form $\operatorname{Spec}(A_i \otimes_B A_j)/J$ hence $U_i \cap U_j$ is indeed an open affine. □

We have the following obvious corollary:

Corollary 3.6.2. *Let X be separated over an affine scheme* $\operatorname{Spec} A$. *Then the intersection of every affine open in X is an affine open.*

Note that this text in algebraic geometry has never once mentioned the notion of a variety, largely because the author was first introduced to algebraic geometry through the language of schemes. However, we now have sufficient language to give the definition of a variety, which are often the most geometric feeling schemes. We note that the definition of a variety varies wildly throughout the literature, and will change in this text when we discuss Abelian varieties.

Definition 3.6.3. Let X be a scheme, then X is a **variety over k** if X is of finite type over a field k, reduced, and separated over $\operatorname{Spec} k$.

Note that every variety is immediately quasi-compact as it is the finite union of affine schemes. Each of these affine schemes is Spec of a finitely generated k-algebra thus every variety is locally Noetherian. In particular, by Corollary 3.4.2 every variety is Noetherian.

Example 3.6.5. The n-dimensional affine plane $\mathbb{A}^n_{\mathbb{C}}$, and projective space $\mathbb{P}^n_{\mathbb{C}}$ are varieties. In general, the closed points of 'nice enough' varieties over $\mathbb{C}$, when equipped with the standard topology induced by that on $\mathbb{C}^n$ have the structure of smooth manifolds. We will make this notion precise later in the text.

Example 3.6.6. Let X be a variety, then every closed subset of $Z \subset X$ is a variety when equipped with the induced reduced subscheme structure. Every reduced closed subscheme of X is isomorphic to such a Z, so every reduced closed subscheme of X is a variety.

Let U an open subscheme of X, then U is a variety. Indeed, open embeddings are separated by Proposition 3.6.2, and are locally of finite type by Example 3.5.2. Since X is Noetherian the open embedding $\iota : U \to X$ is of finite type, thus U. Finally U is reduced as being reduced is a local property.

Suppose that Y is a reduced locally closed subscheme of X, i.e. there exists a morphism $\iota : Y \to X$ such that ι is a locally closed immersion. Then ι factors as an open embedding in to a reduced closed subscheme $Z \subset X$, followed by the closed embedding $Z \hookrightarrow X$. It follows that Y is a variety as it is an open subscheme of the variety Z.

We have the following result:

Theorem 3.6.1. *Let X be a reduced projective k-scheme, then X is a variety. In particular, any closed subset of $\mathbb{P}^n_k$ equipped with the induced reduced subscheme structure is a variety.*

Proof. By Theorem 3.1.1 a projective k scheme is closed subscheme of $\mathbb{P}^n_k$ for some k, hence there exists some closed embedding $X \hookrightarrow \mathbb{P}^n_k$. Since closed embeddings are separated by Proposition 3.6.2, and separated morphisms are closed under composition by Proposition 3.6.4, we have that the natural morphism $X \hookrightarrow \mathbb{P}^n_k \to \operatorname{Spec} k$ making a X a k scheme is separated. Moreover, by Example 3.5.2, $X \hookrightarrow \mathbb{P}^n_k$ is separated, so Proposition 3.5.3 implies that X is separated over k. Since X is assumed to be reduced, we have that X naturally carries the structure of a scheme of variety over k.

Let $X \subset \mathbb{P}^n_k$ be any closed subset, then equipped with the induced reduced subscheme structure, we have that the above discussion applies to X as well, hence X is a variety. □

With Theorem 3.6.1, and Example 3.6.6 in mind we employ the following definitions:

Definition 3.6.4. A scheme X is a **projective variety** if it is a reduced closed subscheme of $\mathbb{P}^n_k$ for some n. In particular, every projective variety is isomorphic to a closed subset of $\mathbb{P}^n_k$ equipped with the induced reduced subscheme structure. A scheme X is a **quasi-projective variety** if it is an open subscheme of $\mathbb{P}^n_k$ for some n.

We note that $\mathbb{A}^n_k$ is a quasi-projective variety, and that every reduced closed subscheme of $\mathbb{A}^n$ is quasi-projective variety. Moreover, every projective variety is quasi-projective, as they $Z \to Z \hookrightarrow X$ is a locally closed immersion. We therefore end this discussion, by remarking that most varieties one comes across in nature are quasi-projective, and the construction of a variety that is not quasi-projective was a research area of great interest until Nagata provided such an example in 1950's.

3.7 Proper Z-Schemes

A compact topological space X is generally one where every open cover has a finite subcover. Throughout this text, we have called this property quasi-compactness, largely because this definition is not restrictive enough. Indeed, the analogue of the complex vector space $\mathbb{C}^n$ in algebraic geometry is $\mathbb{A}^n_{\mathbb{C}}$. Under the usual definition of compactness, $\mathbb{A}^n_{\mathbb{C}}$ is compact as every affine scheme is quasi-compact, but $\mathbb{C}^n$ is most definitely not. Given this, instead we follow the lead of our separatedness condition, and define our analogue of compactness relative to a base scheme.

In topology, a proper map $f : X \to Y$ is one in which the inverse image of a compact set is compact. This is the correct way of thinking of 'relative compactness' in the setting of topological spaces. However, in this sense, when working with schemes, almost every morphism is proper. Indeed, if we deal with Noetherian schemes, which are Noetherian topological spaces, every subset of a scheme is compact, so every map between Noetherian topological spaces is proper in the topological sense. This is not very a helpful condition, so, following our treatment of separatedness, we analyze an equivalent definition of proper maps.

Recall that if X and Y are locally compact Hausdorff spaces, then $f : X \to Y$ being proper is equivalent to f being universally closed. That is, in topology, if $g : Z \to Y$ is another continuous map, then there exists a fibre product:

$$X \times_Y Z = \{(x, z) \in X \times Z : f(x) = g(z)\}$$

equipped with the subspace topology. The map f is then universally closed if f is closed, and the projection $X \times_Y Z \to Z$ is also closed for every topological space Z. It is easy to check that these two descriptions of properness are equivalent in the setting of locally compact, Hausdorff spaces.

In the setting of schemes, the definition of universally closed is the same:

Definition 3.7.1. Let $f : X \to Z$ be a closed morphism of schemes, i.e. f takes closed subsets to closed subsets[59]. Then f is **universally closed** if for every Z-scheme Y the projection $X \times_Z Y \to Y$ is also closed. In other words a closed morphism is universally closed if it closed under base change.

Now, we know what the analogue of Hausdorff is in the category of Z-schemes, so we need a good analogue of what it means for a Z-scheme to be locally compact. However, we have already encountered such an analogue, indeed if X if of finite type over Z, i.e. if $f : X \to Z$ is of finite type, then this morally feels like X being locally compact in the usual sense. This motivates our definition of proper morphisms and 'compactness' in the category of schemes:

Definition 3.7.2. Let $f : X \to Z$ be a morphism of schemes. Then f is a **proper morphism** if f separated, of finite type, and universally closed. We call any Z-scheme $f : X \to Y$ a **proper Z-scheme**, or **proper over Z** if f is proper.

So our usual analogues of compactness, and proper maps in algebraic geometry are proper morphisms and proper Z-schemes respectively We wish to show that proper morphisms are local on target, stable under base change, and closed under composition. It clearly suffices to prove the following:

Lemma 3.7.1. *Universally closed morphism are:*

a) Local on target.

b) Stable under base change.

c) Closed under composition.

[59]Note that this does not mean that f is a closed embedding!

Proof. Let $f : X \to Z$ be a universally closed morphism, and $g : Y \to Z$ be any morphism of schemes. Let $\{V_i = \operatorname{Spec} C_i\}$ be an affine open cover of Z, and $\{U_{ij} = \operatorname{Spec} A_{ij}\}$, $\{W_{ik} = \operatorname{Spec} B_{ik}\}$ be affine open covers of X and Y such that U_{ij} and W_{ik} map into V_i. We want to show that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is universally closed. First note that $f|_{f^{-1}(V_i)}$ is indeed a closed map, as if $S \subset f^{-1}(V_i)$ is a closed subset then $S = T \cap f^{-1}(V_i)$ for some closed $T \subset X$. We have that:

$$f|_{f^{-1}(V_i)}(S) = f(T) \cap V_i$$

so since f is closed, it follows that the restriction is too. Now note that $f^{-1}(V_i) \times_{V_i} Y \cong f^{-1}(V_i) \times_{V_i} g^{-1}(V_i)$. We already know that $\pi_Y : X \to_Z Y \to Y$ is a closed map, so it's restriction to the open set $f^{-1}(V_i) \times_{V_i} g^{-1}(V_i)$ must now also be a closed map, hence $f|_{f^{-1}(V_i)}$ is again universally closed.

Now suppose that $f|_{f^{-1}(V_i)}$ is a universally closed map for all i. We first claim that f is closed. We have that $f(T) \cap V_i$ is closed for all i, hence:

$$Y \setminus f(T) = \bigcup_i V_i \setminus (f(T) \cap V_i)$$

which is an infinite union of open sets and thus open. It follows that if $f|_{f^{-1}(V_i)}$ is universally closed for all i, then $\pi_Y|_{f^{-1}(V_i) \times_{V_i} g^{-1}(V_i)}$ is closed for all i, so the same argument above shows that π_Y is closed, implying $a)$.

To show $b)$, we need to show that $\pi_Y : X \times_Z Y \to Y$ is universally closed. Let $h : Y' \to Y$ be a Y scheme, and note that:

$$\begin{aligned}(X \times_Z Y) \times_Y Y' &\cong X \times_Z (Y \times_Y Y') \\ &\cong X \times_Z Y'\end{aligned}$$

The map $\pi_{Y'} : X \times_Z Y' \to Y'$ is closed, and is equal to the map $\pi_{Y'} : (X \times_Z Y) \times_Y Y' \to Y'$ composed with the above isomorphisms, hence $\pi_{Y'} : (X \times_Z Y) \times_Y Y' \to Y'$ is closed as well, so π_Y is also universally closed.

To show $c)$, let $f : X \to Y$, and $g : Y \to Z$ be universally closed maps. We see that $g \circ f$ is a closed map, so we need only show that it is universally closed. Let Y' be a Z-scheme, then we need to show that $\pi_{Y'} : X \times_Z Y' \to Y'$ is a closed map. We have the following commutative diagram:

$$\begin{tikzcd} X \times_Z Y' \arrow[r, "f \times \mathrm{Id}"] \arrow[d, "\pi_X"] & Y \times_Z Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "\pi_Y"] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

where $f \times \mathrm{Id}$ comes from the following diagram:

$$\begin{tikzcd} X \times_Z Y' \arrow[dr, "f \times \mathrm{Id}"] \arrow[drr, bend left, "\pi_{Y'}"] \arrow[ddr, bend right, "f \circ \pi_X"'] & & \\ & Y \times_Z Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "\pi_Y"] & Y' \arrow[d, "h"] \\ & Y \arrow[r, "g"] & Z \end{tikzcd}$$

The right and outer squares are cartesian, so it follows as that the left square is cartesian as well. We have that f is universally closed, so $f \times \mathrm{Id}$ must be a closed map. It follows that $\pi_{Y'} = \pi_{Y'} \circ f \times \mathrm{Id}$ is the composition of closed maps and is thus closed. Therefore we have that $g \circ f$ is universally closed as desired. $\square$

We now have the following corollary:

Corollary 3.7.1. *Proper morphisms are local on target, stable under base change, and closed under composition.*

Proof. This follows because separated maps, universally closed maps, and maps of finite type are all local on target, stable under base change, and closed under composition. □

Note that if a scheme is proper over a field, i.e. $X \to \operatorname{Spec} k$ is proper for a field k, then X is in a sense 'compact'. We now demonstrate that $\mathbb{A}^n_{\mathbb{C}}$ is not proper over $\mathbb{C}$:

Example 3.7.1. Clearly the map $\mathbb{A}^n_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is closed, separated, and of finite type. We need to show that this morphism is not universally closed. Consider the scheme morphism $\pi : \mathbb{A}^n_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$. This morphism comes from (up to isomorphism) the ring homomorphism $\mathbb{C}[y] \hookrightarrow \mathbb{C}[x_1, \dots, x_{n+1}]$. Consider the closed subset $\mathbb{V}(x_1 \cdots x_{n+1} - 1)$, we claim that:

$$\mathbb{C}[x_1, \dots, x_{n+1}] / \langle x_1 \cdots x_{n+1} - 1 \rangle \cong \mathbb{C}[x_1, \dots, x_n]_{x_1 \cdots x_n}$$

which is an integral domain. Indeed, note that there is a map:

$$\mathbb{C}[x_1, \dots, x_{n+1}] \longrightarrow \mathbb{C}[x_1, \dots, x_n]_{x_1 \cdots x_n}$$

given by $x_i \mapsto x_i$ for $i \leq n$, and $x_{n+1} \mapsto 1/(x_1 \dots x_n)$. This map clearly factors through the quotient hence we have well defined map:

$$\phi : \mathbb{C}[x_1, \dots, x_{n+1}] / \langle x_1 \cdots x_{n+1} - 1 \rangle \to \mathbb{C}[x_1, \dots, x_n]_{x_1 \cdots x_n}$$

Now note that there is map:

$$\mathbb{C}[x_1, \dots, x_n] \longrightarrow \mathbb{C}[x_1, \dots, x_{n+1}] / \langle x_1 \cdots x_{n+1} - 1 \rangle$$

given by the composition of the inclusion map with the map with the projection map. We have that $[x_1 \cdots x_n]$ is invertible in $\mathbb{C}[x_1, \dots, x_{n+1}] / \langle x_1 \cdots x_{n+1} - 1 \rangle$ so the there is a well defined map:

$$\psi : \mathbb{C}[x_1, \dots, x_n]_{x_1 \cdots x_n} \longrightarrow \mathbb{C}[x_1, \dots, x_{n+1}] / \langle x_1 \cdots x_{n+1} - 1 \rangle$$

These are then clearly inverses of one another, so we have that the two rings are isomorphic. As the localization of an integral domain is an integral domain, it follows that $x_1 \cdots x_{n+1} - 1$ is irreducible.

The induced projection map then takes $\langle x_1 \cdots x_{n+1} - 1 \rangle \subset \mathbb{V}(x_1 \cdots x_{n+1} - 1)$ to the zero ideal, which is the generic point in $\mathbb{A}^1_{\mathbb{C}}$. It follows that $\pi(\mathbb{V}(x_1 \cdots x_{n_1} - 1))$ cannot be closed, so the map $\mathbb{A}^n_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not universally closed.

We now see that all closed embedding's are proper:

Example 3.7.2. Let $f : X \to Z$ be a closed embedding, then f is separated, of finite type and closed. We need only show that f is universally closed, but closed embeddings are stable under base change, so $\pi : X \times_Z Y \to Y$ is a closed embedding as well. It follows that π must be a closed map, hence f universally closed, and thus proper.

For our first nontrivial example we show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper, however, we need to be able to characterize the scheme-theoretic fibre of a scheme morphism. In other words, for $f : X \to Y$, we would like to know how to make sense of the preimage of $f^{-1}(y)$ for $y \in Y$ as a scheme.

First note that in the category of topological spaces, if $f : X \to Y$ is continuous map, then we can naturally identify $f^{-1}(p)$ with $\{y\} \times_Y X$, where $\{y\} \hookrightarrow Y$ is the inclusion map. In the category of schemes, we can naturally equip $\{y\}$ with a scheme structure given by $\operatorname{Spec} k_y$, where k_y is the residue field. We define a scheme morphism $g : \operatorname{Spec} k_y \to Y$ by first defining the topological map to be $\eta = \langle 0 \rangle \mapsto y$, and the sheaf morphism $g^\sharp : \mathscr{O}_Y \to g_* \mathscr{O}_{\operatorname{Spec} k_y}$ by first noting that if $y \in U$ then $\mathscr{O}_{\operatorname{Spec} k_y}(g^{-1}(U)) = k_y$, and if $y \notin U$ then $\mathscr{O}_{\operatorname{Spec} k_y}(\emptyset) = \{0\}$. We thus define $g^\sharp$ on open sets by:

$$g^\sharp_U(s) = \begin{cases} 0 \in \{0\} & \text{if } y \notin U \\ [s_y] \in k_y & \text{if } y \in U \end{cases}$$

which trivially commutes with restriction maps[60]. This then motivates our following definition:

Definition 3.7.3. Let $f : X \to Y$ be a scheme, then the for any $y \in Y$, the **scheme theoretic fibre** over y, denoted X_y is given by $\operatorname{Spec} k_y \times_Y X$.

[60]Note that by Corollary 1.3.1 we have that g is a monomorphism, as the stalk map $g_\eta : (\mathscr{O}_Y)_{g(\eta)} \to (\mathscr{O}_{\operatorname{Spec} k_y})_\eta$ is always the projection $(\mathscr{O}_Y)_y \to k_y$

Note that this naturally has the structure of a scheme, so it is mainly important to show that there is a natural identification with elements in the fibre over y and elements in $\operatorname{Spec} k_y \times_Y X$.

Lemma 3.7.2. *Let $f : X \to Y$ be a morphism of schemes. Then there is a natural identification between $\operatorname{Spec} k_y \times_Y X$ with the fibre $f^{-1}(y)$.*

Proof. We have the following diagram:

$$\begin{tikzcd} X_y \arrow[r, "\pi_X"] \arrow[d, "\pi_y"] & X \arrow[d, "f"] \\ \operatorname{Spec} k_y \arrow[r, "g"] & Y \end{tikzcd}$$

We first want to show that the image of π_X is the fibre $f^{-1}(y)$, and then demonstrate that π_X is a homeomorphism onto it's image. Note that if suffices to check this on an affine open cover of X_y, so let $\{V_i = \operatorname{Spec} B_i\}$ be an affine open cover of Y, and $\{U_{ij} = \operatorname{Spec} A_{ij}\}$ an open cover of X such that $f(U_{ij}) \subset V_i$ for all i and j. It follows that:

$$X_y = \bigcup_{ij} \operatorname{Spec} k_y \times_{V_i} U_{ij}$$

We will show that $\pi_X|_{\operatorname{Spec} k_y \times_{V_i} U_{ij}} = \pi_{U_{ij}}$ is a homeomorphism onto $U_{ij} \cap f^{-1}(y) = f|_{U_{ij}}^{-1}(y)$. Moreover, supposing that $y \in V_i$ as otherwise $\operatorname{Spec} k_y \times_{V_i} U_{ij}$ is clearly empty, we can write y as a prime ideal $\mathfrak{p} \subset B_i$, so $k_y = k_{\mathfrak{p}} = B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$. Suppressing the i and j notation for clarity, we have the following diagram:

$$\begin{tikzcd} \operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \otimes_B A \arrow[r, "\pi_U"] \arrow[d, "\pi_y"] & \operatorname{Spec} A \arrow[d, "f|_U"] \\ \operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \arrow[r, "g"] & \operatorname{Spec} B \end{tikzcd}$$

where it is understood that g is now the morphism $\operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \to \operatorname{Spec} B$ induced by the localization map followed by the projection to the residue field. Now note that:

$$\operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \otimes_B A \cong \operatorname{Spec}(B_{\mathfrak{p}} \otimes_B A)/\langle \mathfrak{m}_{\mathfrak{p}} \otimes 1 \rangle$$

and that A is a B algebra via the ring homomorphism $\phi : B \to A$ inducing $f|_U$. We define $A_{\mathfrak{p}}$ to be $\phi(B \setminus \mathfrak{p})^{-1}A$, and claim that:

$$B_{\mathfrak{p}} \otimes_B A \cong A_{\mathfrak{p}}$$

We have a map:

$$\beta : B_{\mathfrak{p}} \otimes_B A \longrightarrow A_{\mathfrak{p}}$$

given on simple tensors by $b/s \otimes a \mapsto \phi(b) \cdot a/\phi(s)$. Moreover, we have a ring homomorphism $A \to B_{\mathfrak{p}} \otimes A$ given by $a \mapsto 1 \otimes a$. For all $\phi(s) \in \phi(B \setminus \mathfrak{p})$, we have $1 \otimes \phi(s)$ is invertible, as $1 \otimes \phi(s) = s/1 \otimes 1$, which has inverse given by $1/s \otimes 1$. It follows that there is ring homomorphism:

$$\begin{aligned} \alpha : A_{\mathfrak{p}} &\to B_{\mathfrak{p}} \otimes A \\ a/\phi(s) &\longmapsto 1/s \otimes a \end{aligned}$$

These maps are clearly inverses of each other so we have that:

$$B_{\mathfrak{p}} \otimes_B A \cong A_{\mathfrak{p}}$$

Now note that under the map β we have that:

$$\beta(\langle \mathfrak{m}_{\mathfrak{p}} \otimes 1 \rangle) = \{a/\phi(s) \in A_{\mathfrak{p}} : a \in \langle \phi(\mathfrak{p}) \rangle\} = \langle \phi(\mathfrak{p})/1 \rangle \subset A_{\mathfrak{p}}$$

so it follows that we have the following isomorphism:

$$\operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \otimes A \cong \operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p})/1 \right\rangle$$

The projection π_U is now induced by the ring homomorphism $A \to A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p})/1 \right\rangle$, and the projection π_y is given by the composition $B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \to B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \otimes_B A \cong A_{\mathfrak{p}}/\left\langle \phi(q)/1 \right\rangle$. In the category of commutative rings, we thus have the following commutative diagram:

$$\begin{tikzcd}
A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p})/1 \right\rangle & A \arrow[l, "\pi\circ\pi_l"'] \\
B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \arrow[u, "\iota"] & B \arrow[l, "\nu"] \arrow[u, "\phi"']
\end{tikzcd}$$

where π_l is the localization map, and π is the quotient map. Let $\mathfrak{p} \in \operatorname{Spec} B$, then:

$$f|_U^{-1}(\mathfrak{p}) = \left\{\mathfrak{q} \in \operatorname{Spec} A : \phi^{-1}(\mathfrak{q}) = \mathfrak{p}\right\}$$

However, clearly from the commutativity of the first diagram, we have that $\pi_U(\operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p}) \right\rangle) \subset f^{-1}|_U(\mathfrak{p})$, so we need to define an inverse map $\eta : f|_U^{-1}(\mathfrak{p}) \to \operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p}) \right\rangle$.

Let $\mathfrak{q} \in \operatorname{Spec} A$ satisfy $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, implying that $\phi(\mathfrak{p}) \subset \mathfrak{q}$. We first show that:

$$\langle \pi_l(\mathfrak{q}) \rangle = \{a/s \in A_{\mathfrak{p}} : a \in \mathfrak{q}\}$$

is a prime ideal. This is clearly an ideal by construction, so suppose that $a/s, c/t \in A_{\mathfrak{q}}$ such that $ac/st \in \langle \pi_l(\mathfrak{q}) \rangle$. It follows that $ac/st = d/r$ such that $d \in \mathfrak{q}$, hence there exists some $u \in \phi(B \smallsetminus \mathfrak{p})$ such that:

$$u(acr - dst) = 0$$

Note that $u, r, s, t \in \phi(B \smallsetminus \mathfrak{p})$, hence $u, r, s, t \notin \phi(\mathfrak{p}) \subset \mathfrak{q}$. We thus have that $acru \in \mathfrak{q}$, so $ac \in \mathfrak{q}$, so either a or c are in $\mathfrak{q}$. Note that $\langle \pi_l(\mathfrak{q}) \rangle$ is not all of $A_{\mathfrak{q}}$, as otherwise we have that $\mathfrak{q} \cap \phi(B \smallsetminus \mathfrak{p}) \neq \emptyset$, which would imply that $\phi^{-1}(\mathfrak{q}) \cap \phi^{-1}(\phi(B \smallsetminus \mathfrak{p})) \neq \emptyset$, so $\mathfrak{p} \cap B \smallsetminus \mathfrak{p} \neq \emptyset$ which is a clear contradiction.

Let $\psi = \pi \circ \pi_l$; since $\langle \pi_l(\mathfrak{q}) \rangle$ clearly contains $\langle \phi(\mathfrak{p})/1 \rangle$, we have that $\pi(\langle \pi_l(\mathfrak{q}) \rangle)$ is a prime ideal of $A_{\mathfrak{p}}$. We thus define $\eta(\mathfrak{q}) \in \operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p})/1 \right\rangle$ by:

$$\eta(\mathfrak{q}) = \{[a/s] : a \in \mathfrak{q}\}$$

which clearly then satisfies $\eta(\mathfrak{q}) = \langle \psi(\mathfrak{q}) \rangle = \pi(\langle \pi_l(\mathfrak{q}) \rangle)$. Let $U_{[a/1]}$ be a distinguished open of $\operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p})/1 \right\rangle$, then we see that:

$$\begin{aligned}
\eta^{-1}(U_{[a/1]}) =& \{\mathfrak{q} \in f|_U^{-1}(\mathfrak{p}) : [a/1] \notin \langle \psi(\mathfrak{q}) \rangle\} \\
=& \{\mathfrak{q} \in f_U^{-1}(\mathfrak{p}) : a \notin \mathfrak{q}\} \\
=& U_a \cap f|_U^{-1}(\mathfrak{p})
\end{aligned}$$

which is open in $f_U^{-1}(\mathfrak{p})$. Since $U_a \cap f|_U^{-1}(\mathfrak{p})$ form a basis we have that η is indeed continuous.

We see that $\psi^{-1}(\eta(\mathfrak{q})) = \mathfrak{q}$, so $\pi_U \circ \eta = \mathrm{Id}$. Now let $\mathfrak{q} \in \operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{q}) \right\rangle$, then:

$$\eta(\psi^{-1}(\mathfrak{q})) = \{[a/s] : a \in \psi^{-1}(\mathfrak{q})\}$$

Suppose that $[a/s] \in \mathfrak{q}$, then $[a/1] \in \mathfrak{q}$, and $a \in \psi^{-1}(\mathfrak{q})$. Now suppose that $[a/s]$ satisfies $a \in \psi^{-1}(\mathfrak{q})$, then $[a/1] \in \mathfrak{q}$ so $[a/s] \in \mathfrak{q}$ as well. It follows that $\eta(\psi^{-1}(\mathfrak{q})) = \mathfrak{q}$ hence $\eta \circ \pi_U = \mathrm{Id}$, and π_U is a homeomorphism onto $f|_U^{-1}(\mathfrak{p})$.

Since the above argument holds for all affine opens $\operatorname{Spec} k_y \times_{V_i} U_{ij}$, it follows that $\pi_X : X_s \to X$ is a homeomorphism onto $f^{-1}(y)$ implying the claim.

$\square$

We can now show that $\mathbb{P}^n_A$ is proper.

Example 3.7.3. Note that $\mathbb{P}^n_A \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} A$, so if $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is proper, we have that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper, as proper morphisms are stable under base change.

We have already shown that $\mathbb{P}^n_{\mathbb{Z}}$ is separated, and it is clearly of finite type, so we need only show that $f : \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is universally closed. Let $g : Y \to \operatorname{Spec}\mathbb{Z}$ be any $\mathbb{Z}$ scheme, then we want to show that $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y \to Y$ is closed. As we have shown, being closed is local on target, so it suffices to show that for any open affine $U = \operatorname{Spec} A \subset Y$ that $\pi : \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} A \cong \mathbb{P}^n_A \to \operatorname{Spec} A$ is a closed map.

Let $Z = \mathbb{V}(I) \subset \mathbb{P}^n_A$ where $I = \langle g_1, g_2, \dots \rangle$ is a homogenous ideal. We need to determine the primes $\mathfrak{p} \in \operatorname{Spec} A$ which lie in $\pi(Z)$. In other words, by the preceding lemma, we want to know for which $\mathfrak{p}$, the fiber $\pi^{-1}(\mathfrak{p}) \cap Z \cong \operatorname{Spec} k_{\mathfrak{p}} \times_A Z = Z_{\mathfrak{p}}$ is non empty. We have that $k_{\mathfrak{p}} = \operatorname{Frac}(A/\mathfrak{p})$ which is an A algebra, therefore, $Z_{\mathfrak{p}} \subset (\mathbb{P}^n_A)_{\mathfrak{p}}$, and $(\mathbb{P}^n_A)_{\mathfrak{p}} = \mathbb{P}^n_A \times_A \operatorname{Spec} k_{\mathfrak{p}} \cong \mathbb{P}^n_{k_{\mathfrak{p}}}$. It follows that $Z_{\mathfrak{p}}$ is a closed subset of $\mathbb{P}^n_{k_{\mathfrak{p}}}$, and that locally

$$Z_{\mathfrak{p}} \cap \operatorname{Spec} k_{\mathfrak{p}} \times_A U_{x_i} = \operatorname{Spec} k_{\mathfrak{p}} \otimes_A (A[x_0, \dots, x_n]_{x_i})/(I_{x_0})_0 \cong \operatorname{Spec}(k_{\mathfrak{p}}[x_0, \dots, x_n]_{x_i})_0/J$$

where J is the ideal generated by the image of $(I_{x_0})_0$ under the map $(A[x_0, \dots, x_n]_{x_i})_0 \to (k_{\mathfrak{p}}[x_0, \dots, x_n]_{x_i})_0$. Hence, $Z_{\mathfrak{p}} = \mathbb{V}(I_{\mathfrak{p}})$, where $I_{\mathfrak{p}} = \langle [g_1], [g_2], \dots \rangle$, and $[g_i]$ is the image of the map:

$$A[x_0, \dots, x_n] \longrightarrow k_{\mathfrak{p}}[x_0, \dots, x_n]$$

induced by the projection $\pi : A \to A/\mathfrak{p}$, followed by the inclusion $A/\mathfrak{p} \hookrightarrow \operatorname{Frac}(A/\mathfrak{p})$. It follows that $Z_{\mathfrak{p}}$ is non empty if and only if $\mathbb{V}(I_{\mathfrak{p}}) \neq \mathbb{V}(\langle x_0, \dots, x_n \rangle)$, hence $\sqrt{I_{\mathfrak{p}}} \not\supset \langle x_0, \dots, x_n \rangle$. Equivalently for all $n > 0$, we have that:

$$\langle x_0, \dots, x_n \rangle^n \not\subset \langle [g_1], [g_2], \dots \rangle$$

If $S = k_{\mathfrak{p}}[x_0, \dots, x_n]$, then non containment is equivalent to the map:

$$\begin{aligned} \bigoplus_i (A[x_0, \dots, x_n])_{d - \deg g_i} &\longrightarrow S_d \\ f_i &\longmapsto [f_i g_i] \end{aligned}$$

not begin surjective for all d. Let $d_0 = \dim_{k_{\mathfrak{p}}} S_d$[61], then this gives us a matrix with coefficients in A, d_0 rows, and potentially infinite columns. All of the $d_0 \times d_0$ minors of this matrix must have determinant zero in $k_{\mathfrak{p}}$, so the determinants lie in $\mathfrak{p}$, and therefore the ideal generated by these determinants, $\tilde{J}$, is contained in $\mathfrak{p}$. It follows that the fibre $Z_{\mathfrak{p}} = \pi^{-1}(\mathfrak{p}) \cap Z$ is non empty if and only $\mathfrak{p}$ lies in $\mathbb{V}(\tilde{J})$.

Now if $\mathfrak{p} \in \pi(Z)$, then $\pi^{-1}(\mathfrak{p}) \subset Z$, hence $Z_{\mathfrak{p}}$ is nonempty so $\mathfrak{p} \in \mathbb{V}(\tilde{J})$, and if $\mathfrak{p} \in \mathbb{V}(\tilde{J})$ then the fibre $Z_{\mathfrak{p}}$ is non empty, so $\mathfrak{p} \in \pi(Z)$. Therefore, $\pi(Z) = \mathbb{V}(\tilde{J})$, hence π is closed map, and $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper as desired.

We have the following corollary:

Corollary 3.7.2. *Let $Z \subset \mathbb{P}^n_A$ be a closed subscheme, then Z is proper over* $\operatorname{Spec} A$.

Proof. The map $Z \to \operatorname{Spec} A$ is given by the closed embedding $\iota : Z \to \mathbb{P}^n_A$, followed by the canonical morphism $\mathbb{P}^n_A \to \operatorname{Spec} A$ from Example 2.3.1, then by Example 3.7.3, we have that this map is proper. By Example 3.7.2, closed embeddings are proper, and by Corollary 3.7.1 proper morphisms are closed under composition. It follows that $Z \to \operatorname{Spec} A$ is proper. $\square$

Recall that if $f : X \to Y$ is a continuous map between Hausdorff topological with X compact, then f is proper. We wish to prove the algebraic geometry analogue of this result; i.e. if X and Y are S-schemes, with X proper over S and Y separated over S, then any morphism $f : X \to Y$ is proper as well. This will follow from the following lemma:

Lemma 3.7.3. *Let X and X' be Y-schemes, and Y a separated Z-scheme. Then the map $X \times_Y X' \to X \times_Z X'$ is a closed embedding.*

[61] Note this that this is finite, as $\dim_{k_{\mathfrak{p}}}$ is equal to the partitions of d.

Proof. This follows from Theorem 2.3.1 as the following diagram is Cartesian:

$$\begin{tikzcd} X \times_Y X' \arrow[r] \arrow[d] & X \times_Z X' \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

It follows that the morphism $X \times_Y X' \to X \times_Z X'$ is the base change of $\Delta_Y : Y \to Y \times_Z Y$, which is a closed embedding. Since closed embeddings are stable under base change the claim follows. □

We can now prove the desired result:

Theorem 3.7.1. *Let X and Y be Z-schemes, with Y separated over Z, and $f : X \to Y$ a Z-scheme morphism. Then the following hold:*

a) If X is universally closed over Z, then f is universally closed.

b) If X is proper over Z, then f is proper.

Proof. Let g and h be the morphisms which make X and Y Z-Schemes respectively. Let α be the unique morphism making the following the diagram commute:

$$\begin{tikzcd} X \arrow[dr, "\alpha"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\mathrm{Id}"'] & & \\ & X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "h"] \\ & X \arrow[r, "g"] & Z \end{tikzcd}$$

It follows that f factors as:

$$X \xrightarrow{\alpha} X \times_Z Y \xrightarrow{\pi_Y} Y$$

We see that π_Y is the base change of a universally closed morphism, and is thus universally closed. It thus suffices to show that α is a universally closed. With $X' = Y$, we claim that, up to isomorphism, α is the top horizontal map making the diagram in Lemma 3.7.3 commute. Indeed, if $X' = Y$ then $X \times_Y Y$ is uniquely isomorphic to X, with projections given by $\mathrm{Id} : X \to X$ and $f : X \to Y$. X then fits into the following Cartesian diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X \times_Z Y \arrow[d, "f \times \mathrm{Id}"] \\ Y \arrow[r, "\Delta"] & Y \times_Z Y \end{tikzcd}$$

By our work in Theorem 2.3.1, the horizontal map is then precisely the one defining α, so by Lemma 3.7.3 α is a closed embedding. Since closed embedding are universally closed by Example 3.7.2, we have proven a).

Now suppose that X is proper over Z, then π_Y is the base change of a proper map and is thus proper. In particular α is a closed embedding which is proper by Example 3.7.2, so the same argument guarantees that f is a proper map implying b). □

Example 3.7.4. Let X be a projective variety, then X is proper by Corollary 3.7.2 so any k morphism $X \to Y$ with Y separated over k is proper. In particular, every k morphism from X to a variety Y is proper.

We end the section with the following general result:

Theorem 3.7.2. *Let P be a property of a morphism of Z-schemes $f : X \to Y$ such that P is closed under composition and stable under base change. Then if $f : X \to Y$ and $g : X' \to Y'$ both satisfy P then the induced morphism $f \times g : X \times_Z X' \to Y \times_Z Y'$ satisfies property P.*

Proof. Let h and h' be the morphisms making Y and Y' Z-schemes, and q and q' the morphisms making X and X' Z-schemes. We have that $f \times g$ comes from the following commutative diagram:

$$\begin{tikzcd}
X \times_Z X' \arrow[dr, "f\times g"] \arrow[drr, bend left, "g\circ\pi_{X'}"] \arrow[ddr, bend right, "f\circ\pi_X"] & & \\
& Y \times_Z Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "\pi_Y"] & Y' \arrow[d, "h'"] \\
& Y \arrow[r, "h"] & Z
\end{tikzcd}$$

It is clear that $f \times g = \mathrm{Id} \times g \circ f \times \mathrm{Id}$, so it suffices to show that $f \times \mathrm{Id}$ and $\mathrm{Id} \times g$ both satisfy proper P. We have the following commutative diagram:

$$\begin{tikzcd}
X \times_Z X' \arrow[r, "f\times\mathrm{Id}"] \arrow[d, "\pi_X"] & Y \times_Z X' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_Y"] & X' \arrow[d, "q'"] \\
X \arrow[r, "f"] & Y \arrow[r, "h"] & Z
\end{tikzcd}$$

The right square is Cartesian, and since $h \circ f = q$, and $\pi_{X'} \circ f \times \mathrm{Id} = \pi_{X'}$ the outer diagram is Cartesian, so the left square is also Cartesian. Since the left square is Cartesian, it follows that $f \times \mathrm{Id}$ is the base change of f, and thus satisfies property P. Now note that we also have the following commutative diagram:

$$\begin{tikzcd}
Y \times_Z X' \arrow[r, "\mathrm{Id}\times g"] \arrow[d, "\pi_{X'}"] & Y \times_Z Y' \arrow[r, "\pi_Y"] \arrow[d, "\pi_{Y'}"] & Y \arrow[d, "h"] \\
X' \arrow[r, "g"] & Y' \arrow[r, "h'"] & Z
\end{tikzcd}$$

The right square is Cartesian, and the outer square satisfies $h' \circ g = q'$, and $\pi_Y \circ \mathrm{Id} \times g = \pi_Y$, so it is Cartesian as well. It follows that the left square is Cartesian, and that $\mathrm{Id} \times g$ is the base change of g, so $\mathrm{Id} \times g$ satisfies property P as well. Since P is closed under composition, we have that $f \times g$ satisfies property P as well. $\square$

It immediately follows that nearly ever property of morphisms we have studied in this chapter is stable under fibre products as in the above discussion.

3.8 Affine Morphisms

In this section we introduce affine morphisms, though it will more fruitful to study special types of affine morphisms as in the next section.

Definition 3.8.1. Let $f : X \to Y$ be a morphism of schemes, then f is affine if for every open affine $V \subset Y$, we have that $f^{-1}(V)$ is also affine.

Example 3.8.1. Any closed embedding is an affine morphism. Any open embedding is an affine morphism.

We prove the following structure result regarding schemes:

Lemma 3.8.1. *Let X be a scheme, and $\mathscr{O}_X(X) = A$. Suppose that $g_1, \ldots, g_n \in A$ generate the unit ideal, and that X_{g_i} is affine for each i, then $X \cong \operatorname{Spec} A$.*

Proof. Recall that:

$$X_{g_i} = \{x \in X : (g_i)_x \notin \mathfrak{m}_x\} \tag{3.8.1}$$

is an open set in X. Moreover, since the g_i generate the unit ideal in A, we have that for every affine open $U \subset X$, $g_i|_U$ generate the unit ideal of $\mathscr{O}_X(U)$. It follows that the distinguished open $U_{g_i|_U} \subset U$ cover U, however by our work in Proposition 2.1.2 we know that:

$$U_{g_i|_U} = X_{g_i} \cap U$$

It follows that X_{g_i} cover X as if $x \in X$, then there is an open affine U containing x, and thus an i such that $x \in U_{g_i|_U}$, hence $x \in X_{g_i}$ and $\bigcup_i X_{g_i} \subset X$.

Set $X_{g_i} = \operatorname{Spec} A_i$, and $X_{ij} = X_{g_i} \cap X_{g_j}$. Since each X_{g_i} is affine, by our work in Proposition 2.1.2, we have that each X_{ij} is a distinguished open in both $\operatorname{Spec} A_i$ and $\operatorname{Spec} A_j$, thus:

$$\operatorname{Spec}(A_i)_{g_j|_{X_i}} \cong X_{ij} \cong \operatorname{Spec}(A_j)_{g_i|_{X_j}}$$

The rings $\mathscr{O}_X(X_{g_j})$ and $\mathscr{O}_X(X_{ij})$ have canonical $\mathscr{O}_X(X)$ module structures given by by the restriction maps $\theta^X_{X_{g_j}}$ and $\theta^X_{X_{ij}}$. There is a natural map

$$\begin{aligned} \alpha : \mathscr{O}_X(X) &\longrightarrow \bigoplus_j \mathscr{O}_X(X_{g_j}) \\ s &\longmapsto (s|_{X_{g_j}}) \end{aligned}$$

where by $(s|_{X_{g_j}})$ we mean $(s|_{X_{g_0}}, \ldots, s|_{X_{g_n}})$. This map is an injection as the X_{g_j} cover X. We define another map:

$$\begin{aligned} \beta : \bigoplus_j \mathscr{O}_X(X_{g_j}) &\longrightarrow \bigoplus_{k<j} \mathscr{O}_X(X_{kj}) \\ (s_j) &\longmapsto (s_{kj}) \end{aligned}$$

where:

$$(s_{kj}) = \left(s_k|_{X_{kj}} - s_j|_{X_{kj}}\right)$$

Note that $\beta \circ \alpha = 0$, as:

$$\beta((s|_{X_{g_j}})) = ((s|_{X_{g_k}})|_{X_kj} - (s|_{X_{g_j}})|_{X_{kj}}) = (s|_{X_{kj}} - s|_{X_{kj}}) = 0$$

Similarly, if $\alpha((s_j)) = 0$, then we have sections $s_j \in \mathscr{O}_X(X_{g_j})$ such that for all k and j:

$$s_j|_{X_{kj}} = s_k|_{X_{kj}}$$

It follows by the sheaf axioms, that there exists an $s \in \mathscr{O}_X(X)$ such that $s|_{X_{g_j}} = s_j$. We have thus shown that $\ker \beta = \operatorname{im} \alpha$, and so we have the following sequence of $\mathscr{O}_X(X)$ modules:

$$0 \longrightarrow \mathscr{O}_X(X) \longrightarrow \bigoplus_j \mathscr{O}_X(X_{g_j}) \longrightarrow \bigoplus_{k<j} \mathscr{O}_X(X_{kj})$$

hence the following exact sequence of A modules:

$$0 \longrightarrow A \longrightarrow \bigoplus_j A_j \longrightarrow \bigoplus_{k<j} A_{kj}$$

We can localize[62] the sequence at g_i, to obtain the exact sequence:

$$0 \longrightarrow A_{g_i} \longrightarrow \bigoplus_j (A_j)_{g_i|_{X_{g_j}}} \longrightarrow \bigoplus_{k<j} (A_{kj})_{g_i|_{X_{kj}}}$$

Note that first morphism, which we denote α_i, is induced by the unique ones which makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\theta^X_{X_{g_j}}"] \arrow[d, "\pi_{g_i}"'] & A_j \arrow[d, "\theta^{X_{g_j}}_{X_{ij}}"] \\ A_{g_i} \arrow[r, "(\alpha_i)_j"] & (A_j)_{g_i|_{X_j}} \end{tikzcd}$$

[62]We take this on a faith for the moment. A precise proof is given in greater generality in Lemma 5.3.1.

where $(\alpha_i)_j$ is the jth component of the map α_i. Moreover, the second morphism is given by:

$$\begin{aligned}\beta_i : \bigoplus_j \mathscr{O}_X(X_{ji}) &\longrightarrow \bigoplus_{k<j} \mathscr{O}_X(X_{kj} \cap X_i)\\ (s_j) &\longmapsto (s_k|_{X_{ki}\cap X_j} - s_j|_{X_{ji}\cap X_k})\end{aligned}$$

Finally, note that $(A_i)_{g_i|_{X_i}}$ is A_i as $g_i|_{X_i}$ is invertible in A_i, so the map $(\alpha_i)_i$ is given by the localization of the restriction map $\theta^X_{X_i}$. We wish to show that $(\alpha_i)_i$ is an isomorphism.

Let $a/g_i^k \in A_{g_i}$ satisfy $(\alpha_i)_i(a/g_i^k) = 0$, then, since g_i maps to an invertible element, we have that $a/1$ also maps to zero. We claim that $\alpha_i(a/1) = 0$; indeed, we have that $(\alpha_i)_i(a/1) = 0$ by assumption, and that:

$$\begin{aligned}(\alpha_i)_j(a/1) =&(\alpha_i)_j(\pi_{g_i}(a))\\ =&(a|_{X_{f_j}})|_{X_{ij}}\\ =&a|_{X_{ij}}\\ =&(a|_{X_{f_i}})|_{X_{ij}}\end{aligned}$$

Since $\theta^{X_{g_i}}_{X_{ii}} = \theta^{X_{g_i}}_{X_{g_i}} = \mathrm{Id}$, it follows that:

$$a|_{X_{f_i}} = (\alpha_i)_i(\pi_{g_i}(a)) = (\alpha_i)_i(a/1) = 0$$

hence $(\alpha_i)_j(a/1) = 0$ for all $j \neq i$ as well. By exactness, have that $a/1 = 0$, hence $(\alpha_i)_i$ is injective.

Now let $s \in A_i$; then $s|_{X_{ij}} \in (A_j)|_{f_i|_{X_j}}$ for all j, hence we have an element $(s_j) \in \bigoplus_j (A_j)|_{f_i|_{X_j}}$. It follows that:

$$\beta_i((s_j)) = (s_k|_{X_{ki}\cap X_j} - s_j|_{X_{ji}\cap X_k})$$

but:

$$s_k|_{X_{ki}\cap X_j} = s|_{X_{ik}}|_{X_{ij}\cap X_j} = s|_{X_{ij}\cap X_j}$$

and similarly for j, hence $\beta_i((s_j)) = 0$. It follows by exactness that there exists some $a/g_i^k \in A_{g_i}$ such that $\alpha_i(a/g_i^k) = (s_j)$, hence $(\alpha_i)_i$ is surjective. Therefore, we have $A_{g_i} \cong A_i$, and so $X_{g_i} \cong \operatorname{Spec} A_{g_i}$.

By Proposition 2.1.2, there is a natural map $f' : X \to \operatorname{Spec} A$ induced by the identity map $A \to \mathscr{O}_X(X)$. Furthermore, since the g_i generate the unit ideal in A, we know that U_{g_i} cover $\operatorname{Spec} A$. The morphism:

$$f'|_{X_{g_i}} : X_{g_i} \longrightarrow U_{g_i}$$

is the one induced by the ring homomorphism:

$$\begin{aligned}A_{g_i} &\longrightarrow \mathscr{O}_X(X_{g_i})\\ a/g_i^k &\longmapsto a|_{X_{g_i}} \cdot (g_i|_{X_{g_i}})^{-k}\end{aligned}$$

however this is precisely $(\alpha_i)_i$, which we just showed was an isomorphism. Since f' restricts to an isomorphism on the inverse image of an open cover of $\operatorname{Spec} A$, we have that $X \cong \operatorname{Spec} A$ □

Proposition 3.8.1. *Affine morphisms are local on target.*

Proof. Suppose that $f : X \to Y$ is an affine morphism, and let $V \subset Y$ be an affine open. We wish to show that the morphism $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is an affine morphism as well. Well, let $W \subset V$ be an affine open, then, in particular, W is an affine open in Y, and $(f|_{f^{-1}(V)})^{-1}(W) = f^{-1}(W)$ which is affine by assumption. It follows that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is an affine morphism as desired.

Let $f : X \to Y$ be a morphism, and let $\{V_i = \operatorname{Spec} B_i\}$ be an affine open cover such that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is an affine morphism. By assumption, each $f^{-1}(V_i)$ is affine so set $f^{-1}(V_i) = \operatorname{Spec} A_i$, and let $V = \operatorname{Spec} B \subset Y$ be an arbitrary open affine of Y. We have that:

$$V = \bigcup_i V_i \cap V$$

By Lemma 2.1.1, each $V_i \cap V$ can be covered by open affines:

$$V_i \cap V = \bigcup_j U_{ij}$$

where U_{ij} is a distinguished open affine in V_i and V. Hence:

$$V = \bigcup_{ij} U_{ij}$$

and each U_{ij} satisfies:

$$(f|_{f^{-1}(V)})^{-1}(U_{ij}) = f^{-1}(U_{ij}) = (f|_{f^{-1}(V_i)})^{-1}(U_{ij})$$

But $f|_{f^{-1}(V_i)} : \operatorname{Spec} A_i \to \operatorname{Spec} B_i$ is a morphism of affine schemes, and U_{ij} is a distinguished open, hence $(f|_{f^{-1}(V_i)})^{-1}(U_{ij})$ is a distinguished open of $\operatorname{Spec} A_i$ and thus an affine open of $f^{-1}(V)$. It follows that $V = \operatorname{Spec} B$ admits a cover of distinguished opens U_{ij} such that $f^{-1}(U_{ij}) \subset f^{-1}(V)$ is an affine open.

We have thus reduced the result to the following problem: if $f : X \to \operatorname{Spec} B$ is a morphism of schemes such that there is a cover of $\operatorname{Spec} B$ by distinguished opens $\{U_{b_i}\}_{i=0}^n$ with $f^{-1}(U_{b_i})$ affine, then X is an affine scheme. Let $\phi : B \to \mathscr{O}_X(X)$ be the unique ring homomorphism inducing f; by our work in Proposition 2.1.2, we know that $f^{-1}(U_{b_i}) = X_{\phi(b_i)}$. Since b_i generate the unit ideal in in B, we have that $\phi(b_i)$ generate the unit ideal in $\mathscr{O}_X(X)$. Therefore, by Lemma 3.8.1 we have that X is affine, hence if $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is an affine morphism, then f is an affine morphism. □

Corollary 3.8.1. *Morphisms between affine schemes are affine. In particular, let $f : X \to Y$ be a morphism of schemes, and $\{U_i\}$ an affine open cover of Y, then f is affine if and only if $f^{-1}(U_i)$ is an affine scheme for all i.*

Proof. Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes. Let $V \subset \operatorname{Spec} B$ be an open affine scheme, then we would like to show that $f^{-1}(V)$ is an affine scheme.

Set $V = \operatorname{Spec} C$, and set $X = f^{-1}(V)$. Then we have have morphism $g : X \to \operatorname{Spec} C$ given by $f|_{f^{-1}(V)}$. We can cover $\operatorname{Spec} C$ with distinguished opens U_{c_i} which are also distinguished opens of $\operatorname{Spec} B$, hence $g^{-1}(U_{c_i})$ are open affines, as f is a morphism of affine schemes. In particular, if $\phi : C \to \mathscr{O}_X(X)$ is the unique morphism inducing g, then $g^{-1}(U_{c_i}) = X_{\phi(c_i)}$. Since c_i generate the unit ideal in C, $\phi(c_i)$ generate the unit ideal in $\mathscr{O}_X(X)$. It follows by Lemma 3.8.1 that X is affine, hence morphisms between affine schemes are affine.

Now let $f : X \to Y$ be an affine morphism; then for any open affine cover $\{U_i\}$, we have that $f^{-1}(U_i)$ is open by definition. Conversely, if $f^{-1}(U_i)$ is an affine scheme, then $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is a morphism of affine schemes, and thus an affine morphism. It follows by Proposition 3.8.1 that f is an affine morphism. □

We of course need to also check that affine morphisms are stable under base change, and that the composition of affine morphisms is affine:

Proposition 3.8.2.

a) *Affine morphisms are stable under base change.*

b) *The composition of affine morphisms is again affine.*

Proof. For a), let $f : X \to Z$ be an affine morphism, and $g : Y \to Z$ be any morphism. Let $\{V_i\}$ be an affine cover of Z, then $\{W_i = f^{-1}(V_i)\}$ is an affine cover for X, and we can obtain an affine open cover of $\{U_{ij}\}$ of Y such that $g(U_{ij}) \subset V_i$ for all j. We need only show that $\pi_Y^{-1}(U_{ij})$ is an affine scheme; indeed we claim that $\pi_Y^{-1}(U_{ij}) \cong W_i \times_{V_i} U_{ij}$, which is manifestly an affine scheme. For ease of notation, set $S = \pi_Y^{-1}(U_{ij})$, then $\pi_Y|_S(S) \subset U_{ij}$, and we have that:

$$f \circ \pi_X|_S(S) = g \circ \pi_Y|_S(S) \subset V_i$$

It follows that:

$$\pi_X|_S(S) \subset f^{-1}(V_i) = W_i$$

We thus have unique morphisms $\pi_{U_{ij}} : S \to U_{ij}$ and $\pi_{W_i} : S \to W_i$ such that $\iota_{U_{ij}} \circ \pi_{U_{ij}} = \pi_y|_S$ and $\iota_{W_i} \circ \pi_{W_i} = \pi_X|_S$. Moreover, these morphisms make the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "\pi_{U_{ij}}"] \arrow[d, "\pi_{W_i}"] & U_{ij} \arrow[d, "g|_{U_{ij}}"] \\ W_i \arrow[r, "f|_{w_i}"] & V_i \end{tikzcd}$$

Now suppose that we have morphisms $p_{W_i} : Q \to W_i$ and $p_{U_{ij}} : Q \to U_{ij}$ which make the relevant diagram commute. Then by composing with open embeddings, we obtain a unique morphism $\phi : Q \to X \times_Z Y$ such that the following diagram commutes:

$$\begin{tikzcd} Q \arrow[dr, "\phi"] \arrow[drr, bend left, "\iota_{U_{ij}} \circ p_{U_{ij}}"] \arrow[ddr, bend right, "\iota_{W_i} \circ p_{W_i}"'] & & \\ & X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

We first claim that $\phi(Q) \subset S$. Indeed, we have that

$$\pi_Y(\phi(Q)) = \iota_{U_{ij}} \circ p_{U_{ij}}(Q) \subset U_{ij}$$

hence:

$$\phi(Q) \subset \pi_Y^{-1}(U_{ij}) = S$$

Therefore, there exists a unique map $\psi : Q \to S$ such that $\iota_S \circ \psi = \phi$. We need to check that that ψ makes the relevant diagram commute. We see that:

$$\begin{aligned} \iota_{U_{ij}} \circ (\pi_{U_{ij}} \circ \psi) =& \pi_Y|_S \circ \psi \\ =& \pi_Y \circ \iota_S \circ \psi \\ =& \pi_Y \circ \phi \\ =& \iota_{U_{ij}} \circ p_{U_{ij}} \end{aligned}$$

and similarly that:

$$\iota_{W_i} \circ (\pi_{W_i} \circ \psi) = \iota_{W_i} \circ p_{W_i}$$

Since open embeddings are monomorphisms, it follows that the following diagram commutes:

$$\begin{tikzcd} Q \arrow[dr, "\psi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & S \arrow[r, "\pi_{U_{ij}}"] \arrow[d, "\pi_{W_i}"] & U_{ij} \arrow[d, "g|_{U_{ij}}"] \\ & W_i \arrow[r, "f|_{w_i}"] & V_i \end{tikzcd}$$

so S satisfies the universal property of $W_i \times_{V_i} U_{ij}$ and is thus affine. It follows by Corollary 3.8.1 that π_Y is an affine morphism, as desired.

For b), let $f : X \to Y$, and $g : Y \to Z$ be affine morphisms, then clearly we have that for any affine open $U \subset Z$, that $f^{-1}(g^{-1}(U))$ is affine; it follows that $g \circ f$ is an affine morphism implying the claim. $\square$

3.9 Finite and Integral Morphisms

In this section, we discuss finite and integral morphisms of schemes. Recall that a morphism of rings $\phi : B \to A$ is finite if it makes A a finitely generated B module. That is, there is a finite set $\{a_1, \ldots, a_n\}$ such that any a can be written as:

$$a = \sum_{i=1}^{n} \phi(b_i) a_i$$

for some $b_i \in B$. Often times the notation ϕ is suppressed and we write $b_i \cdot a_i$. Furthermore, a morphism $\phi : B \to A$, is integral if every element of A is integral over $\phi(B)$. That is, every $a \in A$ is the root of some monic polynomial in $\phi(B)[x]$. If a finite morphism, or an integral morphisms is injective, i.e. an inclusion of rings, then they are called finite extensions, or integral extensions respectively. In either case, we will often suppress the notation $\phi(p)$ for a polynomial in $\phi(B)[x]$, and simply write $p \in B[x]$ with evaluation on A understood to be the one induced by ϕ.

Definition 3.9.1. Let $f : X \to Y$ be a morphism of schemes, then f is **finite** if for every open affine $V \subset Y$ we have that $f^{-1}(V)$ is affine, and the induced morphism $f|_{f^{-1}(V)}$ of affine schemes comes from a finite morphism of rings. Similarly, f is **integral** if for every open affine $V \subset Y$ we have that $f^{-1}(V)$ is affine, and the induced morphism $f|_{f^{-1}(V)}$ of affine schemes comes from an integral morphism of rings.

Note that finite and integral morphisms are examples of affine morphisms. We need to show that finite morphisms are closed under composition before moving forward:

Lemma 3.9.1. *Let $f : X \to Y$ and $g : Y \to Z$ be finite morphisms. Then $g \circ f$ is a finite morphism.*

Proof. This statement clearly reduces to the following: if $\phi : C \to B$, and $\psi : B \to A$ are finite, then $\psi \circ \phi$ is finite. Suppose ψ and ϕ are finite, then there exists $\{b_1, \ldots, b_m\}$ and $\{a_1, \ldots, a_n\}$ which generate B as a C-module and A as a B-module. Let $a \in A$, then there exist β_i such that:

$$a = \sum_{i=1}^{n} \phi(\beta_i) \cdot a_i$$

There exist c_{ij} such that each β_i satisfies:

$$\beta_i = \sum_{j=1}^{m} \psi(c_i j) \cdot b_j$$

hence:

$$a = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi(\psi(c_{ij})) \cdot \phi(b_j) a_i$$

hence the set $\{\phi(b_j) \cdot a_i : 1 \le i \le n, 1 \le j \le m\}$ generates A as a C module, which is finite, so $\psi \circ \phi$ is finite. □

We also demonstrate the following relationship between integral and finite morphisms:

Proposition 3.9.1. *Let $f : X \to Y$ be a finite morphism, then f is integral. If $f : X \to Y$ is integral and locally of finite type, then f is finite.*

Proof. For the first statement, it suffices to show that if $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is finite then it is integral. This then reduces to the case that if $\phi : B \to A$ is a finite morphism then it is integral.

Suppose $\phi : B \to A$ is finite, then A is a finitely generated b module, hence there exists $a_1, \ldots, a_n \in A$ such that for all $a \in A$ there are $b_1, \ldots, b_n \in B$ satisfying:

$$a = b_1 a_1 + \cdots + b_n a_n$$

We want to show that any $a \in A$ is the root of a monic polynomial in $\phi(B)[x]$. First note that we have a surjective map of B-modules:

$$\pi : B^{\oplus n} \longrightarrow A$$
$$(b_1, \ldots, b_n) \longmapsto \sum_{i=1}^{n} a_i b_i$$

and that for any $a \in A$ we have B-module endomorphism $\psi_a \in \mathrm{End}_B(A)$ given by $s \mapsto a \cdot s$. For each i we have:

$$a \cdot a_i = \sum_{ij} b_{ij} a_j$$

for some $b_{ij} \in B$. This gives us an $n \times n$ matrix T with coefficients in B given by:

$$T = \begin{pmatrix} b_{11} & \cdots & b_{n1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{nn} \end{pmatrix}$$

The following diagram then commutes:

$$\begin{tikzcd} B^{\oplus n} \arrow[r, "\pi"] \arrow[d, "T"] & A \arrow[d, "\psi_a"] \\ B^{\oplus n} \arrow[r, "\pi"] & A \end{tikzcd}$$

Let $p \in B[x]$, and consider $p(T)$ and $p(\psi_a)$, where in the latter polynomial p is technically a polynomial in $\phi(B)[x]$ as p is acting on elements of a via the ring homomorphism ϕ. The following diagram then also commutes:

$$\begin{tikzcd} B^{\oplus n} \arrow[r, "\pi"] \arrow[d, "p(T)"] & A \arrow[d, "p(\psi_a)"] \\ B^{\oplus n} \arrow[r, "\pi"] & A \end{tikzcd}$$

as it would commute for any endomorphism of A and it's induced matrix T. Suppose that $p(T)$ is the zero morphism, and let $a \in A$. Then there exists $(b_1, \ldots, b_n) \in B^{\oplus n}$ such that $\pi(b_1, \ldots, b_n) = a'$ so:

$$p(\psi_a)(a') = p(\psi_a) \circ \pi(b_1, \ldots, b_n) = \pi \circ p(T)(b_1, \ldots, b_n) = 0$$

so $p(\psi_a)$ is also zero. If $p(\psi_a) = 0$, then $p(a)$ is also zero as the ring homomorphism $a \mapsto \psi_a$ is injective; it thus suffices to show that there exists a polynomial $p \in B[x]$ such that $p(T) = 0$.

Note that if B is a field then this holds by the Cayley-Hamilton theorem. Consider the surjection:

$$\begin{aligned} F : \mathbb{Z}[x_{ij}] &\longrightarrow B \\ x_{ij} &\longrightarrow b_{ij} \end{aligned}$$

and the inclusion:

$$G : \mathbb{Z}[x_{ij}] \longrightarrow \mathbb{Q}(x_{ij})$$

where $\mathbb{Q}(X_{ij})$ is the field of fractions $\mathrm{Frac}(\mathbb{Z}[x_{ij}])$. We have an induced ring homomorphism:

$$F' : \mathrm{End}_{\mathbb{Z}[x_{ij}]}(\mathbb{Z}[x_{ij}]^n) \longrightarrow \mathrm{End}_B(B^n)$$

which is given by[63]:

$$\begin{pmatrix} p_{11} & \cdots & p_{n1} \\ \vdots & \ddots & \vdots \\ p_{1n} & \cdots & p_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} F(p_{11}) & \cdots & F(p_{n1}) \\ \vdots & \ddots & \vdots \\ F(p_{1n}) & \cdots & F(p_{nn}) \end{pmatrix}$$

and a similar inclusion:

$$G' : \mathrm{End}_{\mathbb{Z}[x_{ij}]}(\mathbb{Z}[x_{ij}]^n) \longrightarrow \mathrm{End}_{\mathbb{Q}(x_{ij})}(\mathbb{Q}(x_{ij})^n)$$

[63] Since both $\mathbb{Z}[x_{ij}]^n$ and B^n are free modules of rank n, their endomorphism rings are $n \times n$ matrices with coefficients in their respective rings.

Let:

$$T' = \begin{pmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix}$$

then $F'(T') = T$. Since $\mathbb{Q}(x_{ij})$ is a field, there is a monic polynomial $q \in \mathbb{Q}(x_{ij})[y]$ such that $q(G'(T')) = 0$. This polynomial is given by $\det(y \cdot I_n - G'(T'))$, where I_n is the $n \times n$ identity matrix. Since each component $G'(T')_{ij} \in \mathbb{Z}[x_{ij}] \subset \mathbb{Q}(x_{ij})$, it follows that $q \in (\mathbb{Z}[x_{ij}])[y] \subset \mathbb{Q}(x_{ij})[y]$, and $q(T') = 0$. We have an induced ring homomorphism $F'' : (\mathbb{Z}[x_{ij}])[y] \to B[y]$, and it follows that:

$$0 = F'(q(T')) = F''(q)(F'(T')) = F''(q)(T)$$

so $p = F''(q)$ is a monic polynomial in $B[y]$ which has T as a root. By our earlier discussion it follows that $p(a) = 0$, and since $a \in A$ was arbitrary the map $\phi : B \to A$ is integral, implying the claim.

For the second statement, it also suffices to show that if $\phi : B \to A$ is an integral morphism which makes A a finitely generated B-algebra then ϕ is finite. Let $\{a_1, \ldots, a_n\}$ generate A as a B algebra. Then morphism:

$$\begin{aligned} B[x_1, \ldots, x_n] &\longrightarrow A \\ x_i &\longrightarrow a_i \end{aligned}$$

is surjective. Moreover, for all $a \in A$, there exists a monic $p \in B[y]$ such that $p(0) = a$. Let $p_i \in B[y]$ satisfy $p_i(0) = a_i$, and let $d_i = \deg(p_i)$, then we claim that the set:

$$\{a_1^{m_1} \cdots a_n^{m_n} : 0 \leq m_i \leq d_i - 1\}$$

generate A as a B module. Let $a \in A$, then we have that:

$$a = \sum_{i_1 \cdots i_n} b_{i_1 \cdots i_n} a_1^{i_1} \cdots a_n^{i_n}$$

for some $b_{i_1 \cdots i_n}$, then we need only show that each $i_j \leq d_i - 1$. We prove this by induction on n; if $n = 1$ then we have that a can be written as:

$$a = \sum_i b_i a_1^i$$

Now $\phi(p)(a_1) = 0$, so:

$$a_1^{d_1} = -(b_{d_1 - 1} a_1^{d_1 - 1} + \cdots + b_0) \tag{3.9.1}$$

We need only show that any $a_1^{d_1 + m}$ for $m \geq 0$ is in the B-span of $\{a_1^i : 0 \leq i \leq d_i - 1\}$. The base case $m = 0$ is proven, now suppose $m - 1$th case so that:

$$\begin{aligned} a_1^{d_1 + m} =& (a_1^{d_1 + m - 1}) \cdot a_1 = a_1 \left(b'_{d_1 - 1} \cdot a_1^{d_1 - 1} + \cdots + b'_0\right) \\ =& a_1^{d_1} b'_{d_1 - 1} + \cdots + a_1 \cdot b'_0 \end{aligned}$$

Since $a_1^{d_1}$ can be written as in (3.9.1), when $n = 1$ we have that A is a finitely generated B module. Now supposing the $n - 1$th case, we have that the sub algebra $A' \subset A$ generated by $\{a_1, \ldots, a_{n-1}\}$ is a finite B module. Since $\phi(B) \subset A'$, we have that A is integral over A', and A is clearly finitely generated over A' by a_n, hence A by the $n = 1$ case we have that A is a finite A' module. By Lemma 3.9.1, it follows that A is a finitely generated module with generators given by:

$$\{a_1^{m_1} \cdots a_n^{m_n} : 0 \leq m_i \leq d_i - 1\}$$

as desired. □

Corollary 3.9.1. *Let $\phi : B \to A$ be a ring homomorphism, and $a_1, a_2 \in A$ be integral over B. Then, $a_1 + a_2$, $a_1 \cdot a_2$, and $b \cdot a_i$ are integral elements over B.*

Proof. Let $A' \subset B$ be the B algebra generated by a_1 and a_2. The same induction argument in the second part of Proposition 3.9.1 then shows that A' is a finite B module[64], and thus A' is integral over B implying the claim. □

Example 3.9.1. Consider the map $\mathbb{Q} \to \bar{\mathbb{Q}}$, where $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. This is integral by construction, but is not finite as $\bar{\mathbb{Q}}$ is not a finite dimensional $\mathbb{Q}$-vector space. Indeed, suppose that $\bar{\mathbb{Q}}$ is n dimensional as a $\mathbb{Q}$ vector space, and consider the polynomial $x^{n+1} - 2$; this polynomial as $n+1$ roots over $\mathbb{C}$ all of which must lie in $\bar{\mathbb{Q}} \setminus \mathbb{Q}$. These roots are all linearly independent hence $\bar{\mathbb{Q}}$ contains an $n+1$ dimensional $\mathbb{Q}$-linear subspace, and $\mathbb{Q} \to \bar{\mathbb{Q}}$ can't be finite.

Note that when dealing with varieties over a fixed field k, then every morphism is of finite type[65]. Indeed, let A and B be finitely generated k algebras, with generating sets $\{b_1, \dots, b_n\}$ and $\{a_1, \dots, a_m\}$. if $\phi : B \to A$ is a morphism, then consider the induced morphism:

$$\begin{aligned} \phi' : B[x_1, \dots, x_m] &\longrightarrow A \\ x_i &\longmapsto a_i \end{aligned}$$

which on B acts by ϕ. This map is surjective as $k \subset B$, hence if:

$$a = p(a_1, \dots, a_m)$$

for some $p \in k[x_1, \dots, x_m]$, then $p \in B[x_1, \dots, x_n]$, and $\phi'(p) = a$. It follows that A is a finitely generated B algebra so any morphism of varieties must be of finite type. It follows that in this setting we have that integral morphisms and finite morphisms between varieties are the same.

We now proceed with the rest of our standard results:

Proposition 3.9.2. *The following hold:*

a) Integral morphisms are stable under composition.

b) Finite and integral morphisms are stable under base change.

c) Finite and integral morphism are local on target.

Proof. As in Lemma 3.9.1, *a*) clearly reduces to the following: if $\phi : C \to B$, and $\psi : B \to A$ are integral, then $\psi \circ \phi$ is integral. Suppose that ϕ and ψ are integral; let $a \in A$, then there exists a monic polynomial $p \in B[x]$ such that $p(a) = 0$. Set:

$$p(x) = b_0 + b_1 x + \cdots + x^n$$

and let $B' \subset B$ be the C algebra generated by $\{b_0, \dots, b_n\}$. Note that B' is integral over C as B is integral over C hence B' is a finite C module. Let $A' \subset A$ be the B' algebra generated by a, then A' is obviously finitely generated over B', and integral over B' by Corollary 3.9.1, so Proposition 3.9.1 show that A' is finite over B'. We thus have the composition:

$$C \to B' \to A'$$

is a composition of finite morphisms and is thus finite. It follows by Proposition 3.9.1 that $C \to A'$ is integral, thus there exists a monic polynomial $q \in C[y]$ such that $q(a) = 0$. Since $a \in A$ was arbitrary we have that $C \to A$ is integral as well.

We now have that *b*) reduces as: if $\phi : B \to A$ is finite/integral, and and $\psi : B \to C$ is any morphism, then the induced map $C \to A \otimes_B C$ is finite/integral. Suppose that ϕ is finite, and let $\{a_1, \dots, a_n\}$ generate A as a finite B-module. We claim that $S = \{a_1 \otimes 1, \dots, a_n \otimes 1\}$ generates $A \otimes_B C$ as a C module. Indeed, since $A \otimes_B C$ is generated as an abelian group by simple tensors, it suffices to show that

[64]This is because the argument only uses that the generators are integral.

[65]Not locally of finite type, because every variety is quasi-compact, hence we can take every open cover to be finite

any $a \otimes c$ lies in the C span of S. Well, for some $b_i \in B$:

$$\begin{aligned} a \otimes c &= \left(\sum_i a_i b_i\right) \otimes c \\ &= \sum_i (a_i b_i) \otimes c \\ &= \sum_i a_i \otimes (b_i c) \\ &= \sum_i (a_i \otimes 1) \cdot (1 \otimes b_i c) \end{aligned}$$

as desired[66].

Now suppose that ψ is an integral morphism, then by Corollary 3.9.1 we need only show that $a \otimes 1$ is integral over C. We know there exists a monic polynomial $p \in B[x]$, so consider it's image in $C[x]$, which we also denote by p. This polynomials image $A \otimes_B C[x]$ is given by:

$$(1 \otimes p)(x) = (1 \otimes b_0) + \cdots (1 \otimes b_n)x^n$$

then:

$$\begin{aligned} (1 \otimes p)(a \otimes 1) =& (1 \otimes b_0) + \cdots (1 \otimes 1)(a^n \otimes 1) \\ =& a \otimes b_0 + \cdots + a^n \otimes b_n \\ =& b_0 \otimes 1 + a^n \otimes 1 \\ =& (p(a)) \otimes 1 \\ =& 0 \end{aligned}$$

so $C \to A \otimes_B C$ is an integral, implying b).

For c), suppose that $f : X \to Y$ is an integral/finite morphism, and U is any affine open of Y, and set $V = f^{-1}(U)$. We need to show that $f|_V : V \to U$ is integral/finite. Note that by Proposition 3.8.1, we have that $f|_V : V \to U$ is an affine morphism, and that any open affine ove U is an affine open of Y, hence $(f|_V)|_{(f|_V)^{-1}(U)} = f|_{f^{-1}(U)}$ must come from an integral/finite morphism of rings by the definition of integral/finite morphisms. It follows that $f|_V$ is integral/finite.

Let $\{U_i = \operatorname{Spec} A_i\}$ be an open affine cover of Y, and $\{V_i = f^{-1}(U_i) = \operatorname{Spec} B_i\}$ be the corresponding open cover of X. Suppose that each $f : X \to Y$ is a morphism with each $f|_{V_i}$ integral/finite, and let $U = \operatorname{Spec} A \subset Y$ be an affine open of Y. Since f is affine by Proposition 3.8.1 we know that $f^{-1}(U) = \operatorname{Spec} B$ is affine. By Lemma 2.1.1, we can cover $\operatorname{Spec} A$ with open sets which are simultaneously distinguished in $\operatorname{Spec} A$ and $\operatorname{Spec} A_i$ for some i, hence there exists a distinguished open cover $\{U_{a_j}\}$ of $\operatorname{Spec} A$ such that the induced morphism $f^{-1}(U_{a_j}) \to U_{a_j}$ is integral/finite. Let $\phi : A \to B$ be the ring homomorphism induces $f|_{f^{-1}(U)}$, then we have reduced the problem to the following situation: let $\{a_1, \ldots, a_n\} \subset A$ generate the unit ideal, and the induced map $\phi_j : A_{a_j} \to B_{\phi(a_j)}$ be integral/finite, then ϕ is integral/finite.

First suppose that each ϕ_j is finite; then there exist $s_{1_j}, \ldots, s_{n_j} \in B_{\phi(a_j)}$ which generate $B_{\phi(a_j)}$ as an A_{a_j} module. We can write each s_{i_j} as:

$$s_{i_j} = \frac{b_{i_j}}{\phi(a_j)^{k_{i_j}}}$$

for some $b_{i_j} \in B$, some $k_{i_j} \in \mathbb{N}$. Since $1/a_j \in A_{a_j}$, it follows that we can take our generators to be of the form:

$$s_{i_j} = \frac{b_{i_j}}{1}$$

for all i_j. This gives us a finite set $\{b_{i_j}\} \subset B$, which we claim generates B as an A module; let $N = |\{b_{i_j}\}$, and consider the morphism of A modules:

$$\begin{aligned} \psi : A^{\oplus N} &\longrightarrow B \\ (a_{i_j}) &\longmapsto \sum_i \sum_j a_{i_j} b_{i_j} \end{aligned}$$

[66] Recall that the canonical C module structure on $A \otimes_B C$ is given by $c\dot{(}a \otimes c') = (1 \otimes c) \cdot (a \otimes c')$.

Let $\pi : B \to C$ be cokernel of this map, then since cokernels commute with localization[67], we have that the induced map $\pi_{a_j} : B_{\phi(a_j)} \to C_{\pi(\phi(a_j))}$ is the cokernel of:

$$A_{a_j}^{\oplus N} \longrightarrow B_{\phi(a_j)}$$

which is surjective hence $C_{\pi(\phi(a_j))} = 0$ for all j. Now let $c \in C$, then $c/1 \in C_{\pi(\phi(a_j))} = 0$, hence there exists some m_j such that $\pi(\phi(a_j))^{m_j} c = 0$. The a_j generate the unit ideal, so $a_j^{m_j}$ generate the unit ideal as well, hence $1 = \sum_j a_j^{m_j} \alpha_j$, therefore:

$$c = 1 \cdot c = \sum_j \pi(\phi(a_j^{m_j} \alpha_j)) \cdot c = 0$$

hence $C = 0$ and so ψ is surjective.[68]

Now suppose that each ϕ_j is integral. Let $b \in B$, then for all j, $b/1 \in B_{\phi(a_j)}$ is the root of a monic polynomial $p_j \in A_{a_j}[x]$. Note that $A_{a_j}[x] = (A[x])_{a_j}$; let:

$$p_j = x^{n_j} + \frac{b_{n_j-1}}{a_j^{k_{n_j-1}}} x^{n_j-1} + \cdots + \frac{b_0}{a_j^{k_0}}$$

There exists a M_j such that:

$$a_j^{M_j} p_j = a_j^{M_j} x^{n_j} + \frac{b'_{n_j-1}}{1} x^{n_j-1} + \cdots + \frac{b'_0}{1}$$

There thus exists a $p'_j \in A[x]$ such that $p'_j/1 \in (A[x])_{a_j}$ is equal to $a_j^{M_j} p_j$. Since $\phi(a_j)^{M_j} p_j(b) = 0$ it follows that there is an L_j such that $\phi(a_j)^{L_j+M_j} p_j(b) = 0$. Moreover, if we set $q_j = a_j^{L_j} \cdot p'_j$, then $q_j(b) = 0$, and $q_j/1 = a_j^{M_j+L_j} p_j$. Let N be the maximum degree of the q_j, and let $m_j = N - n_j$. Now again, we have that the set $\{a_1^{K_1}, \ldots, a_n^{K_n}\}$ generates the unit ideal, hence there are h_j such that:

$$1 = \sum_j h_j a_j^{K_j}$$

so we define $q \in A[x]$ by:

$$q = \sum_j h_j x^{m_j} q_j$$

Note that each $x^{m_j} q_j$ has degree N, and that the degree N term of q is given by:

$$q = \sum_j h_j x^{m_j} a_j^{K_j} x^{n_j} = x^N \sum_j h_j a_j^{K_j} = x^N$$

hence q is a monic polynomial in $A[x]$. We claim that $q(b) = 0$, however this is clear as $q_j(b) = 0$ for all j. It follows that $\phi : A \to B$ is integral, implying the claim.

□

Our goal is to now further justify the the nomenclature 'finite morphism' in the sense that we wish to prove that these maps have finite fibres. Let $f : X \to Y$ be a finite morphism, and recall that the scheme theoretic fibre of $y \in Y$ is given by:

$$X_y = \operatorname{Spec} k_y \times_Y X$$

Note that if $U = \operatorname{Spec} A \subset Y$ is an affine scheme containing y then we have the following isomorphism:

$$X_y \cong \operatorname{Spec} k_y \times_U f^{-1}(U)$$

If f is finite then it is affine as well, and so with $f^{-1}(U) = \operatorname{Spec} B$, it suffices to show that:

$$X_y \cong \operatorname{Spec}(k_y \otimes_A B)$$

is a finite topological space which ultimately amounts to showing that $k_y \otimes_A B$ has finitely many prime ideals. To do so we will need to develop the theory of Artinian rings, a class of rings which satisfy a condition dual to the Noetherian one.

[67] See xyz

[68] If this feels like like there is some sheaf business going on here, that's because there is!

Definition 3.9.2. Let A be a commutative ring, then A is commutative if every strictly decreasing chain of ideals:

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

terminates.

One quickly sees that being Artinian is a much less reasonable finiteness condition than being Noetherian. Indeed, let $A = \mathbb{Z}$, then the following chain never terminates:

$$2\mathbb{Z} \supset 4\mathbb{Z} \supset 8\mathbb{Z} \supset \cdots$$

so $\mathbb{Z}$ is not Artinian. Furthermore, in contrast to Theorem 3.4.1, we have that $A[x_1, \ldots, x_n]$ is never Artinian as the following chain never terminates:

$$\langle x_i \rangle \supset \langle x_i^2 \rangle \supset \cdots \supset \langle x_i^n \rangle \supset \cdots$$

Example 3.9.2. Let $A = k^n$ with the ring structure given the canonical product ring structure. Then we have that every ideal is a vector subspace and the length of any chain of ideals is bounded above by $n+1$, hence must be finite. It follows that A is Artinian (and Noetherian). Moreover, any finite k-algebra is Artinian, and any ring that is finite as a set is also Artinian, i.e. $\mathbb{Z}/n\mathbb{Z}$.

The following is an analogue of Lemma 3.4.2:

Lemma 3.9.2. *Let A be a Artinian, then the following hold:*

a) If S is any multiplicatively closed subset then $S^{-1}A$ is Artinian.

b) If $I \subset A$ is an ideal then A/I is Artinian.

Proof. For *a*) let:

$$J_1 \supset J_2 \supset \cdots$$

be a strictly descending chain of ideals in $S^{-1}A$. If $\pi : A \to S^{-1}A$ is the localization map, then we have that:

$$\pi^{-1}(J_1) \supset \pi^{-1}(J_2) \supset \cdots$$

is chain of ideals in A. For some n this must terminate, hence for all $m \geq n$ we have that $\pi^{-1}(J_m) = \pi^{-1}(J_n)$. It now suffices to show that $\langle \pi(\pi^{-1}(J_m)) \rangle = J_m$ for any m. Clearly, we have the inclusion $\langle \pi(\pi^{-1}(J_m)) \rangle \subset J_m$; let $a/s \in J_m$, then $a/1 \in J_m$,and $a \in \pi^{-1}(J_m)$. It follows that $a/1 \in \pi(\pi^{-1}(J_m))$, hence $a/s \in \langle \pi(\pi^{-1}(J_m)) \rangle$ implying the equality.

For *b*), we employ the same argument; however since $\pi : A \to A/I$ is surjective we automatically have the equality $\langle \pi(\pi^{-1}(J_m)) \rangle = J_m$. ☐

The above gives us the following strange result:

Proposition 3.9.3. *Let A be Artinian, then every $\mathfrak{p} \in \operatorname{Spec} A$ is maximal. In particular, A is an integral domain if and only if it is a field.*

Proof. Let A be Artinian, and $\mathfrak{p} \in \operatorname{Spec} A$, then by Lemma 3.9.2 we have that $A/\mathfrak{p}$ is an Artinian integral domain. Let $[a] \in A/\mathfrak{p}$ be nonzero and consider the following chain:

$$\langle [a] \rangle \supset \langle [a]^2 \rangle \cdots$$

which must stabilize, hence for some n we have that $\langle [a]^n \rangle = \langle [a]^{n+1} \rangle$. This implies that $[a]^n \in \langle [a]^{n+1} \rangle$ so there exists $[b] \in A/\mathfrak{p}$ such that $[a]^{n+1}[b] = [a]^n$, thus:

$$[a]^n([a] \cdot [b] - [1]) = 0 \Rightarrow [a] \cdot [b] - 1 = 0$$

as $[a]$ is assumed nonzero. It follows that $[b] = [a]^{-1}$ hence every nonzero element of $A/\mathfrak{p}$ is invertible so $A/\mathfrak{p}$ is a field implying that $\mathfrak{p}$ is maximal. In particular, if A is an integral domain then $\langle 0 \rangle$ is prime and thus maximal so A is a field. ☐

We now need the following general lemma:

Lemma 3.9.3. *Let A be a commutative ring, and $\mathfrak{q}, \mathfrak{p}_i \in \operatorname{Spec} A$ for $1 \leq i \leq n$. Then $\bigcap_i \mathfrak{p}_i \subset \mathfrak{q}$ if and only if for some i we have $\mathfrak{p}_i \subset \mathfrak{q}$.*

Proof. We proceed by induction, the base case $n = 1$ is trivial, and if $\mathfrak{p}_i \subset \mathfrak{q}$ for some i, then clearly we have that $\bigcap_i \mathfrak{p}_i \subset \mathfrak{q}$. Assuming the $n-1$th case, we have that:

$$\left(\bigcap_{i=1}^{n-1} \mathfrak{p}_i\right) \cap \mathfrak{p}_n \subset \mathfrak{q}$$

If $\bigcap_{i=1}^{n-1} \mathfrak{p}_i \subset \mathfrak{q}$, we are done by induction, so assume that $\bigcap_{i=1}^{n-1} \mathfrak{p}_i \not\subset \mathfrak{q}$. Let $a \in \mathfrak{p}_n$, then by assumption there exists some $b \in \left(\bigcap_{i=1}^{n-1} \mathfrak{p}_i\right)$ such that $b \notin \mathfrak{q}$. It follows that $a \cdot b \in \left(\bigcap_{i=1}^{n-1} \mathfrak{p}_i\right) \cap \mathfrak{p}_n$ which lies in $\mathfrak{q}$, however $\mathfrak{q}$ is prime hence either $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$, thus again by assumption we have that $a \in \mathfrak{q}$. It follows that $\mathfrak{p}_n \subset \mathfrak{q}$.

□

Proposition 3.9.4. *Let A be Artinian, then $\operatorname{Spec} A$ is a finite topological space and carries the discrete topology*[69]*.*

Proof. Suppose that $\operatorname{Spec} A$ has infinitely many maximal ideals, then we can choose some infinite sequence $\{\mathfrak{m}_i\}_{i=1}^{\infty}$ of pairwise distinct maximal ideals. Consider the following chain:

$$\mathfrak{m}_1 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \supset \cdots$$

We claim that this chain is strictly decreasing and never stabilizes, implying A is not Artinian. Suppose:

$$\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \cap \mathfrak{m}_{n+1}$$

then we have that:

$$\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \subset \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \cap \mathfrak{m}_{n+1} \subset \mathfrak{m}_{n+1}$$

It follows that one of the $\mathfrak{m}_i$ is contained $\mathfrak{m}_{n+1}$ by Lemma 3.9.3, hence $\mathfrak{m}_i = \mathfrak{m}_{n+1}$ as these are all maximal ideals. However this is impossible as all maximal ideals are pairwise distinct by assumption, so A is not Artinian.

Supposing A is Artinian, we have by the above that $\operatorname{Spec} A$ has only finitely many maximal ideals. Since every prime ideal is maximal, by Proposition 3.9.3 we have that $\operatorname{Spec} A$ is a finite topological space equal to $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ where each $\mathfrak{m}_i$ is a maximal ideal. We see that $\mathbb{V}(\mathfrak{m}_i) = \{\mathfrak{m}_i\}$ so the singleton sets are closed, hence every subset of $\operatorname{Spec} A$ is closed, so every subset of $\operatorname{Spec} A$ is open implying that $\operatorname{Spec} A$ carries the discrete topology. □

We can now show that finite morphisms have finite fibres as initially discussed:

Corollary 3.9.2. *let $f : X \to Y$ be a finite morphism, then for all $y \in Y$, the fibre $X_y = \operatorname{Spec} k_y \times_Y X$ is a finite topological space.*

Proof. From our earlier discussion, if $U = \operatorname{Spec} A \subset Y$ contains y, and $\operatorname{Spec} B = f^{-1}(U)$, then we have that:

$$X_y \cong \operatorname{Spec}(k_y \otimes_A B)$$

Since f is finite, we have that B is a finite A algebra, hence by Proposition 3.9.2 we have that $k_y \otimes_A B$ is a finite k_y algebra. Example 3.9.2 then implies that $k_y \otimes_A B$ is Artinian, hence $\operatorname{Spec}(k_y \otimes_A B)$ is a finite topological space with the discrete topology by Proposition 3.9.4 as desired. □

[69] Recall that in the discrete topology every subset is open

3.10 Finite Morphisms are Proper

We now end our discussion on integral and finite morphisms by connecting them to the other classes of morphisms discusses in this chapter. In particular we wish to show that integral morphisms are precisely those morphisms which are affine and universally closed, and finite morphisms are precisely those morphisms which are affine and proper. To do so, as usual, we will need to prove a slew of results from commutative algebra. Namely, this section could just as easily be called Lying Over, Going Up, and Nakayama's Lemma as we our desired results will be applications of these lemmas.

We begin with Nakayama's Lemma; it comes in many flavors, and we prove five of them:

Lemma 3.10.1. *Let A be a ring, $I \subset A$ an ideal, and M a finitely generated A module. The following then hold:*

a) If $IM = M$ then there exists and $a \in A$ such that $[a] = [1] \in A/I$, and $a \cdot M = 0$.

b) If $IM = M$, and

$$I \subset \bigcap_{\mathfrak{m} \in |\operatorname{Spec} A|} \mathfrak{m}$$

then $M = 0$.

c) Let N' and N be A-modules with $M, N \subset N'$, and suppose that I is contained in all maximal ideals of A as in b). Then if $N' = N + IM$, $N' = N$.

d) Let $f : N \to M$ be an A module morphism and suppose I is contained in all maximal ideals of A. Then if $\bar{f} : N/IN \to M/IM$ is surjective, f is surjective.

e) Suppose I is contained in all maximal ideals of A, and let $\pi : M \to M/IM$ be the natural surjection. If the image $\{f_1, \ldots, f_n\} \subset M$ generates M/IM then $\{f_1, \ldots, f_n\}$ generate M.

Proof. We start with *a*); note that:

$$IM = \{i \cdot m : i \in I, m \in M\}$$

Choose generators $f_1, \ldots, f_n$ of M, then we claim that the map:

$$\begin{aligned} \alpha : I^n &\longrightarrow M \\ (b_1, \ldots, b_n) &\longmapsto \sum_i b_i f_i \end{aligned}$$

is surjective. Let $m \in M$, then since $IM = M$ we have that $m = i \cdot n$ for some $i \in I$ some $n \in N$. However, $n = \sum_i a_i f_i$ as the f_i generate M, hence $m = \sum_i (ia_i) f_i$, and each $ia_i \in I$ implying the initial claim. In particular, we can write each generator as:

$$f_i = \sum_j c_{ij} f_j$$

for some $c_{ij} \in I$. Consider the matrix with coefficients in A given by:

$$S = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$$

which determines a morphism $A^n \to A^n$. Let $\beta : A^n \to M$ be the natural surjection[70] and set:

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

[70] Defined the same as α, just on all of A^n.

where the 1 is in the ith position, and note that $\beta(e_i) = f_i$. Then:

$$\beta \circ S(e_i) = \beta\left(\sum_j c_{ij} e_j\right) = \sum_j c_{ij} f_j = f_i$$

Now observe that for all i:

$$\beta \circ (\mathrm{Id} - S)(e_i) = 0$$

hence $\beta \circ (\mathrm{Id} - S)$ is identically zero. Define $a \in A$ by:

$$a = \det(\mathrm{Id} - S) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(\delta_{1\sigma(1)} - c_{1\sigma(1)}) \cdots (\delta_{n\sigma(n)} - c_{n\sigma(n)})$$

where S_n is the symmetric group. Note that $[a] = [1] \in A/I$ as if σ is not the identity then under the projection $\pi : A \to A/I$, we have

$$\pi(\delta_{i\sigma(i)} - c_{i\sigma(i)}) = \pi(c_{i\sigma(i)}) = 0$$

and if σ is the identity, then:

$$\pi(\delta ii - c_{ii}) = \pi(1 - c_{ii}) = [1]$$

Moreover, recall that for any matrix T there exists an adjugate matrix $\mathrm{adj}(T)$ satisfying:

$$\mathrm{adj}(T) \cdot T = T \cdot \mathrm{adj}(T) = \det(T) \cdot \mathrm{Id}$$

hence for all i, we have that:

$$\begin{aligned} a \cdot f_i =& a \cdot \beta(e_i) \\ =& \beta(a \cdot e_i) \\ =& \beta \circ (\det(\mathrm{Id} - S)\mathrm{Id})(e_i) \end{aligned}$$

Now note that for any matrix T, we have that:

$$\beta \circ T(e_i) = \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ f_i \\ \vdots \\ 0 \end{pmatrix}$$

hence:

$$(\mathrm{Id} - S) \cdot \begin{pmatrix} 0 \\ \vdots \\ f_i \\ \vdots \\ 0 \end{pmatrix} = 0$$

and so:

$$\beta \circ (\det(\mathrm{Id} - S))(e_i) = \mathrm{adj}(\mathrm{Id} - S) \cdot (\mathrm{Id} - S) \cdot \begin{pmatrix} 0 \\ \vdots \\ f_i \\ \vdots \\ 0 \end{pmatrix} = 0$$

implying that $a \cdot f_i = 0$ as desired. In particular, since a annihilates each generator, we have that $a \cdot M = 0$, implying a).

For b) suppose in addition that:

$$I \subset \bigcap_{\mathfrak{m} \in |\operatorname{Spec} A|} \mathfrak{m}$$

then with a as defined in a), we claim that a is invertible. Indeed, there exists $i \in I$ such that $a = 1 + i$, and this $i \in \mathfrak{m}$ for all $\mathfrak{m} \in |\operatorname{Spec} A|$. Consider the ideal $\langle 1 + i \rangle$, then if this ideal is not all of A, there must be some $\mathfrak{m} \in |\operatorname{Spec} A|$ such that $\langle 1 + i \rangle \subset \mathfrak{m}$. However, $\iota \in \mathfrak{m}$ as well so $1 \in \mathfrak{m}$ which is a contradiction. It follows that $\langle i + 1 \rangle = A$ hence a invertible. Let $m \in M$, then $a \cdot m = 0$ by construction, but:

$$0 = a^{-1} \cdot (a \cdot m) = m$$

hence $M = 0$ implying b).

For c), suppose that $N' = N + IM$, then note this implies that $N' = N + M$ as if $n' \in N'$ then we have $n' = n + i \cdot m$ for $n \in N$, $i \in I$, and $m \in M$. However $i \cdot m \in M$ hence $N' \subset N + M$. Since N and M are submodules of N' it follows that $N' = N + M = N + IM$. In particular, we have that N'/N is finitely generated, as if $\{f_1, \ldots, f_k\}$ generate M, then we claim that $\{[f_1], \ldots, [f_k]\}$ generate N'/N. Indeed, let $[n'] \in N'/N$, then any class representative n' can be written as $n + m$ for $n \in N$ and $m \in M$. Any $m \in M$ can be written as:

$$m = \sum_i a_i f_i$$

hence:

$$[n'] = \sum_i a_i [f_i] + [n] = \sum_i a_i [f_i]$$

Moreover we claim that $I(N'/N) = N'/N$; clearly we have $I(N'/N) \subset (N'/N)$, so let $[n'] \in N'/N$. Then any class representative n' can be written as $n + i \cdot m$, hence $[n'] = [i \cdot m] = i \cdot [m]$ so $[n'] \in I(N'/N)$. It follows by b) that $N'/N = 0$, implying the claim.

For d), let $f : N \to M$ be an A module homomorphism. Note that $\bar{f} : N/IN \to M/IM$ is induced by $\pi \circ f : N \to M/IM$ and factors uniquely through the quotient as $IN \subset \ker(\pi \circ f)$. Obviously, we have that $M = \operatorname{im}(f) + IM$, and M is finitely generated hence by c) we have that $\operatorname{im}(f) = M$ and f is surjective, as desired.

For e), I be as in b), and consider the natural projection $\pi : M \to M/IM$. We have that M/IM is finitely generated by $\{[f_1], \ldots, [f_n]\}$. Let $N \subset M$ be the submodule generated by $f_1, \ldots, f_n$, then we claim that $M = N + IM$. Let $m \in M$, and consider $[m]$. Then:

$$[m] = \sum_i a_i [f_i] = \left[\sum_i a_i f_i \right]$$

It follows that there exists $\beta \in IM$ such that:

$$m = \sum_i a_i f_i + \beta$$

hence $m \in N + \mathfrak{m}M$. Since M is finitely generated, it follows by c) that $M = N$ hence $\{f_1, \ldots, f_n\}$ generate M. ☐

We need the following lemma for both Lying Over and Going Up

Lemma 3.10.2. *Let $\phi : B \to A$ be an integral morphism, $I \subset A$, $J \subset B$ ideals, and $T \subset B$ a multiplicatively closed set. Then the following hold:*

a) The morphism $B \to A/I$ is integral.

b) The morphism $B/J \to A/\langle \phi(J) \rangle$ is integral.

c) The morphism $T^{-1}B \to \phi(T)^{-1}A$ is integral.

Proof. To show a), recall that the composition of integral morphisms is integral, so it suffices to show that $\pi : A \to A/I$ is integral. Let $[a] \in A/I$, then $p(x) = x - a \in A[x]$ is a monic polynomial with $[a]$ as a root, hence π is integral.

For b) let $J \subset B$ be an ideal, then the morphism $\psi : B/J \to A/\langle \phi(J) \rangle$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d, "\pi_B"] & A \arrow[d, "\pi_A"] \\ B/J \arrow[r, "\psi"] & A/\langle \phi(J) \rangle \end{tikzcd}$$

Let $[a] \in A/\langle \phi(J) \rangle$, then $a \in \pi_A^{-1}([a])$, and there is a polynomial $p \in B[x]$:

$$p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$$

of which a is a root. There is then a polynomial $q \in B/J[x]$ given by:

$$q(x) = x^n + [b_{n-1}]x^{n-1} + \cdots + b_0$$

We see that:

$$\begin{aligned} q([a]) =& [a]^n + [b_{n-1}][a]^{n-1} + \cdots + b_0 \\ =& [a^n + b_{n-1}a^{n-1} + \cdots + b_0] \\ =& [p(a)] \\ =& 0 \end{aligned}$$

hence ψ is integral.

For c), the morphism $\psi : T^{-1}B \to \phi(T)^{-1}A$ is the unique one which makes the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d, "\pi_B"] & A \arrow[d, "\pi_A"] \\ T^{-1}B \arrow[r, "\psi"] & \phi(T)^{-1}A \end{tikzcd}$$

It suffices to show that $a/1$ and $1/\phi(t)$ are the roots of monic polynomials in $T^{-1}B[x]$ by Corollary 3.9.1. Let $a/1 \in \phi(T)^{-1}A$, then there exists $a \in A$ which maps to $a/1$ under π_A. Let $p \in B[x]$ be given by:

$$p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$$

and satisfy $p(a) = 0$. Define $q \in T^{-1}B[x]$ by:

$$q(x) = x^n + \frac{b_{n-1}}{1}x^{n-1} + \cdots + \frac{b_0}{1}$$

then:

$$q(a) = \frac{p(a)}{1} = 0$$

as desired. For $1/\phi(t)$ we claim that:

$$q(x) = x - \frac{1}{t} \in T^{-1}B[x]$$

satisfies $q(1/\phi(t)) = 0$. However, this clear as:

$$q(1/\phi(t)) = \frac{1}{\phi(t)} - \psi\left(\frac{1}{t}\right)$$

which by the definition of ψ reduces to:

$$\frac{1}{\phi(t)} - \frac{1}{\phi(t)} = 0$$

as desired. $\square$

As the following example shows,

Example 3.10.1. If $S \subset A$ is multiplicatively closed, and $\phi : B \to A$ is a morphism of rings, then the natural map $\psi : B \to S^{-1}A$ given by $\psi = \pi \circ \phi$ is not in general integral even if ϕ is. Indeed, if this were true then every localization would be integral as the identity map is integral; as a counter example take the localization map $\mathbb{C}[t] \to \mathrm{Frac}(\mathbb{C}[t])$, then since $\mathbb{C}[t]$ is an integrally closed domain it follows that if $\alpha \in \mathrm{Frac}(\mathbb{C}t)$ is integral then $\alpha \in \mathbb{C}[t]$. However $t^{-1} \notin \mathbb{C}[t]$ hence t^{-1} can't be integral over $\mathbb{C}[t]$ so the map $\mathbb{C}[t] \to \mathrm{Frac}(\mathbb{C}[t])$ is not integral.

With our many flavours of Nakayama's lemma at hand, as well as Lemma 3.10.2 we can now prove the Lying Over, and Going Up result, beginning with the former:

Lemma 3.10.3. *Let $\phi : B \to A$ be an integral extension of rings, then induced map on schemes $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is surjective.*

Note that this is called 'Lying Over' because it implies that for any $\mathfrak{p} \in \operatorname{Spec} B$ we can find a prime $\mathfrak{q} \in \operatorname{Spec} A$ which maps to it.

Proof. Given $\mathfrak{p} \in \operatorname{Spec} B$, we simply need to show that the fibre $f^{-1}(\mathfrak{p}) = \operatorname{Spec} A \times_B \operatorname{Spec} k_{\mathfrak{p}}$ is non empty. By Lemma 3.7.2, we have that:

$$f^{-1}(\mathfrak{p}) \cong \operatorname{Spec} A_{\mathfrak{p}} / \langle \phi(\mathfrak{p})/1 \rangle$$

where $A_{\mathfrak{p}} = \phi(B \setminus \mathfrak{p})^{-1}A$. It follows that $f^{-1}(\mathfrak{p})$ is empty if and only if $\langle \phi(\mathfrak{p})/1 \rangle = A_{\mathfrak{p}}$, as the only ring without a maximal ideal is the 0 ring.

The localization map $\phi_{\mathfrak{p}} : B_{\mathfrak{p}} \to A_{\mathfrak{p}}$ is an integral morphism by Lemma 3.10.2. In particular, if $b/s \in B_{\mathfrak{p}}$, and $\phi(b)/\phi(s) = 0 \in A_{\mathfrak{p}}$, then there exists some $\phi(t) \in \phi(B \setminus \mathfrak{p})$ such that

$$\phi(bt) = 0$$

This implies $b \cdot t = 0$, but then $b/s = 0 \in B_{\mathfrak{p}}$. It follows that $\phi_{\mathfrak{p}}$ is injective as well. Let $\mathfrak{m}_{\mathfrak{p}}$ be the unique maximal ideal in $B_{\mathfrak{p}}$, then by the commutativity of the diagram:

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d, "\pi_{\mathfrak{p}}"] & A \arrow[d, "\pi_{\mathfrak{p}}"] \\ B_{\mathfrak{p}} \arrow[r, "\phi_{\mathfrak{p}}"] & A_{\mathfrak{p}} \end{tikzcd}$$

we have that $\langle \phi_{\mathfrak{p}}(\mathfrak{m}_{\mathfrak{p}}) \rangle = \langle \phi(\mathfrak{p})/1 \rangle$. Indeed, suppose that $a/s \in \langle \phi(\mathfrak{p})/1 \rangle$, then by definition, we have that $a \in \phi(\mathfrak{p})$, and $s \in \phi(B \setminus \mathfrak{p})$. There is then a unique $b \in \mathfrak{p}$, and $t \in B \setminus \mathfrak{p}$ such that $\phi_{\mathfrak{p}}(b/t) = \phi(b)/\phi(t) = a/s$. Similarly, if $a/s \in \langle \phi_{\mathfrak{p}}(\mathfrak{m}_{\mathfrak{p}}) \rangle$, then $a/s = \phi(b)/\phi(t)$ for some unique $b \in \mathfrak{p}$, and $t \in B \setminus \mathfrak{p}$, hence $a/s \in \langle \phi(\mathfrak{p})/1 \rangle$.

The condition that $\langle \phi_{\mathfrak{p}}(\mathfrak{m}_{\mathfrak{p}}) \rangle = A_{\mathfrak{p}}$ is now more aptly written as $\mathfrak{m}_{\mathfrak{p}} \cdot A_{\mathfrak{p}} = A_{\mathfrak{p}}$. For the sake of contradiction, suppose this holds, then we have that $1 \in A_{\mathfrak{p}}$ can be written as:

$$1 = \sum_{i=1}^{n} m_i \cdot g_i \tag{3.10.1}$$

with $m_i \in \mathfrak{m}_{\mathfrak{p}}$, and $g_i \in A_{\mathfrak{p}}$. Take the subalgebra $A' \subset A_{\mathfrak{p}}$ generated by $\{g_1, \ldots, g_n\}$, then A' is integral over $B_{\mathfrak{p}}$ and finitely generated, hence a a finite $B_{\mathfrak{p}}$ module by Proposition 3.9.1. We then have that (3.10.1) implies $1 \in \mathfrak{m}_{\mathfrak{p}} \cdot A'$, hence $\mathfrak{m}_{\mathfrak{p}} \cdot A' = A'$. However, $\mathfrak{m}_{\mathfrak{p}}$ is the only maximal ideal of $B_{\mathfrak{p}}$, so by Nakayama's lemma[71], we have that $A' = 0$, contradicting the injectivity of $\phi_{\mathfrak{p}}$. □

Going Up is now a borderline immediate consequence of Lying Over:

Lemma 3.10.4. *Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be an integral morphism, and $\mathfrak{p} \subset \mathfrak{p}' \in \operatorname{Spec} B$. Let $\mathfrak{q} \in \operatorname{Spec} A$ satisfy $f(\mathfrak{q}) = \mathfrak{p}$, then there exists $\mathfrak{q}' \in \operatorname{Spec} A$ containing $\mathfrak{q}$ such that $f(\mathfrak{q}') = \mathfrak{p}'$.*

Note that this is called 'Going Up' as it implies that we can lift chains of prime ideals.

[71] Part *b*) of Lemma 3.10.1.

Proof. Let $\phi : B \to A$ be the ring homomorphism which induces f; in particular, ϕ makes A integral over B. With $\mathfrak{p}, \mathfrak{p}'$, and $\mathfrak{q}$ as stated, consider the induced map $\phi' : B \to A/\mathfrak{q}$. Note that by Lemma 3.10.2 this map is integral. We claim that $\ker \phi' = \mathfrak{p}$. Indeed, let $b \in \mathfrak{p}$, then $\phi'(p) = [\phi(b)]$, but $\phi(b) \in \mathfrak{q}$ as $\phi(\mathfrak{p}) \subset \mathfrak{q}$. Now suppose that $\phi'(b) = 0$, then $\phi(b) \in \mathfrak{q}$, so $b \in \phi^{-1}(\mathfrak{q}) = \mathfrak{p}$. It follows that the map $B/\mathfrak{p} \to A/\mathfrak{q}$ is injective; in particular it is integral by Lemma 3.10.2 as $(A/\mathfrak{q})/\phi'(\mathfrak{p}) = (A/\mathfrak{q})/\langle 0 \rangle = A/\mathfrak{q}$. By Lying Over we have that that the induced map $\operatorname{Spec} A/\mathfrak{q} \to \operatorname{Spec} B/\mathfrak{p}$[72] is surjective, so there exists a prime $\mathfrak{q}'$ containing $\mathfrak{q}$ which maps to $\mathfrak{p}'$ as desired. □

Example 3.10.2. Let $N : \tilde{X} \to X$ be the normalization map of an integral scheme X. We claim that N is integral, and surjective. First note by the proof in Theorem 3.3.1, where we define N on an affine cover $\operatorname{Spec} A_i$, of X, that $N^{-1}(\operatorname{Spec} A_i) \cong \operatorname{Spec} \bar{A}_i$, so N is affine by Proposition 3.8.1. On this open cover, N is given by the $A \hookrightarrow \bar{A}$ which is integral extension by definition, hence N is integral by Proposition 3.9.2. In particular, by Lemma 3.10.3 we have that $\operatorname{Spec} \bar{A}_i \to \operatorname{Spec} A_i$ is surjective for all i, hence N is surjective.

If $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism satisfying:

> For any $\mathfrak{p} \subset \mathfrak{p}' \in \operatorname{Spec} B$, and $\mathfrak{q} \in \operatorname{Spec} A$ with $f(\mathfrak{q}) = \mathfrak{p}$, there exists a $\mathfrak{q}' \in \operatorname{Spec} A$ containing $\mathfrak{q}$ such that $f(\mathfrak{q}') = \mathfrak{p}'$.

we say that Going Up holds for f. In particular, Going Up is equivalent to f being a closed map:

Proposition 3.10.1. *Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism, then f is closed if and only if Going Up holds for f.*

Proof. Suppose that $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is closed, and let $\phi : B \to A$ be the ring homomorphism inducing f. Let $\mathfrak{p} \subset \mathfrak{p}' \in \operatorname{Spec} B$, and $\mathfrak{q} \in \operatorname{Spec} A$ satisfying $f(\mathfrak{q}) = \mathfrak{p}$. Consider $\mathbb{V}(\mathfrak{q})$, then $f(\mathbb{V}(\mathfrak{q}))$ is closed, and contains $\mathfrak{p}$, hence $f(\mathbb{V}(\mathfrak{q}))$ contains the closure of $\mathfrak{p}$, $\mathbb{V}(\mathfrak{p})$. Since $\mathfrak{p}'$ is contained in $\mathbb{V}(\mathfrak{p})$, we have that $\mathfrak{p}' \in f(\mathbb{V}(\mathfrak{q}))$ hence there exists some $\mathfrak{q}' \in \mathbb{V}(\mathfrak{q})$ such that $f(\mathfrak{q}') = \mathfrak{q}$. It follows that Going Up holds for f.

Now suppose that going up holds for f, and let $\mathbb{V}(I) \subset \operatorname{Spec} A$ be a closed subset. Note that since $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is integral, we have that Going Up holds for $\operatorname{Spec} A/I \to \operatorname{Spec} A$, thus clearly Going Up holds for $\operatorname{Spec} A/I \to \operatorname{Spec} B$. It thus suffices to show that if Going Up holds for $f : \operatorname{Spec} A \to \operatorname{Spec} B$ then $f(\operatorname{Spec} A)$ has closed image.

Let $Z = f(\operatorname{Spec} A)$, and let $\mathfrak{p} \in \bar{Z}$. Then for any open set containing $\mathfrak{p}$ we must have that $U \cap Z \neq \emptyset$, as other wise U^c is a closed set containing Z, and thus contains $\bar{Z}$. However, $\mathfrak{p} \notin U^c$ so $\mathfrak{p} \notin \bar{Z}$, a contradiction. Hence, for all $g \notin \mathfrak{p}$, we have that $U_g \cap Z \neq \emptyset$. In particular, since $U_g \cap Z = f(U_{\phi(g)})$, we have that $U_{\phi(g)}$ is not empty for $g \notin \mathfrak{p}$.

This implies that $A_{\mathfrak{p}} = \phi(B \setminus \mathfrak{p})^{-1} A$ is not the zero ring. Indeed, if $A_{\mathfrak{p}}$ is the zero ring that $1 = 0$, hence there would exist some $g \in B \setminus \mathfrak{p}$ such that $\phi(g) = 0$, but that would imply that $U_{\phi(g)}$ is empty, a contradiction. We now consider the composition:

$$\operatorname{Spec} A_{\mathfrak{p}} \to \operatorname{Spec} A \to \operatorname{Spec} B$$

where the first map is induced by the localization map $\pi : A \to A_{\mathfrak{p}}$. Now let $\tilde{\mathfrak{q}} \in \operatorname{Spec} A_{\mathfrak{p}}$, and consider $\mathfrak{p}' = f(\pi^{-1}(\tilde{\mathfrak{q}}))$; we claim that $\mathfrak{p}' \subset \mathfrak{p}$. Suppose the contrary, then there exists a $g \in \mathfrak{p}'$ such that $g \notin \mathfrak{p}$. It follows that $\phi(g)/1 \in \tilde{\mathfrak{q}}$, but if $g \notin \mathfrak{p}$, then $\phi(g) \in \phi(B \setminus \mathfrak{p})$, so $\tilde{\mathfrak{q}} = A_{\mathfrak{p}}$, a contradiction.

In particular, we have shown that there exists $\mathfrak{p}' \subset \mathfrak{p} \in \operatorname{Spec} B$, and $\mathfrak{q}' = \pi^{-1}(\tilde{\mathfrak{q}})$ satisfying $f(\mathfrak{q}') = \mathfrak{p}'$. Since Going Up holds for f, it follows that there exists a $\mathfrak{q} \in \operatorname{Spec} A$ satisfying $\mathfrak{q}' \subset \mathfrak{q}$ and $f(\mathfrak{q}) = \mathfrak{p}$. Therefore, if $\mathfrak{p} \in \bar{Z}$, we have $\mathfrak{p} \in Z$, so Z is closed, implying the claim. □

Lemma 3.10.5. *Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be induced by $\phi : B \to A$. Then, the closure of the image, $\operatorname{cl}(f(\operatorname{Spec} A))$ is equal to $\mathbb{V}(\ker \phi)$.*

Proof. Set $Z = \operatorname{cl}(f(\operatorname{Spec} A))$. First, let $\mathfrak{p} \in f(\operatorname{Spec} A)$, then $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ for some $\mathfrak{q} \in \operatorname{Spec} A$. Since $0 \in \mathfrak{q}$, we have that $\phi^{-1}(0) = \ker \phi \subset \mathfrak{p}$, hence $\mathfrak{p} \in \mathbb{V}(\ker \phi)$. It follows that $f(\operatorname{Spec} A) \subset \mathbb{V}(\ker \phi)$ hence $Z \subset \mathbb{V}(\ker \phi)$. Now by definition:

$$Z = \bigcap_{f(\operatorname{Spec} A) \subset \mathbb{V}(I)} \mathbb{V}(I)$$

[72] Which is topologically equivalent to the map $f|_{\mathbb{V}(\mathfrak{q})}$.

If $f(\operatorname{Spec} A) \subset \mathbb{V}(I)$, then $I \subset \phi^{-1}(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} A$. Let $b \in I$, then $\phi(b) \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$, so $\phi(b) \in \sqrt{\langle 0 \rangle}$, i.e. there exists some n such that $\phi(b)^n = 0$. This however, implies that $b \in \sqrt{\ker \phi}$, hence $I \subset \sqrt{\ker \phi}$, and we have that $\mathbb{V}(I) \supset \mathbb{V}(\ker \phi)$. It follows that $\mathbb{V}(\ker \phi) = Z$ as desired. □

Lemma 3.10.6. *Let $f : X \to Z$ be a surjective*[73] *morphism of schemes, and $g : Y \to Z$ any other morphism. Then the base change $X \times_Z Y \to Y$ is surjective.*

Proof. Let $y \in Y$, then we need to show that the fibre:

$$\pi_Y^{-1}(y) = \operatorname{Spec} k_y \times_Y (Y \times_Z X)$$

is not empty. Note that:

$$\operatorname{Spec} k_y \times_Y (Y \times_Z X) \cong \operatorname{Spec} k_y \times_Z X$$

Let $z = g(y)$, then we also have that:

$$f^{-1}(z) \times_{k_z} \operatorname{Spec} k_y \cong (X \times_Z k_z) \times_{k_z} \operatorname{Spec} k_y \cong g^{-1}(y)$$

where the morphism making $\operatorname{Spec} k_y$ a k_z scheme comes from composing the stalk map $g_y : (\mathscr{O}_Z)_z \to (\mathscr{O}_Y)_y$ with the projection $\pi_z : (\mathscr{O}_Y)_y \to k_y$. Since g is a morphism of locally ringed spaces, this gives rise to a field morphism $k_z \to k_y$, which we take to induce the structural morphism of $\operatorname{Spec} k_y$ as a k_z scheme.

Now since $f^{-1}(z)$ is not empty, we have that there is a non empty affine open $U = \operatorname{Spec} A \subset f^{-1}(z)$. It thus suffices to show that $\operatorname{Spec} A \otimes_{k_z} k_y$ is nonempty. We claim that $A \otimes_{k_z} k_y$ is a nonzero ring, indeed since $A \neq 0$ we have that A is a non zero k_z vector space. Any k_z basis then extends to a k_y basis for $A \otimes_{k_z} k_y$ of the same cardinality, hence $A \otimes_{k_z} k_y$ cannot be the zero vector space. Since every ring has a maximal ideal, it follows that that $\pi_Y^{-1}(y)$ is non empty implying the claim. □

We now prove the first major result of the section:

Theorem 3.10.1. *Let $f : X \to Y$ be a morphism of schemes. Then f is integral if and only if f is affine, and universally closed.*

Proof. Suppose f is integral, then f is automatically affine, so it suffices to show f is universally closed. Since f is integral, it's base change is integral by Proposition 3.9.2, so it suffices to show that an integral morphism is closed. Clearly, it then suffices to show this in the case $X = \operatorname{Spec} A$, and $Y = \operatorname{Spec} B$, but this follows from the fact Going Up holds for integral morphism, and Proposition 3.10.1.

Now suppose that f is affine and universally closed. It again clearly suffices to show that f is integral in the case where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, so let $\phi : B \to A$ be the morphism inducing f. We want to show that for all $a \in A$, there exists a monic polynomial $p \in B[x]$ such that p lies in the kernel of the map $\operatorname{ev}_a : B[x] \to A$, given by sending x to a. Consider the composition:

$$\psi : B[x] \to A[x] \to A[x]/\langle ax - 1 \rangle$$

Let $\beta \in \ker \psi$, then with $\beta = \sum_i b_i x^i$, there exists some polynomial $q \in A[x]$ such that:

$$\sum_i \phi(b_i) x^i = (ax - 1)q$$

Let $q = \sum_i c_i x^i$, then in particular we must have that:

$$\phi(b_i) = a \cdot c_{i-1} - c_i$$

If $\deg q = d$, we claim that:

$$p = \sum_{i=0}^{d} b_i x^{d+1-i} \in \ker \operatorname{ev}_a$$

---

[73] Set theoretically.

We rewrite the sum as follows:

$$\begin{aligned}\sum_{i=0}^{d}\phi(b_i)x^{d+1-i} =& \sum_{i=0}^{d}(a\cdot c_{i-1}-c_i)x^{d+1-i}\\ =& a\sum_{i=0}^{d}c_{i-1}x^{d+1-i} - x\sum_{i=0}c_ix^{d-i}\end{aligned}$$

Now note that the $c_{-1}=0$, so we we can rewrite the first sum as:

$$\begin{aligned}\sum_{i=0}^{d}\phi(b_i)x^{d+1-i} =& a\sum_{i=0}^{d}c_ix^{d-i} - x\sum_{i=0}c_ix^{d-i}\\ =& (a-x)\cdot\sum_{i=0}c_ix^{d-i}\end{aligned}$$

which certainly maps to zero under the morphism $B[x]\to A$ sending x to a, hence $p\in\ker\mathrm{ev}_a$. Moreover, if $b_0=1$ then p is monic, which would imply A is integral over B. It thus suffices to show that $\ker\psi$ contains a β satisfying $b_0=1$.

We claim this is equivalent to $\operatorname{Spec}B[x]/(\ker\psi+\langle x\rangle)$ being empty. Certainly, if $\beta\in\ker\psi$ with $b_0=1$ then $\ker\psi+\langle x\rangle=B[x]$. If $\ker\psi+\langle x\rangle=B[x]$, then that means $1\in\ker\psi+\langle x\rangle$ hence:

$$1=\beta+xg$$

for $\beta\in\ker\psi$, and $g\in B[x]$. However this clearly implies that $b_0=1$.

Note that the morphism $\varphi: B[x]\to A[x]$ is induced by the following diagram:

$$\begin{tikzcd}
A\otimes_{\mathbb{Z}}\mathbb{Z}[X] & & \\
& B\otimes_{\mathbb{Z}}\mathbb{Z}[X] \arrow[ul, "\exists!\varphi"] & \mathbb{Z}[X] \arrow[l, "\iota_{\mathbb{Z}[x]}"'] \arrow[ull, bend right, "\mathrm{Id}"'] \\
& B \arrow[u, "\iota_B"] \arrow[uul, bend left, "\iota_A\circ\phi"] & \mathbb{Z} \arrow[l] \arrow[u]
\end{tikzcd}$$

By Theorem 3.7.2, we have that φ induces a unique morphism $f':\operatorname{Spec}A[x]\to\operatorname{Spec}B[x]$ which is universally closed. We claim that $f'(\mathbb{V}(ax-1))=\mathbb{V}(\ker\psi)$; note that $f'|_{\mathbb{V}(ax-1)}$ is induced by ψ, hence the closure of the image of $f'|_{\mathbb{V}(ax-1)}$ is equal to $\mathbb{V}(\ker\psi)$ by Lemma 3.10.5. However, f' is a closed map, so it's restriction to any closed set is a closed map, hence $f'(\mathbb{V}(ax-1))=\mathbb{V}(\ker\psi)$ as desired.

We claim that:

$$(B[x]/\ker\psi)\otimes_{B[x]}B\cong B[x]/(\ker\psi+\langle x\rangle)$$

Indeed, the morphism $\mathrm{ev}_0: B[x]\to B$ is what makes B a $B[x]$ algebra, hence by our work in Lemma 3.1.2:

$$(B[x]/\ker\psi)\otimes_{B[x]}B\cong B/\langle\mathrm{ev}_0(\ker\psi)\rangle$$

It thus suffices to show that:

$$B/\langle\mathrm{ev}_0(\ker\psi)\rangle\cong B[x]/(\ker\psi+\langle x\rangle)$$

Consider the composition:

$$B\hookrightarrow B[x]\to B[x]/(\ker\psi+\langle x\rangle)$$

and note that if $b\in\langle\mathrm{ev}_0(\ker\psi)\rangle$, then:

$$b=\sum_i b_ip_i(0)$$

where $p_i \in \ker \psi$. If we consider b as an element in $b[x]$, then b is in $\ker \psi + \langle x \rangle$ as it is given by:

$$\sum_i b_i p_i - \sum_i b_i (p_i - p_i(0))$$

where clearly each $p_i - p_i(0) \in \langle x \rangle$. It follows that this factors through the quotient to give us a well defined homomorphism:

$$F : B/\langle \mathrm{ev}_0(\ker \psi)\rangle \longrightarrow B[x]/(\ker \psi + \langle x \rangle)$$

Now consider the composition:

$$B[x] \to B \to B/\langle \mathrm{ev}_0(\ker \psi)\rangle$$

If $p \in \ker \psi + \langle x \rangle$, then p can be written as:

$$p = q + xp'$$

where $q \in \ker \psi$, and $p' \in B[x]$. It follows that $q(0) \in \langle \mathrm{ev}_0(\ker \psi)\rangle$ hence this map also factors through the quotient to yield a well defined homomorphism:

$$G : B[x]/(\ker \psi + \langle x \rangle) \longrightarrow B/\langle \mathrm{ev}_0(\ker \psi)\rangle$$

Now let $[p] \in B[x]/(\ker \psi + \langle x \rangle)$, then:

$$G([p]) = [p(0)] \in B/\langle \mathrm{ev}_0(\ker \psi)\rangle$$

while:

$$F([p(0)]) = [p(0)] \in B[x]/(\ker \psi + \langle x \rangle)$$

However:

$$[p] - [p(0)] \in \langle x \rangle$$

so $F \circ G = \mathrm{Id}$. Clearly $G \circ F = \mathrm{Id}$, so the two are isomorphic as desired. It follows that the following diagram is Cartesian:

$$\begin{tikzcd}
\operatorname{Spec} B[x]/(\ker \psi + \langle x \rangle) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
\operatorname{Spec} B[x]/\ker \psi \arrow[r] & \operatorname{Spec} B[x]
\end{tikzcd}$$

Moreover, we claim that the following diagram is commutative:

$$\begin{tikzcd}
\operatorname{Spec} B \otimes_{B[x]} A[x]/\langle ax-1 \rangle \arrow[r] \arrow[d] & \operatorname{Spec} B[x]/(\ker \psi + \langle x \rangle) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
\operatorname{Spec} A[x]/\langle ax-1 \rangle \arrow[r] & \operatorname{Spec} B[x]/\ker \psi \arrow[r] & \operatorname{Spec} B[x]
\end{tikzcd}$$

The right square is Cartesian, so we need only show the left square commutes, but this is equivalent to the following diagram commuting:

$$\begin{tikzcd}
B \otimes_{B[x]} A[x]/\langle ax-1 \rangle & B[x]/(\ker \psi + \langle x \rangle) \arrow[l, "\iota_B"'] \\
A[x]/\langle ax-1 \rangle \arrow[u, "\iota_A"] & B[x]/\ker \psi \arrow[l] \arrow[u]
\end{tikzcd}$$

However, $0 \in B$ is equal to $\mathrm{ev}_0(x)$, hence in $B \otimes_{B[x]} A[x]/\langle ax-1\rangle$:

$$0 \otimes [a] = \mathrm{ev}_0(x) \otimes 1 = 1 \otimes [ax] = 1 \otimes 1$$

so $0 = 1$, and $B \otimes_{B[x]} A[x]/\langle ax-1\rangle$ is the zero ring. It follows that the left square trivially commutes, so by Lemma 2.3.4 the left square is Cartesian. Now note, that the morphism:

$$\operatorname{Spec} A[x]/\langle ax-1\rangle \to \operatorname{Spec} B[x]/\ker\psi$$

is surjective as it is given by $f'|_{\mathbb{V}(ax-1)}$ with restricted image, so by Lemma 3.10.6 we have that the morphism:

$$\operatorname{Spec} B \otimes_{B[x]} A[x]/\langle ax-1\rangle \to \operatorname{Spec} B[x]/(\ker\psi + \langle x\rangle)$$

is also surjective. However, $\operatorname{Spec} B \otimes_{B[x]} A[x]/\langle ax-1\rangle$ is empty, hence $\operatorname{Spec} B[x]/(\ker\psi + \langle x\rangle)$ is also empty, so by our earlier remarks $\operatorname{Spec} A \to \operatorname{Spec} B$ is integral as desired. ☐

We now proceed with showing that all finite morphisms are proper, though much of the leg work has already been covered. We first need the following immediate result:

Lemma 3.10.7. *Let $f : X \to Y$ be affine, then f is separated.*

Proof. Since the property of being separated is local on target, and f is affine, it suffices to show this in the case $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. However this clear by Example 3.6.2, hence f is separated. ☐

The above borderline immediately implies the following:

Theorem 3.10.2. *Let $f : X \to Y$ be a morphism. Then f is finite if and only if it is affine and proper.*

Proof. Suppose f is finite, then f is automatically affine, and integral. It follows that f is separated by Lemma 3.10.7, and universally closed by Theorem 3.10.1. Moreover, f is of finite type as every finite morphism is automatically finite[74]. It follows that f is affine and proper.

Now suppose f is affine and proper, then f is affine and universally closed so it is integral by Theorem 3.10.1. Since f is of finite type, we then obtain that f is finite by Proposition 3.9.1, implying the claim. ☐

[74]In particular if A is finitely generated as B module, then it is finitely generated as a B algebra by the same generating set.

Varieties I: A Rosetta Stone

$\mathscr{O}_X$ Modules I: Towards Vector Bundles

5.1 Definitions and Examples over Ringed Spaces

In Section 1.2 and Section 1.3 we broadly discussed sheafs of rings, abelian groups, and sets over topological spaces. In this chapter, we will extend these ideas to the category of modules over a commutative ring. Let $\mathscr{F}$ be a sheaf of abelian groups over X , then prescribing an A-module structure on $\mathscr{F}(U)$ for each $U \subset X$ gives us a sheaf of a A-modules. However, what we would really like, is for the A-module structure to vary with respect to a sheaf of rings on X, i.e. we want $\mathscr{F}(U)$ to be an $\mathscr{O}_X(U)$ module for all $U \subset X$. We define this precisely now:

Definition 5.1.1. Let $(X, \mathscr{O}_X)$ be a ringed space, and $\mathscr{F}$ a presheaf on X. Then $\mathscr{F}$ is a **presheaf of $\mathscr{O}_{\mathbf{X}}$ modules** if: There exists a sheaf morphism:

$$m_{\mathscr{F}} : \mathscr{O}_X \times \mathscr{F} \longrightarrow \mathscr{F}$$

which makes $\mathscr{F}(U)$ an $\mathscr{O}_X(U)$ module for each $U \subset X$. A **sheaf of $\mathscr{O}_{\mathbf{X}}$ modules** or a **$\mathscr{O}_{\mathbf{X}}$ module** is a presheaf of $\mathscr{O}_X$ modules that is also a sheaf. A **morphism of presheaves of $\mathscr{O}_{\mathbf{X}}$ modules** is a presheaf morphism $F : \mathscr{F} \to \mathscr{G}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{O}_X \times \mathscr{F} \arrow[r, "m_{\mathscr{F}}"] \arrow[d, "\mathrm{Id} \times F"] & \mathscr{F} \arrow[d, "F"] \\
\mathscr{O}_X \times \mathscr{G} \arrow[r, "m_{\mathscr{G}}"] & \mathscr{G}
\end{tikzcd}$$

A **morphism of sheaves of $\mathscr{O}_{\mathbf{X}}$ modules** is a morphism in the underlying category of presheaves of $\mathscr{O}_X$ modules. We denote the category of presheaves of $\mathscr{O}_X$ modules, and the category of sheaves of $\mathscr{O}_X$ modules by $\mathrm{Mod}_{\mathscr{O}_X}$ and $\mathrm{Mod}_{\mathscr{O}_X}$ respectively. At times, we will refer to sheaves of $\mathscr{O}_X$ modules simply as '$\mathscr{O}_X$ modules'.

Example 5.1.1. Letting $E \to X$ be a smooth vector bundle over a smooth manifold X, by Example 1.2.2 we have that $\Gamma(-, E)$ is a sheaf on M; we wish to show that this is a C^∞ module. For each open U, we define:

$$m_U : C^\infty(U) \times \Gamma(U, E) \to \Gamma(U, E)$$

to be the usual multiplication of a smooth function with a smooth section of E over U. If $(f, \phi) \in C^\infty(U) \times \Gamma(U, E)$, we need to show that:

$$f|_U \cdot \phi|_U = (f \cdot \phi)|_U$$

This is however true by construction, because f is an honest to god function on U with values in $\mathbb{R}$, and ϕ is an honest to map $U \to E|_U$. Moreover, a vector bundle morphism over X $F : E \to E'$ induces a morphism of the underlying C^∞ modules.

One readily checks that that $\mathrm{Mod}_{\mathscr{O}_X}$ and $\mathrm{Mod}_{\mathscr{O}_X}$ form abelian categories, and that the proof of Theorem 1.2.1 holds essentially verbatim when one replaces the words 'abelian group' with '$\mathscr{O}_X$ module'. We can also sheafify $\mathscr{O}_X$ modules, glue $\mathscr{O}_X$ modules and their morphisms, and take stalks at a point x

to get $(\mathscr{O}_X)_x$ modules. These facts are all borderline immediate given the content covered in Section 1.2 and Lemma 5.1.1, so we elect to not reprove these results in this section as there are more pressing matters at hand. The main being that given a continuous map $f : X \to Y$, the inverse image functor $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$, does not send $\mathscr{O}_Y$ modules to $\mathscr{O}_Y$ modules, but to $f^{-1}\mathscr{O}_Y$ modules. We will instead need to construct a different functor, called the pullback functor, which will combine tensor products, and the inverse image functor. We begin with showing that sheafification commutes with finite products:

Lemma 5.1.1. *Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves on X, then $(\mathscr{F}\times\mathscr{G})^\sharp$ is canonically isomorphic to $\mathscr{F}^\sharp\times\mathscr{G}^\sharp$. In particular, if $f : Y \to X$ is a continuos map, then $f^{-1}(\mathscr{F}\times\mathscr{G}) \cong f^{-1}\mathscr{F}\times f^{-1}\mathscr{G}$.*

Proof. One might imagine there is a slick proof of this fact exploiting the universal property of products, and sheafification, but as far as we can tell, there is no avoiding a direct computation with the definition of sheafification, and stalks, hence we show the isomorphism directly.

First note, that clearly we have a canonical isomorphism $(\mathscr{F}\times\mathscr{G})_x \cong \mathscr{F}_x\times\mathscr{G}_x$, hence we can consider elements of $(\mathscr{F}\times\mathscr{G})^\sharp$ to be sequence (s_x), where $s_x\in\mathscr{F}_x\times\mathscr{G}_x$. Let $\pi_{\mathscr{F}x}, \pi_{\mathscr{G}x}$ denote the projections on the level of stalks $\mathscr{F}_x\times\mathscr{G}_x\to\mathscr{F}_x$, $\mathscr{F}_x\times\mathscr{G}_x\to\mathscr{G}_x$ respectively induced by the projection morphisms on presheaves. Then for all U, we claim that the map:

$$\begin{aligned}(\mathscr{F}\times\mathscr{G})^\sharp(U) &\longrightarrow \prod_{x\in U}\mathscr{F}_x\times\prod_{x\in U}\mathscr{G}_x\\ (s_x)&\longmapsto ((\pi_{\mathscr{F}x}(s_x)),(\pi_{\mathscr{G}_x}(s_x)))\end{aligned}$$

as image in $\mathscr{F}^\sharp(U)\times\mathscr{G}^\sharp(U)$. Since doing the following for $\mathscr{F}$ will be the same as doing it for $\mathscr{G}$, we need only show that for each $x\in U$, there exists an open neighborhood V of x, and a section $t\in\mathscr{F}(U)$ such that $t_y = \pi_{\mathscr{F}y}(s_y)$ for all $y\in V$. This is however clear; since $(s_x)\in(\mathscr{F}\times\mathscr{G})^\sharp(U)$, we have that there exists an open neighborhood V of x and a section $t\in\mathscr{F}(U)\times\mathscr{G}(U)$ such that $t_y = s_y$. Now note that for all $y\in V$:

$$\pi_{\mathscr{F}}(t)_y = \pi_{\mathscr{F}_y}(t_y) = \pi_{\mathscr{F}y}(s_y)$$

so we have obtained a map:

$$F : (\mathscr{F}\times\mathscr{G})^\sharp(U) \longrightarrow \mathscr{F}^\sharp(U)\times\mathscr{G}^\sharp(U)$$

which clearly commutes restricts. This is also clearly an isomorphism on stalks, so F is an isomorphism as desired, which must be unique by abstract nonsense.

To prove the second claim, by the first it suffices to provide an isomorphism:

$$f_p^{-1}(\mathscr{F}\times\mathscr{G})\cong f_p^{-1}\mathscr{F}\times f_p^{-1}\mathscr{G}$$

Our work in Proposition 1.3.5 demonstrates that $\mathscr{F}\mapsto f_p\mathscr{F}$ is a functor $\mathrm{PSh}(X)\to\mathrm{PSh}(Y)$, and so there are projection maps:

$$f_p^{-1}\pi_{\mathscr{F}} : f_p^{-1}(\mathscr{F}\times\mathscr{G})\to f_p^{-1}\mathscr{F} \qquad f_p^{-1}\pi_{\mathscr{G}} : f_p^{-1}(\mathscr{F}\times\mathscr{G})\to f_p^{-1}\mathscr{G}$$

and so by the universal property of the product, these determine a morphism:

$$F : f_p^{-1}(\mathscr{F}\times\mathscr{G}) \longrightarrow f_p^{-1}\mathscr{F}\times f_p^{-1}\mathscr{G}$$

given on an open set U by:

$$s\longmapsto (f_p^{-1}\pi_{\mathscr{F}}(s), f_p^{-1}\pi_{\mathscr{F}}\mathscr{G}(s))$$

To check that this is an isomorphism, it suffices to check that this is an isomorphism on stalks. Recall that there are natural isomorphisms:

$$f_p^{-1}(\mathscr{F}\times\mathscr{G})_y\cong(\mathscr{F}\times\mathscr{G})_{f(y)}\cong\mathscr{F}_{f(y)}\times\mathscr{G}_{f(y)}$$

and so if $s_y = [U, s]$, for $y\in U\subset Y$, then:

$$(f_p^{-1}\pi_{\mathscr{F}})_y(s_y) = [U, f_p^{-1}\pi_{\mathscr{F}}(s)]$$

However, if $s = [V, t]$, for $t \in \mathscr{F}(V) \times \mathscr{G}(V)$, and $f(U) \subset V$, then:

$$(f_p^{-1}\pi_{\mathscr{F}})([V, t]) = [V, \pi_{\mathscr{F}}(t)]$$

Under the isomorphism $(f_p^{-1}\mathscr{F})_y \cong \mathscr{F}_{f(y)}$ we have that:

$$[U, f_p^{-1}\pi_{\mathscr{F}}(s)] \longmapsto [V, \pi_{\mathscr{F}}(t)] = (\pi_{\mathscr{F}})_{f(y)}(t_y)$$

and similarly for $\pi_{\mathscr{G}}$. It follows that up to isomorphism, the stalk map:

$$(f_p^{-1}F)_y : f_p^{-1}(\mathscr{F} \times \mathscr{G})_y \longrightarrow f_p^{-1}\mathscr{F}_y \times f_p^{-1}\mathscr{G}_y$$

is the given by the map:

$$\begin{aligned}(\mathscr{F} \times \mathscr{G})_{f(y)} &\longrightarrow \mathscr{F}_{f(y)} \times \mathscr{G}_{f(y)} \\ t_{f(y)} &\longmapsto ((\pi_{\mathscr{F}})_{f(y)}(t_{f(y)}), (\pi_{\mathscr{G}})_{f(y)}(t_{f(y)}))\end{aligned}$$

which is an obvious isomorphism, implying the claim. □

Not only does Lemma 5.1.1 guarantee that the $\mathrm{Mod}_{\mathscr{O}_X}$ and $\mathrm{Mod}_{\mathscr{O}_X}$ behave mostly as expected, but it also allows us to quickly demonstrate the following failure:

Corollary 5.1.1. *Let $f : X \to Y$ be a morphism of ringed space, $\mathscr{F}$ an $\mathscr{O}_X$ module on X, and $\mathscr{G}$ an $\mathscr{O}_Y$ module on Y. Then $f_*\mathscr{F}$ is an $\mathscr{O}_Y$ module, and $f^{-1}\mathscr{G}$ is an $f^{-1}\mathscr{O}_Y$ module.*

Proof. We give $f_*\mathscr{F}$ the structure of an $\mathscr{O}_Y$ module by setting:

$$\begin{aligned}m_U : \mathscr{O}_Y(U) \times (f_*\mathscr{F})(U) &\longrightarrow (f_*\mathscr{F})(U) \\ (s, \phi) &\longmapsto f_U^\sharp(s) \cdot \phi\end{aligned}$$

Since $f_U^\sharp \in (f_*\mathscr{O}_X)(U) = \mathscr{O}_X(f^{-1}(U))$, this makes $(f_*\mathscr{F})(U)$ an $\mathscr{O}_Y(U)$ module. Moreover, since the restriction maps on $f_*\mathscr{F}$ are inherited from those on $\mathscr{F}$, and thus respect multiplication, and since $f_U^\sharp$ commutes with restriction maps, the collection m_U determines a sheaf morphism, hence $f_*\mathscr{F}$ is an $\mathscr{O}_Y$ module.

We now need to construct a morphism:

$$f^{-1}\mathscr{O}_Y \times f^{-1}\mathscr{G} \longrightarrow f^{-1}\mathscr{G}$$

By Lemma 5.1.1, and Proposition 1.3.5, we have that the defining map:

$$\mathscr{O}_Y \times \mathscr{G} \to \mathscr{G}$$

induces a morphism:

$$f^{-1}\mathscr{O}_Y \times f^{-1}\mathscr{G} \to f^{-1}\mathscr{G}$$

which one each open set $U \subset X$ will make $f^{-1}\mathscr{G}(U)$ an $f^{-1}\mathscr{O}_Y(U)$ module as desired. □

Recall that if M_1 and M_2 are A modules, we can form their tensor product $M_1 \otimes_A M_2$. This tensor product satisfies the following universal property: for every A bilinear map $M_1 \oplus M_2 \to N$, there exists a unique A-linear map $M_1 \otimes_A M_2 \to N$ making the following diagram commute:

$$\begin{tikzcd} M_1 \oplus M_2 \arrow[r] \arrow[d] & N \\ M_1 \otimes_A M_2 \arrow[ur] & \end{tikzcd}$$

With this recollection in mind, we form the following definition:

Definition 5.1.2. A $\mathscr{O}_{\mathbf{X}}$ **bilinear** morphism of presheaves or sheaves of $\mathscr{O}_X$ modules, is a morphism of presheaves/sheaves $\mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{G}$, such that for each $U \subset X$, $\mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \to \mathscr{G}(U)$ is a bilinear map. We define the **tensor product presheaf** by:

$$(\mathscr{F} \otimes^p_{\mathscr{O}_X} \mathscr{G})(U) = \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$$

Obviously, we need to check that this is a presheaf, and while we are at it, we might as well prove some desirable properties of the the presheaf.

Lemma 5.1.2. *Let $\mathscr{F}_1, \mathscr{F}_2$ be presheaves (or sheaves) of $\mathscr{O}_X$ modules, then the following hold:*

i) The tensor product presheaf $\mathscr{F} \otimes^p_{\mathscr{O}_X} \mathscr{G}$ is a presheaf.

ii) The tensor product presheaf satisfies the universal property of the tensor product in $\mathrm{Mod}_{\mathscr{O}_X}$.

iii) For all $x \in X$, there is a natural isomorphism $(\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2)_x \cong (\mathscr{F}_1)_x \otimes_{\mathscr{O}_{X,x}} (\mathscr{F}_2)_x$.

Proof. We obviously start with i). Let $V \subset U$ be open sets of X , we need to write down restriction maps:

$$\theta^U_V : \mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \longrightarrow \mathscr{F}_1(V) \otimes_{\mathscr{O}_X(V)} \mathscr{F}_2(V)$$

Denote the restrictions maps for $\mathscr{F}_1$ and $\mathscr{F}_2$ by $(\theta_1)^U_V$ and $(\theta_2)^U_V$, then we have bilinear map:

$$\begin{gathered} \mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \longrightarrow \mathscr{F}_1(V) \oplus \mathscr{F}_2(V) \longrightarrow \mathscr{F}_1(V) \otimes_{\mathscr{O}_X(V)} \mathscr{F}_2(V) \\ (s,t) \longmapsto (s|_V, t|_V) \longmapsto (s|_V) \otimes (t|_V) \end{gathered}$$

and so by the universal property of the tensor product, we get well defined restriction maps:

$$\begin{aligned} \theta^U_V : \mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) &\longrightarrow \mathscr{F}_1(V) \otimes_{\mathscr{O}_X(V)} \mathscr{F}_2(V) \\ s \otimes t &\longmapsto (s|_V) \otimes (t|_V) \end{aligned}$$

which obviously satisfy $\theta^V_W \circ \theta^U_V = \theta^U_W$, making the tensor product presheaf, a presheaf.

For ii), we first need a bilinear sheaf morphism $\mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2$. For each U, we have bilinear morphism:

$$\begin{aligned} \otimes^p_U : \mathscr{F}_1(U) \oplus \mathscr{F}_2(U) &\longrightarrow \mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \\ (s,t) &\longrightarrow s \otimes t \end{aligned}$$

We need only check that $\theta^U_V \circ \otimes^p_U = \otimes^p_V \circ \theta^U_V$, however this clear as:

$$\theta^U_V \circ \otimes^p_U(t,s) = \theta^U_V(t \otimes s) = t|_V \otimes s|_V = \otimes^p_V(t|_V, s_V) = \otimes^p_V \circ \theta^U_V(t,s)$$

so the assignment $U \mapsto \theta_U$ defines a sheaf morphism. Suppose that $F : \mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{G}$ is a a bilinear sheaf morphism, then for each $U \subset X$, there is a unique Ψ_U which makes the following diagram commute:

$$\begin{tikzcd} \mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[r] \arrow[d] & \mathscr{G}(U) \\ \mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \arrow[ur] & \end{tikzcd}$$

We need to show that $\theta^U_V \circ \Psi_U = \Psi_V \circ \theta^U_V$. Consider the following diagram:

$$\begin{tikzcd} \mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[r, "\theta^U_V \circ F_U"] \arrow[d] & \mathscr{G}(V) \\ \mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \arrow[ur] & \end{tikzcd}$$

and note that:

$$\begin{aligned} \Psi_V \circ \theta^U_V \circ \otimes^p_U =& \Psi_V \circ \otimes^p_V \circ \theta^U_V \\ =& F_V \circ \theta^U_V \\ =& \theta^U_V \circ F_U \end{aligned}$$

while:

$$\theta^U_V \circ \Psi_U \circ \otimes^p_U = \theta^U_V \circ F_U$$

so both $\theta^U_V \circ \Psi_U$ and $\Psi_V \circ \theta^U_V$ make the diagram commute implying equality. It follows that $\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2$ satisfies the universal property of the tensor product in the category of presheaves.

For $iii)$, we want to show $(\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2)_x$ satisfies the universal property of the the tensor product. The morphism $\otimes^p : \mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2$ yields a stalk map $\otimes^p_x : (\mathscr{F}_1)_x \oplus (\mathscr{F}_2)_x \to (\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2)_x$. Now suppose that $F : (\mathscr{F}_1)_x \oplus (\mathscr{F}_2)_x \to M$ is an $\mathscr{O}_{X,x}$ bilinear map. Now this is equivalent to the data in the following diagram:

$$\begin{tikzcd}
\mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[rr, "\theta^U_V"] \arrow[dr, "\psi_U"] \arrow[ddr, bend right, "\phi_U"'] & & \mathscr{F}_1(V) \oplus \mathscr{F}_2(V) \arrow[dl, "\psi_V"'] \arrow[ddl, bend left, "\phi_V"] \\
& (\mathscr{F}_1)_x \oplus (\mathscr{F}_2)_x \arrow[d, "F"] & \\
& M &
\end{tikzcd}$$

where the ϕ_U is bilinear for each U[75]. In particular, each ϕ_U, gives a unique $\varphi_U : \mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \to M$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[r, "\phi_U"] \arrow[d, "\otimes^p_U"] & M \\
\mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \arrow[ur, "\varphi_U"'] &
\end{tikzcd}$$

We claim that these commute with restriction maps; indeed, consider the following diagram:

$$\begin{tikzcd}
\mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[r, "\phi_V \circ \theta^U_V"] \arrow[d, "\otimes^p_U"] & M \\
\mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \arrow[ur, "\varphi_U"'] &
\end{tikzcd}$$

then:

$$\varphi_U \circ \otimes^p_U = \phi_U = \phi_V \circ \theta^U_V$$

so the diagram commutes. We also have that:

$$\begin{aligned}
\varphi_V \circ \theta^U_V \circ \otimes^p_U =& \varphi_V \circ \otimes^p_V \circ \theta^U_V \\
=& \phi_V \circ \theta^U_V
\end{aligned}$$

so by uniqueness of the morphism, we have that:

$$\varphi_U = \varphi_V \circ \theta^U_V$$

[75] This follows because it is true on the level of sets, and since F is bilinear, and ψ_U is linear, the ϕ_U must be bilinear. In particular they are defined by $\phi_U = F \circ \psi_U$.

giving us a unique F' which makes the following diagram commute:

$$\begin{tikzcd}
\mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \arrow[rr, "\theta^U_V"] \arrow[dr, "\psi'_U"] \arrow[ddr, bend right, "\varphi_U"'] & & \mathscr{F}_1(V) \otimes_{\mathscr{O}_X(V)} \mathscr{F}_2(V) \arrow[dl, "\psi'_V"'] \arrow[ddl, bend left, "\varphi_V"] \\
& (\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2)_x \arrow[d, "\exists! F'"] & \\
& M &
\end{tikzcd}$$

We thus now need only check that $F' \circ \otimes^p_x = F$. It suffices to show that:

$$\begin{tikzcd}
\mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[rr, "\theta^U_V"] \arrow[dr, "\psi_U"] \arrow[ddr, bend right, "\phi_U"'] & & \mathscr{F}_1(V) \oplus \mathscr{F}_2(V) \arrow[dl, "\psi_V"'] \arrow[ddl, bend left, "\phi_V"] \\
& (\mathscr{F}_1)_x \oplus (\mathscr{F}_2)_x \arrow[d, "F' \circ \otimes^p_x"] & \\
& M &
\end{tikzcd}$$

Note that $\otimes^p_x$ is given by the following diagram:

$$\begin{tikzcd}
\mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[rr, "\theta^U_V"] \arrow[dr, "\psi_U"] \arrow[ddr, bend right, "\psi'_U \circ \otimes_U"'] & & \mathscr{F}_1(V) \oplus \mathscr{F}_2(V) \arrow[dl, "\psi_V"'] \arrow[ddl, bend left, "\psi'_V \circ \otimes_U"] \\
& (\mathscr{F}_1)_x \oplus (\mathscr{F}_2)_x \arrow[d, "\otimes_x"] & \\
& (\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2)_x &
\end{tikzcd}$$

It follows that:

$$F' \circ \otimes^p_x \circ \psi_U = F' \circ \psi' \circ \otimes^p_U = \varphi_U \circ \otimes^p_U = \phi_U$$

implying the claim. $\square$

The next obvious step is to construct a tensor product in the category $\text{Mod}_{\mathscr{O}_X}$, and there is essentially one way to do this:

Definition 5.1.3. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be $\mathscr{O}_X$ modules[76], then the **tensor product of $\mathscr{O}_\mathbf{X}$ modules** is:

$$\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2 := (\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2)^\sharp$$

We now wish to check that this is actually the tensor product in the category $\text{Mod}_{\mathscr{O}_X}$, i.e. that $\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2$ is satisfies the universal property.

Lemma 5.1.3. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be $\mathscr{O}_X$ modules, then $\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2$ satisfies the universal property of the tensor product in* $\text{Mod}_{\mathscr{O}_X}$*. Moreover, there is a natural isomorphism $(\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2)_x \cong (\mathscr{F}_1)_x \otimes_{\mathscr{O}_{X,x}} (\mathscr{F}_2)_x$*

Proof. The second statement is an obvious consequence of Lemma 1.2.4. We obtain a morphism $\otimes : \mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2$, by setting $\otimes = \text{sh} \circ \otimes^p$. Now suppose that $F : \mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{G}$ is a bilinear $\mathscr{O}_X$

[76] I.e. sheaves of $\mathscr{O}_X$ modules! This is the last reminder of this nomenclature.

morphism. Then by Lemma 5.1.2 there exists a unique $\mathscr{O}_X$-linear morphism $\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2 \to \mathscr{G}$. By the universal property of sheafification, there is then a unique $\mathscr{O}_X$ linear morphism $\tilde{F}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}_1 \oplus \mathscr{F}_2 \arrow[rr, "F"] \arrow[d, "\otimes^p"] & & \mathscr{G} \\
\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2 \arrow[urr, "\exists! F'"] \arrow[rr, "\mathrm{sh}"] & & \mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2 \arrow[u, "\exists! \tilde{F}"]
\end{tikzcd}$$

In particular, we have that $\tilde{F} \circ \otimes = F$, and is the unique map making this diagram commute, hence $\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2$ satisfies the universal property as desired. □

Now note that if $f : X \to Y$ is a morphism of ringed spaces, we have a sheaf morphism $\hat{f}$: $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$, which clearly makes $\mathscr{O}_X$ an $f^{-1}\mathscr{O}_X$ module. It follows that we can take the tensor product $f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$, which can now be viewed as an $\mathscr{O}_X$ module. Indeed, for each open set U, the map given on simple tensors:

$$\mathscr{O}_X(U) \times (f^{-1}\mathscr{F} \otimes^p_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X)(U) \longrightarrow (f^{-1}\mathscr{F} \otimes^p_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X)(U)$$
$$(s, \phi \otimes t) \longmapsto \phi \otimes (st)$$

commutes with restriction maps, and thus defines a morphism of sheaves. This morphism of sheaves clearly makes $f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$ a presheaf of $\mathscr{O}_X$ modules. We then compose with sheafification to obtain a morphism:

$$\mathscr{O}_X \times f^{-1}\mathscr{F} \otimes^p_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \longrightarrow f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$$

and using the universal property of sheafification obtain a sheaf morphism:

$$\mathscr{O}_X \times f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \longrightarrow f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$$

which makes $f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$ an $\mathscr{O}_X$ module.

Definition 5.1.4. Let $f : X \to Y$ be morphism of ringed spaces, and $\mathscr{F}$ an $\mathscr{O}_Y$ module on Y. Then the **pull back of** $\mathscr{F}$ is the $\mathscr{O}_X$ module $f^*\mathscr{F}$, defined by:

$$f^*\mathscr{F} = f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$$

Note that $f^* : \mathrm{Mod}_{\mathscr{O}_Y} \to \mathrm{Mod}_{\mathscr{O}_X}$ is obviously a functor by the fact that f^{-1} is a functor, and the universal property of the tensor product. Moreover, the stalk at x is canonically given by:

$$(f^*\mathscr{F})_x \cong \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{O}_{X,x}$$

In the next section, using a sheaf theoretic version of the tensor-hom adjunction for modules, we will be able to show that f^* is left adjoint f_* in the category of $\mathscr{O}_X$ modules. For now we continue to prove general statements regarding pullbacks and tensor products.

Morally tangential to the pullback, is a sheaf theoretic extension of scalars. In particular, if $\mathscr{O}_X \to \mathscr{O}'_X$ is a morphism of a sheaf of rings, and $\mathscr{F}$ is an $\mathscr{O}_X$ module, we can make $\mathscr{F}$ and $\mathscr{O}_{X'}$ module via:

$$\mathscr{F}' = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}'_X$$

We now prove the following basic statements regarding tensor products:

Proposition 5.1.1. *Let $\mathscr{F}, \mathscr{G}$ and $\mathscr{H}$ be $\mathscr{O}_X$ modules, $\mathscr{H}'$ an $\mathscr{O}'_X$ module, and $\mathscr{G}$ is also an $\mathscr{O}'_X$ module. Then there are unique isomorphisms:*

a) $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \cong \mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{F}$

b) $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) \otimes_{\mathscr{O}'_X} \mathscr{H}' \cong \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{G} \otimes_{\mathscr{O}'_X} \mathscr{H}')$

c) $(\mathscr{F} \oplus \mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{H} \cong (\mathscr{F} \oplus \mathscr{H}) \otimes_{\mathscr{O}_X} (\mathscr{F} \oplus \mathscr{G})$

d) $\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{F} \cong \mathscr{F}$.

Proof. Since sheafification is a functor $\mathrm{PMod}_{\mathscr{O}_X} \to \mathrm{Mod}_{\mathscr{O}_X}$, it suffices to show each statement this for the presheaf tensor product. This is however clear, as for each open $U \subset X$ we have the unique isomorphisms a), b), c) and d) just of the underlying $\mathscr{O}_X(U)$ modules $\mathscr{F}(U), \mathscr{G}(U), \mathscr{H}(U)$. By the universal property, all of these isomorphisms, will have to commute with restriction maps, and so they yield $\mathscr{O}_X$ linear isomorphisms of presheaves of $\mathscr{O}_X$ modules, implying the claim. □

We wish to prove similar properties for the pull back, but we need the following lemma:

Lemma 5.1.4. *Let $\mathscr{F}$, $\mathscr{G}$ be $\mathscr{O}_Y$ modules, and $f : X \to Y$ a morphism of locally ringed spaces. Then there is a canonical isomorphism:*

$$f^{-1}(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) \cong f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{G}$$

Proof. Note that there exists a bilinear map $\mathscr{F} \oplus \mathscr{G} \to \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}$ given by the tensor product. Applying the f^{-1} we get the following natural bilinear morphism:

$$f^{-1}\mathscr{F} \oplus f^{-1}\mathscr{G} \longrightarrow f^{-1}(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G})$$

which on stalks is given up to isomorphism by the tensor product map:

$$\otimes_{f(x)} : \mathscr{F}_{f(x)} \oplus \mathscr{G}_{f(x)} \longrightarrow \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)}$$

By the universal property of the tensor product, there is then $f^{-1}\mathscr{O}_Y$ linear module morphism:

$$F : f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{G} \longrightarrow f^{-1}(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G})$$

By the universal property of the tensor product, the map on stalks must then be the unique one making the following diagram commute:

$$\begin{tikzcd}
\mathscr{F}_{f(x)} \oplus \mathscr{G}_{f(x)} \arrow[rr, "\otimes_{f(x)}"] \arrow[d, "\otimes_{f(x)}"] & & \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)} \\
\mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)} \arrow[urr] & &
\end{tikzcd}$$

which must be the identity. It follows that F is an isomorphism on stalks and thus an isomorphism. □

We can now easily prove the following:

Proposition 5.1.2. *Let $\mathscr{F}$, and $\mathscr{G}$ be a $\mathscr{O}_Y$ modules, and $f : X \to Y$ a morphism of ringed spaces. Then we have the following natural isomorphisms:*

a) $f^*\mathscr{O}_Y \cong \mathscr{O}_X$

b) $f^*(\mathscr{F} \oplus \mathscr{G}) \cong f^*\mathscr{F} \oplus f^*\mathscr{G}$

c) $f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) \cong f^*\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}$

Moreover, if $\iota : U \to X$ is an open embedding, then $\iota^*\mathscr{F} \cong \mathscr{F}|_U$.

Proof. For a), we have that by d) of Proposition 5.1.1:

$$f^*\mathscr{O}_Y = f^{-1}\mathscr{O}_Y \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \cong \mathscr{O}_X$$

For b), we have that by Lemma 5.1.1, and c) of Proposition 5.1.1:

$$\begin{aligned}
f^*(\mathscr{F} \oplus \mathscr{G}) =& f^{-1}(\mathscr{F} \oplus \mathscr{G}) \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \\
\cong& (f^{-1}\mathscr{F} \oplus f^{-1}\mathscr{G}) \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \\
\cong& (f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X) \oplus (f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X) \\
\cong& f^*\mathscr{F} \oplus f^*\mathscr{G}
\end{aligned}$$

For c), by Lemma 5.1.3, and a), d) and b) of Proposition 5.1.1:

$$\begin{aligned}
f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) =& f^{-1}(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \\
\cong& (f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{G}) \otimes_{f^{-1}\mathscr{O}_Y} (\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{O}_X) \\
\cong& f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \left[(f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{O}_X\right] \\
\cong& (f^{-1} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X) \otimes_{\mathscr{O}_X} f^*\mathscr{G} \\
\cong& f^*\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}
\end{aligned}$$

For the final statement we have that by Corollary 1.3.2, d) of Proposition 5.1.1:

$$\begin{aligned}\iota^*\mathscr{F} &= \iota^{-1}\mathscr{F} \otimes_{\iota^{-1}\mathscr{O}_X} \mathscr{O}_U \\ &\cong \mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{O}_U \\ &\cong \mathscr{F}|_U\end{aligned}$$

□

We now provide an elementary proof that the tensor product functor is right exact $-\otimes_A M$.

Lemma 5.1.5. *Let M be an A module, and :*

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

be an exact sequence of A modules, then the following sequence is exact:

$$N_1 \otimes_A M \longrightarrow N_2 \otimes_A M \longrightarrow N_3 \otimes_A M \longrightarrow 0$$

Proof. Let f_i denote the morphism $N_i \to N_{i+1}$, and $f_i \otimes \mathrm{Id}_M$ the induced map:

$$N_i \otimes_A M \to N_{i+1} \otimes_A M$$

We first show that $f_2 \times \mathrm{Id}$ is still surjective. Let:

$$\beta = \sum_i n_i \otimes m_i \in N_3 \otimes M$$

then each $n_i = f(n_i')$ for some $n_i' \in N_2$, so the element:

$$\alpha : \sum_i n_i' \otimes m \in N_2 \otimes M$$

satisfies:

$$f_2 \otimes \mathrm{Id}(\alpha) = \beta$$

implying that $f_2 \otimes \mathrm{Id}$ is surjective as desired.

It is clear that $\mathrm{im}\, f_1 \otimes \mathrm{Id} \subset \ker(f_2 \otimes \mathrm{Id})$. Suppose that:

$$\beta = \sum_i n_i \otimes m_i \in \ker(f_2 \otimes \mathrm{Id})$$

and recall that since the original sequence is exact, $N_3 \cong N_2/f_1(N_1)$. Note there is a canonical isomorphism[77]

$$N_2/\,\mathrm{im}\, f_1 \otimes_A M \cong (N_2 \otimes_A M)/(\mathrm{im}(f_1 \otimes \mathrm{Id}))$$

as then $\beta \in \ker(f_2 \otimes \mathrm{Id})$ implies that up to some canonical isomorphism, $[\beta] = 0 \in (N_2 \otimes_A M)/(\mathrm{im}(f_1 \otimes \mathrm{Id}))$, so $\beta \in \mathrm{im}(f_1 \otimes \mathrm{Id})$ implying exactness. □

Using the above, we wish to extend this right exactness to a statement about $\mathscr{O}_X$ and $\mathscr{O}_Y$ modules:

Proposition 5.1.3. *Let*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

be an exact sequence of $\mathscr{O}_Y$ modules. Then for any $\mathscr{G}$ the following sequence is exact:

$$\mathscr{F}_1 \otimes_{\mathscr{O}_Y} \mathscr{G} \longrightarrow \mathscr{F}_2 \otimes_{\mathscr{O}_Y} \mathscr{G} \longrightarrow \mathscr{F}_3 \otimes_{\mathscr{O}_Y} \mathscr{G} \longrightarrow 0$$

In particular, $f^* : \mathrm{Mod}(Y) \to \mathrm{Mod}(X)$ is a right exact functor.

[77] After noting that clearly $\mathrm{im}\, f \otimes \mathrm{Id} = \mathrm{im}\, \iota \otimes \mathrm{Id}$, where $\iota : \mathrm{im}\, f_1 \to N_2$ is the inclusion map.

Proof. We will leverage Proposition 1.2.9 throughout this proof. The second statement will follow from the first, as if

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

is an exact sequence, then:

$$0 \longrightarrow f^{-1}\mathscr{F}_1 \longrightarrow f^{-1}\mathscr{F}_2 \longrightarrow f^{-1}\mathscr{F}_3 \longrightarrow 0$$

must be exact, as on stalks it is given up to isomorphism by:

$$0 \longrightarrow (\mathscr{F}_1)_{f(x)} \longrightarrow (\mathscr{F}_2)_{f(x)} \longrightarrow (\mathscr{F}_3)_{f(x)} \longrightarrow 0$$

which are exact because the original sequence was exact. This holds for all $x \in X$, hence Proposition 1.2.9 implies that inverse image sequence is an exact sequence of $f^{-1}\mathscr{O}_Y$ modules. It follows that if the tensor product is right exact then:

$$f^*\mathscr{F}_1 \longrightarrow f^*\mathscr{F}_2 \longrightarrow f^*\mathscr{F}_3 \longrightarrow 0$$

is an exact sequence so f^* is a right exact functor.

To see that $- \otimes_{\mathscr{O}_Y} \mathscr{G}$ is right exact, note that on stalks we have:

$$(\mathscr{F}_1)_y \otimes_{\mathscr{O}_{Y,y}} \mathscr{G}_y \longrightarrow (\mathscr{F}_2)_y \otimes_{\mathscr{O}_{Y,y}} \mathscr{G}_y \longrightarrow (\mathscr{F}_3)_y \otimes_{\mathscr{O}_{Y,y}} \mathscr{G}_y \longrightarrow 0$$

which is exact by Lemma 5.1.5. This holds for all $y \in Y$ so Proposition 1.2.9 implies the claim. □

We fix the notation that for any indexing set I, $\mathscr{F}^I$ is the direct sum over:

$$\mathscr{F}^I = \bigoplus_{i \in I} \mathscr{F}$$

In other words we want to take infinite coproducts, and not infinite direct products.[78] We now list some full subcategories of Mod_X with the following barrage of definitions:

Definition 5.1.5. Let $\mathscr{F}$ be an $\mathscr{O}_X$ module, then $\mathscr{F}$ is a **quasicoherent $\mathscr{O}_X$ module** if for every $x \in X$, there exists an open neighborhood U of x, and indexing sets I and J such that we have an exact sequence:

$$\mathscr{O}_U^I \longrightarrow \mathscr{O}_U^J \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

We say that $\mathscr{F}$ is a **finite type** if for every point $x \in X$ there exists a neighborhood U of x such that there is a surjection:

$$\mathscr{O}_U^n \to \mathscr{F}|_U$$

for some $n \in \mathbb{N}$. We say that $\mathscr{F}$ is a **coherent $\mathscr{O}_{\mathbf{X}}$ module** if $\mathscr{F}$ is of finite type, and for any open set $U \subset X$, and every finite set $\{s_1, \ldots, s_m\} \subset \mathscr{F}(U)$, the kernel of the induced map:

$$\mathscr{O}_U^m \to \mathscr{F}|_U$$

is of finite type. Finally, $\mathscr{F}$ is said to be **locally free** for every point in x there exists an open neighborhood U such that

$$\mathscr{F}|_U \cong \mathscr{O}_U^I$$

If I is finite, we say that $\mathscr{F}$ is **finite locally free** , and if we can always choose I to have cardinality n we say that $\mathscr{F}$ is **locally free of rank n**.

We are particularly interested in $\mathscr{O}_X$ modules which are quasicoherent, coherent, or locally free of rank n, which at times we will call vector bundles. We denote their respective categories by $\mathrm{QCoh}_{\mathscr{O}_X}$, $\mathrm{Coh}_{\mathscr{O}_X}$, and $\mathrm{Vec}_{\mathscr{O}_X}$.

[78] In the category of abelian groups, these do not agree when I is infinite. In particular, the infinite coproduct consists of infinite sequences where all but finitely terms are nonzero, and the infinite product is all infinite sequences.

Example 5.1.2. We briefly provide some justification for the term vector bundle. If $\pi : E \to X$ is an honest to god vector bundle over a smooth manifold, and $\{U_i\}$ is a trivializing cover so that:

$$\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^n$$

then there exists a local frame $\{e_1, \dots, e_r\}$ over $\pi^{-1}(U_i)$. In particular, this local frame induces an isomorphism of sheaves:

$$\Gamma(-, E)|_{U_i} \cong \mathscr{O}_{U_i}^r$$

hence locally free sheaves of $\mathscr{O}_X$ modules are our scheme analogue of vector bundles.

One might hope that we have the following chain of implications:

$$\mathscr{F} \text{ is locally free of rank } n \Rightarrow \mathscr{F} \text{ is coherent} \Rightarrow \mathscr{F} \text{ is quasicoherent}$$

however, as the next example shows, not every finite locally free module need be coherent. In fact the following example shows that there exist locally ringed spaces where $\mathscr{O}_X$ is not even coherent over itself!

Example 5.1.3. Let:

$$X = \operatorname{Spec} A = \operatorname{Spec} \left(k[x, y_1, y_2, \dots] / \langle \{xy_i\}_{i=1}^{\infty} \rangle\right)$$

and take $[x] \in \mathscr{O}_X(X)$. Then the induced map:

$$\phi : \mathscr{O}_X \longrightarrow \mathscr{O}_X$$

given on opens by:

$$\begin{aligned} \mathscr{O}_X(U) &\longrightarrow \mathscr{O}_X(U) \\ s|_U &\longmapsto s|_U \cdot [x]|_U \end{aligned}$$

cannot have kernel of finite type. Indeed, if this were true, then for all $\mathfrak{p} \in X$, we would have that the stalk $(\ker \phi)_{\mathfrak{p}}$ is finitely generated $\mathscr{O}_{X,\mathfrak{p}}$ module. Let $\mathfrak{p} = \langle [x], [y_1], \dots \rangle$, then $\mathscr{O}_{X,\mathfrak{p}} \cong A_{\mathfrak{p}}$, and $\phi_{\mathfrak{p}} : A_{\mathfrak{p}} \to A_{\mathfrak{p}}$ is given by:

$$\frac{[a]}{[s]} \longmapsto \frac{[a] \cdot [x]}{[s]}$$

We claim that $[y_i]/1 \in A_{\mathfrak{p}}$ is nonzero for all i. Indeed, if it were then there is some $[s] \notin \mathfrak{p}$ such that:

$$[s] \cdot [y] = 0 \Rightarrow s_i y_i \in \langle \{xy_i\}_{i=1}^{\infty} \rangle$$

implying that $s_i y_i$ has a factor of x, so $[s]$ has a factor of $[x]$ in it as well. In particular, the $[y_i]/1$ are nonzero in $A_{\mathfrak{p}}$, and obviously lie in $\ker \phi_{\mathfrak{p}}$ for all i, so $\langle [y_1]/1, \dots \rangle \subset \ker \phi_{\mathfrak{p}}$. If $[a]/[s] \in \ker \phi_{\mathfrak{p}}$, then there exists some $[t] \notin \mathfrak{p}$ such that :

$$[t] \cdot [x] \cdot [a] = 0 \Rightarrow t \cdot (xa) \in \langle \{xy_i\}_{i=1}^{\infty} \rangle$$

However, since $[t] \notin \mathfrak{p}$, we have that t cannot not have a factor of x or y_i in it. It follows that $xa \in \langle \{xy_i\}_{i=1}^{\infty} \rangle$, and thus a must be a sum of elements which have factors of y_i in them. It follows that $\ker \phi_{\mathfrak{p}} = \langle [y_1]/1, \dots \rangle$, and so $\ker \phi_{\mathfrak{p}}$ is not a finitely generated $\mathscr{O}_{X,\mathfrak{p}}$ module.

The desired implication is fixed if we require $\mathscr{O}_X$ to be coherent over itself. Indeed we have the following lemma:

Lemma 5.1.6. *Suppose that $\mathscr{O}_X$ is coherent over itself, then $\mathscr{F}$ is a coherent $\mathscr{O}_X$ module if and only if it is of finite presentation, i.e. for every x there exists an open neighborhood U, such that the following sequence is exact for some m and n:*

$$\mathscr{O}_U^n \longrightarrow \mathscr{O}_U^m \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

Proof. Suppose $\mathscr{F}$ is coherent, then for every x there is an open neighborhood of x such that $\mathscr{F}|_U$ is finitely generated. In other words, if:

$$\phi : \mathscr{O}_U^n \to \mathscr{F}|_U$$

is the surjection, we have an exact sequence:

$$0 \longrightarrow \ker\phi \longrightarrow \mathscr{O}_U^m \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

Since $\mathscr{F}$ is coherent though, we have that $\ker\phi$ is of finite type, hence there is a neighborhood of x and open neighborhood V, which must be contained in U[79], such that we have a surjection:

$$\mathscr{O}_V^n \to \ker\phi|_V$$

It follows that we have the following exact sequence:

$$\mathscr{O}_V^n \longrightarrow \mathscr{O}_V^m \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

where the first map is the projection onto $\ker\phi|_V$ composed with the inclusion of $\ker\phi|_V$ into $\mathscr{O}_V^m$.

Now suppose that $\mathscr{F}$ is finitely presented Let $\{V_i\}$ be a cover of X such that we have an exact sequence:

$$\mathscr{O}_{V_i}^n \xrightarrow{\beta_i} \mathscr{O}_{V_i}^m \xrightarrow{\alpha_i} \mathscr{F}|_{V_i} \longrightarrow 0$$

We claim that $\mathscr{F}|_{V_i}$ is the cokernel of β_i. However this clear as there is a unique morphism:

$$\operatorname{coker}\alpha_i \to \mathscr{F}|_{V_i}$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{O}_{V_i}^n \arrow[r, "\alpha_i"] & \mathscr{O}_{V_i}^m \arrow[rr, "\beta_i"] \arrow[dr, "\pi"'] & & \mathscr{F}|_{V_i} \arrow[r] & 0 \\
& & \operatorname{coker}\alpha_i \arrow[ur, "\theta"'] & &
\end{tikzcd}$$

On stalks we have the following diagram up to isomorphism:

$$\begin{tikzcd}
\mathscr{O}_{V_i,x}^n \arrow[r, "\alpha_{i,x}"] & \mathscr{O}_{V_i,x}^m \arrow[rr, "\beta_{i,x}"] \arrow[dr, "\pi_x"'] & & \mathscr{F}_x \arrow[r] & 0 \\
& & \operatorname{coker}\alpha_{i,x} \arrow[ur, "\theta_x"'] & &
\end{tikzcd}$$

Since $\beta_{i,x}$ is surjective we have that θ_x is surjective. Note that the $\ker\pi_x = \operatorname{im}\alpha_{i,x}$ by definition, and $\operatorname{im}\alpha_{i,x} = \ker\beta_{i,x}$ by assumption. We have $\ker\beta_{i,x} = \pi_x^{-1}(\ker\theta_x)$, so:

$$\ker\pi_x = \operatorname{im}\alpha_{i,x} = \pi_x^{-1}(\ker\theta_x)$$

In particular, $\pi_x^{-1}(0) = \pi_x^{-1}(\ker\theta_x)$, implying that:

$$0 = \ker\theta_x$$

because π_x is surjective. It follows that θ_x is an isomorphism, so θ is an isomorphism.

Accepting for the moment that $\mathrm{Coh}_{\mathscr{O}_X}$ is an abelian category,[80] it follows that $\mathscr{F}|_{V_i}$ is coherent for each i. Now let $\{s_1, \dots, s_l\} \subset \mathscr{F}(U)$, and $\phi : \mathscr{O}_U^l \to \mathscr{F}|_U$ the associated map. Then for each i, we have that $\phi|_{U\cap V_i} : \mathscr{O}_{U\cap V_i} \to \mathscr{F}|_{U\cap V_i}$ must have kernel of finite type as each $\mathscr{F}|_{V_i}$ is coherent. Since the V_i cover X, it follows that $\ker\phi$ must be of finite type hence $\mathscr{F}$ is coherent as desired. ☐

We have the immediate corollary:

Corollary 5.1.2. *Let X be a locally ringed space such that $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$ module. Then we have the following chain of implications:*

$$\mathscr{F} \text{ is locally free of rank n} \Rightarrow \mathscr{F} \text{ is coherent} \Rightarrow \mathscr{F} \text{ is quasicoherent}$$

[79]This is because $\ker\phi$ is only a sheaf on U.
[80]We prove this in Theorem 5.1.1.

Proof. By Lemma 5.1.6 every coherent $\mathscr{O}_X$ module is finitely presented and thus quasicoherent. Now suppose that $\mathscr{F}$ is locally free of rank n, then $\mathscr{F}$ is locally isomorphic to $\mathscr{O}_U^n$, so again accepting for the moment that $\mathrm{Coh}_{\mathscr{O}_X}$ is an abelian category, we have that there exists an open cover on which $\mathscr{F}|_U$ is coherent. The same argument at the end of Lemma 5.1.6 implies that $\mathscr{F}$ is coherent. □

Now $\mathrm{Vec}_{\mathscr{O}_X}$ has no hope of being an abelian category as the kernel of a vector bundle homomorphism between manifolds, is only a vector bundle when the map has constant rank on each fibre. Furthermore, $\mathrm{QCoh}_{\mathscr{O}_X}$ will be an abelian category when X is a scheme, but there are locally ringed spaces for which this is not true; we will not spend time delving into counter examples. What is always true is that the category of coherent modules over a ringed space is always abelian, a statement we will prove in this section. Before embarking on this endeavor, we first prove a few key results about the categories $\mathrm{QCoh}_{\mathscr{O}_X}$ and $\mathrm{Vec}_{\mathscr{O}_X}$. We will need the following lemma:

Lemma 5.1.7. *Suppose we have exact sequences of $\mathscr{O}_X$ modules:*

$$\mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

$$\mathscr{G}_1 \longrightarrow \mathscr{G}_2 \longrightarrow \mathscr{G}_3 \longrightarrow 0$$

Then there exists an exact sequence of the form:

$$(\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{G}_2) \oplus (\mathscr{F}_2 \otimes_{\mathscr{O}_X} \mathscr{G}_1) \longrightarrow \mathscr{F}_2 \otimes_{\mathscr{O}_X} \mathscr{G}_2 \longrightarrow \mathscr{F}_3 \otimes_{\mathscr{O}_X} \mathscr{G}_3 \longrightarrow 0$$

Proof. Denote by f_i the maps $\mathscr{F}_i \to \mathscr{F}_{i+1}$, and by g_i the maps $\mathscr{G}_i \to \mathscr{G}_{i+1}$. We construct the first map in the claimed exact sequence, which we denote by β, to be the direct sum of $f_1 \otimes \mathrm{Id}$ and $\mathrm{Id} \otimes g_2$, and the second map to be the unique map $f_2 \otimes g_2$. All of these maps come from the obvious diagrams.

To show that this sequence is exact, it suffices to show this on stalks, and so we need only prove this in the category of A-modules. So replacing $\mathscr{F}_i$ with M_i, and $\mathscr{G}_i$ with N_i, and denoting the maps by the same notation, we want to show that the following sequence is exact:

$$(M_1 \otimes_A N_2) \oplus (M_2 \oplus N_1) \xrightarrow{\quad\beta\quad} M_2 \otimes_A N_2 \xrightarrow{f_2 \otimes g_2} M_3 \otimes_A N_3 \longrightarrow 0$$

The map $f_2 \otimes g_2$ is surjective: if $m_3 \otimes n_3 \in M_3 \otimes_A N_3$, then there exists some $m_2 \in M_2$ and $n_3 \in N_2$ such that $f_2(m_2) = m_3$ and $g_2(n_2) = n_3$, hence

$$f_2 \otimes g_2(m_2 \otimes n_2) = f_2(n_2) \otimes g_2(n_2) = n_3 \otimes m_3$$

Since f_2 and g_2 are surjective, we have that:

$$M_3 \cong M_2/\operatorname{im} f_1 \qquad \text{and} \qquad N_3 \cong N_2/\operatorname{im} g_1$$

hence:

$$M_3 \otimes_A N_3 \cong M_2/\operatorname{im} f_1 \otimes_A N_2/\operatorname{im} g_1 \cong (M_2 \otimes_A N_2)/(\operatorname{im}(f_1 \otimes \mathrm{Id}) + \operatorname{im}(\mathrm{Id} \otimes g_1))$$

The submodule we are quotienting out by is precisely $\operatorname{im}\beta$, hence the sequence is exact by the same argument in Lemma 5.1.5 □

Proposition 5.1.4. *Let $f : X \to Y$ be a morphism of locally ringed spaces, and $\mathscr{F}$ an $\mathscr{O}_Y$ module. The following hold:*

i) *The categories $\mathrm{QCoh}_{\mathscr{O}_X}$, and $\mathrm{Vec}_{\mathscr{O}_X}$ are additive.*

ii) $\mathrm{QCoh}_{\mathscr{O}_X}$ *and* $\mathrm{Vec}_{\mathscr{O}_X}$ *are closed under tensor products.*

iii) *If $\mathscr{F}$ is quasicoherent, then so is $f^*\mathscr{F}$.*

iv) *If $\mathscr{F}$ is locally free of finite rank n then so is $f^*\mathscr{F}$.*

Moreover, and if $f : Y \to X$ is a morphism of locally ringed spaces, pulling back induces functors $f^* : \mathrm{QCoh}_{\mathscr{O}_X} \to \mathrm{QCoh}_{\mathscr{O}_Y}$ *and* $f^* : \mathrm{Vec}_{\mathscr{O}_X} \to \mathrm{Vec}\mathscr{O}_Y$.

Proof. Since each category is a full subcategory, and the 0 object obviously lies in each, we need only show that the direct sums stay in their respective categories.

Suppose that $\mathscr{F}$ and $\mathscr{G}$ are locally free of rank n, then for each $x \in X$ there exists $U, V \subset X$ containing x such that:

$$\mathscr{F}|_U \cong \mathscr{O}_U^n \qquad \text{and} \qquad \mathscr{G}|_V \cong \mathscr{O}_V^m$$

It is then obvious that on $U \cap V$:

$$(\mathscr{F} \oplus \mathscr{G})|_{U\cap V} \cong \mathscr{F}|_{U\cap V} \oplus \mathscr{G}|_{U\cap V} \cong \mathscr{O}_{U\cap V}^n \oplus \mathscr{O}_{U\cap V}^m \cong \mathscr{O}_{U\cap V}^{n+m}$$

so $\mathscr{F} \oplus \mathscr{G}$ is locally free of rank $n + m$.

Supposing that $\mathscr{F}$ and $\mathscr{G}$ are quasicoherent, we can via a similar argument above, find an open set U on which there exists indexing sets I, J, K, and L such that the following sequences are exact:

$$\mathscr{O}_U^I \longrightarrow \mathscr{O}_U^J \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

$$\mathscr{O}_U^K \longrightarrow \mathscr{O}_U^L \longrightarrow \mathscr{G}|_U \longrightarrow 0$$

hence the following sequence is exact:

$$\mathscr{O}_U^{I\cup K} \longrightarrow \mathscr{O}_U^{J\cup L} \longrightarrow (\mathscr{F} \oplus \mathscr{G})|_U \longrightarrow 0$$

implying that both $\mathrm{QCoh}_{\mathscr{O}_X}$ and $\mathrm{Vec}\mathscr{O}_X$ are additive proving i).

Let $\mathscr{F}$ and $\mathscr{G}$ be locally free of rank n and m respectively. Finding an open set U on which both are trivial, and letting $\iota : U \to X$ be the open embedding, we have that by inductively applying part c) of Proposition 5.1.1:

$$\begin{aligned}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})|_U \cong\ & \iota^{-1}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) \\ \cong\ & \iota^{-1}\mathscr{F} \otimes_{\iota^{-1}\mathscr{O}_X} \iota^{-1}\mathscr{G} \\ \cong\ & \mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U \\ \cong\ & \mathscr{O}_U^n \otimes_{\mathscr{O}_U} \mathscr{O}_U^m \\ \cong\ & \mathscr{O}_U^{n+m}\end{aligned}$$

as desired. We note that if $\mathscr{F}$ and $\mathscr{G}$ are not of finite rank, i.e. $\mathscr{F}|_U \cong \mathscr{O}_U^I$ and $\mathscr{G}|_U \cong \mathscr{O}_U^J$ then over U there is an isomorphism:

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})|_U \cong \mathscr{O}_U^{I\times J}$$

Indeed, this is true on the level of stalks, so the induced map will be an isomorphism.

Supposing that $\mathscr{F}$ and $\mathscr{G}$ are quasicoherent, and finding an open set on which we have the exact sequences:

$$\mathscr{O}_U^I \longrightarrow \mathscr{O}_U^J \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

$$\mathscr{O}_U^K \longrightarrow \mathscr{O}_U^L \longrightarrow \mathscr{G}|_U \longrightarrow 0$$

By Lemma 5.1.7, we have the following short exact sequence

$$(\mathscr{O}_U^I \otimes_{\mathscr{O}_U} \mathscr{O}_U^L) \oplus (\mathscr{O}_U^J \otimes_{\mathscr{O}_U} \mathscr{O}_U^K) \longrightarrow \mathscr{O}_U^J \otimes_{\mathscr{O}_U} \mathscr{O}_U^L \longrightarrow (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})|_U \longrightarrow 0$$

hence $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is quasicoherent by the preceding result regarding tensor products of locally free/free sheaves proving ii).

For iii) let $\mathscr{F}$ be an $\mathscr{O}_Y$ module which is locally free. Then if $\mathscr{F}|_U \cong \mathscr{O}_U^n$ we have by part a) and b) of Proposition 5.1.2 :

$$f^*\mathscr{F}|_{f^{-1}(U)} \cong f^*\mathscr{O}_U^n \cong$$

where $\mathscr{O}_{f^{-1}(U)}$ is $\mathscr{O}_X$ restricted to $f^{-1}(U)$.

For iv), if $\mathscr{F}$ is a quasicoherent $\mathscr{O}_Y$ module, we have an exact sequence:

$$\mathscr{O}_U^I \longrightarrow \mathscr{O}_U^J \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

for some open $U \subset Y$. Since f^* is right exact by Proposition 5.1.3, we have by part a) of Proposition 5.1.2 that the following sequence is exact:

$$\mathscr{O}_{f^{-1}(U)}^I \longrightarrow \mathscr{O}_{f^{-1}(U)}^J \longrightarrow f^*\mathscr{F}|_{f^{-1}(U)} \longrightarrow 0$$

implying the claim. □

The goals for the the rest of this section are as follows: we wish to prove that $\mathrm{Coh}_{\mathscr{O}_X}$ is an abelian category, that the tensor products of coherent $\mathscr{O}_X$ modules are coherent, and that the pullback of coherent $\mathscr{O}_Y$ modules is coherent under suitable conditions, namely that both $\mathscr{O}_Y$ and $\mathscr{O}_X$ are coherent over themselves. We begin with proving that $\mathrm{Coh}_{\mathscr{O}_X}$ is an abelian category, an exercise we break into stages. We begin with showing kernels and cokernels are coherent.

Lemma 5.1.8. *Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism of coherent $\mathscr{O}_X$ modules, then* $\ker f$ *and* $\operatorname{coker} f$ *are both coherent.*

Proof. Note that both $\mathscr{F}$ and $\mathscr{G}$ are of finite type; in particular, each x there exists a U such that:

$$\pi : \mathscr{O}_U^n \to \mathscr{F}|_U$$

is surjective. Since $\mathscr{G}$ is coherent, the kernel of the composition

$$f \circ \pi : \mathscr{O}_U^n \to \mathscr{G}|_U$$

is of finite type. We claim that the image of:

$$\pi \circ \iota : \ker(f \circ \pi) \hookrightarrow \mathscr{O}_U^n \to \mathscr{F}|_U$$

is $\ker f$. In particular, we claim that:

$$\ker(f|_U \circ \pi) \to \mathscr{F}|_U \to \mathscr{G}|_U$$

is exact at $\mathscr{F}|_U$. It suffices to prove this on stalks; clearly the composition is zero so that $\operatorname{im} \pi_x \circ \iota_x \subset \ker f_x$. Suppose that $s_x \in \ker f_x$, then by surjectivity there exists a $t_x \in \mathscr{O}_{U,x}^n$ such that $\pi_x(t_x) = s_x$, so $s_x \in \operatorname{im} \pi_x$. In particular, $t_x \in \ker f_x \circ \pi_x$ by definition, hence $s_x \in \operatorname{im} \pi_x \circ \iota_x$. We thus have a surjection:

$$\ker(f|_U \circ \pi) \longrightarrow \ker f|_U$$

and since $\ker(f|_U \circ \pi)$ is of finite type, we have that for all $x \in U$ there is some open neighborhood V of x and a surjection:

$$\mathscr{O}_V^m \to \ker(f \circ \pi)|_V \to \ker f|_V$$

so $\ker f$ is of finite type. Now $\ker f$ is a finite type sub $\mathscr{O}_X$ module of $\mathscr{F}$; let $\{s_1, \dots, s_n\} \in \ker f(U)$, then the induced morphism:

$$\phi : \mathscr{O}_U^m \to \ker f|_U$$

must have kernel of finite type because $\ker f|_U$ injects into $\mathscr{F}|_U$. It follows that $\ker f$ is a coherent $\mathscr{O}_X$ module.

Now consider $\operatorname{coker} f$; since $\mathscr{G}$ surjects onto $\operatorname{coker} f$ we have that $\operatorname{coker} f$ must be of finite type. Let $\{s_1, \dots, s_n\} \subset (\operatorname{coker} f)(U)$, and:

$$\phi : \mathscr{O}_U^n \to \operatorname{coker} f|_U$$

the induced morphism. Let $x \in U$, and consider $s_{1,x}, \dots, s_{n,x} \in \operatorname{coker} f_x$; since $\operatorname{coker} f_x \cong \mathscr{G}_x / \operatorname{im} f_x$, we have lifts $t_{1,x}, \dots, t_{n,x} \in \mathscr{G}_x$. By taking $2n$ intersections we obtain an open neighborhood of x, V, with sections $s_1',\dots,s_n' \in \operatorname{coker} f(V)$ and lifts $t_1,\dots,t_n \in \mathscr{G}(V)$ such that $\pi(s_i') = t_i$. By restricting to a smaller open set if necessary, we may assume that there is a surjection $\xi : \mathscr{O}_V^m \to \operatorname{im} f|_V$. We have that $t_1,\dots,t_n$, and ξ determine a surjection:

$$\beta : \mathscr{O}_V^n \oplus \mathscr{O}_V^m \longrightarrow \mathscr{G}|_V$$

We thus can construct the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_V^m \arrow[r, "\iota_m"] \arrow[d, "\xi"] & \mathscr{O}_V^m \oplus \mathscr{O}_V^n \arrow[r, "\pi_n"] \arrow[d, "\beta"] & \mathscr{O}_V^n \arrow[r] \arrow[d, "\phi|_vanishes"] & 0 \\
0 \arrow[r] & \operatorname{im} f|_V \arrow[r, "\iota"] & \mathscr{G}|_V \arrow[r, "\pi"] & \operatorname{coker} f|_V \arrow[r] & 0
\end{tikzcd}$$

The snake lemma, which applies in any abelian category, implies and exact sequence of the form:

$$0 \longrightarrow \ker \xi \longrightarrow \ker \beta \longrightarrow \ker \phi|_V \longrightarrow \operatorname{coker} \xi \longrightarrow \cdots$$

However, ξ is a surjection, hence we have that $\ker \beta$ surjects onto $\ker \phi|_V$ as $\operatorname{coker} \xi = 0$ by Proposition 1.2.8. It follows that since $\ker \beta$ is of finite type as $\mathscr{G}$ is coherent, that $\ker \phi|_V$ must be of finite type as well, implying the claim. □

We have the following corollary:

Corollary 5.1.3. *Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism between sheaves of $\mathscr{O}_X$ modules, where $\mathscr{F}$ is of finite type, and $\mathscr{G}$ is coherent. Then $\ker f$ is of finite type.*

Proof. This follows by noticing that the part of the proof in Lemma 5.1.8 showing that $\ker f$ was of finite type, depended only on $\mathscr{F}$ be of finite type. □

The task of showing that $\mathrm{Coh}_{\mathscr{O}_X}$ has direct sums is surprisingly delicate as far as we can tell. In fact, it seems that the best path towards a proof of this is via the following lemma:

Lemma 5.1.9. *Let:*

$$0 \longrightarrow \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \longrightarrow 0$$

be a short exact sequence of $\mathscr{O}_X$ modules. If any two of the three are coherent, then so is the third.

Proof. Note that if $\mathscr{G}$ and $\mathscr{H}$ are coherent, then $\mathscr{F}$ is the kernel of f and thus coherent by Lemma 5.1.8. If $\mathscr{F}$ and $\mathscr{G}$ are coherent, then $\mathscr{H}$ is the cokernel of the morphism $\mathscr{F} \to \mathscr{G}$, and thus coherent by Lemma 5.1.8.

Now suppose that $\mathscr{F}$ and $\mathscr{H}$ are coherent. We first show that $\mathscr{G}$ is finite type; since $\mathscr{F}$ and $\mathscr{H}$ are finite type, we can find a common open set U such that $\mathscr{O}_U^n$ and $\mathscr{O}_U^m$ surject onto $\mathscr{F}|_U$ and $\mathscr{H}|_U$ respectively. Taking U to be small enough, the same argument in Lemma 5.1.8 demonstrates that we can take lifts of the sections which define the map $\mathscr{O}_U^m$. It follows that we obtain a morphism $\mathscr{O}_u^n \oplus \mathscr{O}_U^m \to \mathscr{F}|_U$ which manifestly makes the following diagrams commute:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_U^n \arrow[r] \arrow[d] & \mathscr{O}_U^n \oplus \mathscr{O}_U^m \arrow[r] \arrow[d] & \mathscr{O}_U^m \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}|_U \arrow[r] & \mathscr{G}|_U \arrow[r] & \mathscr{H}|_U \arrow[r] & 0
\end{tikzcd}$$

It suffices to check surjectivity of the middle morphisms on stalks, however this then follows from the surjectivity part of the five lemma, implying that $\mathscr{G}$ is of finite type.

Now let $\{s_1,\dots,s_n\} \subset \mathscr{G}(U)$ define the morphism:

$$\phi : \mathscr{O}_U^n \to \mathscr{G}|_U$$

Then $\ker g|_U \circ \phi$ is of finite type as $\mathscr{H}|_U$ is coherent. We thus have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathscr{O}_U^n \arrow[r, "\mathrm{Id}"] \arrow[d, "\phi"] & \mathscr{O}_U^n \arrow[r] \arrow[d, "g|_U \circ \phi"] & 0 \\
0 \arrow[r] & \mathscr{F}|_U \arrow[r, "f|_U"] & \mathscr{G}|_U \arrow[r, "g|_U \circ \phi"] & \mathscr{H}_U \arrow[r] & 0
\end{tikzcd}$$

and so the snake lemma once again implies an exact sequence of the form:

$$0 \longrightarrow \ker \phi \longrightarrow \ker g|_U \circ \phi \longrightarrow \operatorname{coker} 0 \longrightarrow \cdots$$

However, $\operatorname{coker} 0 = \mathscr{F}$, hence $\ker \phi$ is the kernel of the morphism $\ker g|_U \circ \phi \to \mathscr{F}|_U$. The claim now follows from Corollary 5.1.3 $\square$

We now prove the main result of the section:

Theorem 5.1.1. *Let $(X, \mathscr{O}_X)$ be a ringed space, then $\mathrm{Coh}_{\mathscr{O}_X}$ is an abelian category.*

Proof. First note that $\mathrm{Coh}_{\mathscr{O}_X}$ is additive; indeed Lemma 5.1.9 implies that if $\mathscr{F}$ and $\mathscr{G}$ are coherent, then $\mathscr{F} \oplus \mathscr{G}$ are coherent, because we have the following exact sequence:

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} \oplus \mathscr{G} \longrightarrow \mathscr{G} \longrightarrow 0$$

Moreover, Lemma 5.1.8 implies that kernels and cokernels of coherent modules are coherent.

We need to show that monomorphisms are kernels, and epimorphisms are cokernels. However, Theorem 1.2.1, shows that if $f : \mathscr{F} \to \mathscr{G}$ is a monomorphism between coherent $\mathscr{O}_X$ modules, then $(\mathscr{F}, f)$ is the kernel of:

$$\pi : \mathscr{G} \to \operatorname{coker} f$$

Since $\mathscr{G}$ is coherent, and $\operatorname{coker} f$ is coherent by Lemma 5.1.8, we have that f is the kernel of a morphism between coherent $\mathscr{O}_X$ modules, as desired. Similarly, if $f : \mathscr{F} \to \mathscr{G}$ is an epimorphism, then $(\mathscr{G}, f)$ is the cokernel of $\iota : \ker f \to \mathscr{F}$, which is a morphism of coherent $\mathscr{O}_X$ module by Lemma 5.1.8. It follows that epimorphisms are cokernels, and so $\mathrm{Coh}_{\mathscr{O}_X}$ is an abelian category. $\square$

We end this section with the following result:

Proposition 5.1.5. *Let $f : X \to Y$ be a morphism of locally ringed spaces. The following hold:*

i) $\mathrm{Coh}_{\mathscr{O}_X}$ *is closed under tensor products.*

ii) If $\mathscr{O}_X$ and $\mathscr{O}_Y$ are coherent modules, then f^* is a functor $\mathrm{Coh}_{\mathscr{O}_Y} \to \mathrm{Coh}_{\mathscr{O}_X}$.

Proof. Note that clearly finitely presented $\mathscr{O}_X$ modules are closed under tensor products by part *ii*) of Proposition 5.1.4. Let $\mathscr{F}$ be of finite presentation, and $\mathscr{G}$ be coherent, then by right exactness of the tensor product, for some U we have:

$$\mathscr{O}_U^n \otimes_{\mathscr{O}_U} \mathscr{G}|_U \longrightarrow \mathscr{O}_U^m \otimes_{\mathscr{O}_U} \mathscr{G}|_U \longrightarrow \mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U \longrightarrow 0$$

By parts *a*) and *c*) of Proposition 5.1.1, this can be rewritten as:

$$\mathscr{G}^n|_U \longrightarrow \mathscr{G}^m|_U \longrightarrow \mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U \longrightarrow 0$$

By Theorem 5.1.1, $\mathrm{Coh}_{\mathscr{O}_X}$ forms an abelian category, hence the first two terms are coherent $\mathscr{O}_X$ modules. It follows that $\mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U$ is a cokernel of a morphism between coherent sheaves and is thus coherent. Since all such U cover X, we have that $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is coherent.

By Lemma 5.1.6, since $\mathscr{O}_Y$ is coherent, we have that $\mathscr{F}$ being a coherent $\mathscr{O}_Y$ module is equivalent to $\mathscr{F}$ being of finite presentation. It follows by part *iv*) of Proposition 5.1.4 that $f^*\mathscr{F}$ is locally of finite presentation as well. Since $\mathscr{O}_X$ is coherent, the same lemma proves that $f^*\mathscr{F}$ is coherent, implying the claim. $\square$

5.2 Tensor-Hom Adjunction for $\mathscr{O}_X$ Modules

In this section we continue to assume that $(X, \mathscr{O}_X)$ is an arbitrary ringed space, and develop the an analogue of the tensor hom adjunction in the category of $\mathscr{O}_X$ modules. This will allow us to easily prove that f^* is the left adjoint of the direct image functor. We begin with a review of the statement and proof in the category of A-modules.

Theorem 5.2.1. *Let M and N be A modules, and P and N be B modules. There is a natural isomorphism of abelian groups:*

$$\mathrm{Hom}_B(M \otimes_A N, P) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P))$$

Before proving this statement recall that the B module structure on $M \otimes_A N$ is given by:

$$b \cdot (m \otimes n) = m \otimes (bn)$$

and that the A module structure on $\mathrm{Hom}_B(N, P)$ is given by:

$$\begin{aligned}(a \cdot \phi) : N &\longrightarrow P \\ n &\longmapsto \phi(a \cdot n)\end{aligned}$$

We now begin the proof:

Proof. We first construct a map:

$$\Psi : \mathrm{Hom}_B(M \otimes_A N, P) \longrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P))$$

Let $f \in \mathrm{Hom}_B(M \otimes_A N, P)$, and let $\otimes : M \oplus N \to M \otimes_A N$ be the tensor map. Let $\tilde{f} = f \circ \otimes$, then we claim that $\tilde{f}$ B linear in the second component. It is clear that the additivity condition holds; let $m \in M$, $n \in N$, and $b \in B$, then we have the following:

$$\tilde{f}(m, bn) = f(m \otimes bn) = f(b \cdot (m \otimes n)) = b \cdot f(m \otimes n)$$

as desired. For each m, we thus get a map $m \lrcorner \tilde{f}$ defined by:

$$(m \lrcorner \tilde{f})(n) = \tilde{f}(m, n)$$

which is B-linear. We want to see that the assignment $m \mapsto m \lrcorner \tilde{f}$ is A linear; let $m_1, m_2 \in M$ and $a_1, a_2 \in A$, then for all n in N:

$$\begin{aligned}(a_1 m_1 + a_2 m_2) \lrcorner \tilde{f}(n) =& \tilde{f}(a_1 m_1 + a_2 m_2, n) \\ =& f(a_1 m_1 \otimes n + a_2 m_2 \otimes n) \\ =& f(a_1 m_1 \otimes n) + f(a_2 m_2 \otimes n) \\ =& f(m_1 \otimes a_1 n) + f(m_2 \otimes a_2 n) \\ =& m_1 \lrcorner \tilde{f}(a_1 n) + m_2 \lrcorner \tilde{f}(a_2 n) \\ =& a_1 \cdot (m_1 \lrcorner \tilde{f})(n) + a_2 \cdot (m_2 \lrcorner \tilde{f})(n)\end{aligned}$$

It follows that we have obtained a map:

$$\begin{aligned}\Psi : \mathrm{Hom}_B(M \otimes_A N, P) &\longrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P)) \\ f &\longrightarrow (m \mapsto m \lrcorner \tilde{f})\end{aligned}$$

This is clearly a morphism of abelian groups, and is functorial/natural in N, and so the morphism is natural.

Suppose that $\Psi(f) = 0$, then for all m we have that $m \lrcorner \tilde{f} = 0$. In particular, for all simple tensors $m \otimes n$, we would have that:

$$f(m \otimes n) = (m \lrcorner \tilde{f})(n) = 0$$

Since f is a group homomorphism, it follows that f is identically zero on $M \otimes_A N$ and is thus the zero morphism. This shows that Ψ is injective.

Now let $\phi \in \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P))$; then we obtain a map:

$$\begin{aligned} g : M \oplus N &\longrightarrow P \\ (m,n) &\longmapsto (\phi(m))(n) \end{aligned}$$

Note that g satisfies the following:

$$g(am,n) = (\phi(am))(n) = (a \cdot \phi(m))(n) = \phi(m)(an) = g(m,na)$$

and is additive in both entries. By the construction of the tensor product[81], these are the minimal requirements to get a well defined group homomorphism:

$$f : M \otimes_A N \to P$$

which satisfies $f \circ \otimes = g$, and is obviously B-linear. Clearly, the assignment $m \mapsto m \lrcorner \tilde{f}$ is then equal to the map ϕ. It follows that Ψ is an isomorphism implying the claim. □

The first stumbling block in extending the above result to the category $\mathscr{O}_X$ modules, is that $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is not a sheaf, so an expression of the form:

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathrm{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H}))$$

makes no sense. We fix this with the following definition:

Definition 5.2.1. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of $\mathscr{O}_X$ modules, then the **Hom sheaf**[82], denoted $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$, is the sheaf defined on opens by:[83]

$$\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})(U) = \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U)$$

Note that since $\mathscr{O}_U$ modules form an abelian category, we have that this is a priori a presheaf of abelian groups.

Lemma 5.2.1. *Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$ modules, then $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is a sheaf of $\mathscr{O}_X$ modules.*

Proof. We first show that $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is a sheaf; the restriction maps are the obvious ones sending a natural transformation to the restricted natural transformation. These obviously satisfy the presheaf conditions. Let $F \in \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U)$, and $\{U_i\}$ be a cover for U such that $F|_{U_i}$ is the zero morphism. In particular, this implies that that the stalk map F_x is zero for all $x \in U$, hence F is the zero morphism.

Now suppose that $F_i \in \mathrm{Hom}_{\mathscr{O}_{U_i}}(\mathscr{F}|_{U_i}, \mathscr{G}|_{U_i})$ so that $F_i|_{U_i \cap U_j} = F_j|_{U_i \cap U_j}$, then Proposition 1.2.11 implies that the F_i glue[84] together to yield a unique morphism $\mathscr{F}|_U \to \mathscr{G}|_U$ which restricts to F_i on U_i. It follows that $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is a sheaf.

We define a sheaf morphism

$$\mathscr{O}_X \times \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$$

as follows: let $U \subset X$ be arbitrary, then $(s, F) \in \mathscr{O}_X(U) \times \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U)$ is sent to the sheaf morphism $s \cdot F$, defined on opens $V \subset U$ by:

$$\begin{aligned} (s \cdot F)_V : \mathscr{F}(V) &\longrightarrow \mathscr{G}(V) \\ t &\longmapsto s|_V \cdot F_V(t) \end{aligned}$$

This clearly commutes with restrictions and so defines an element in $\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U)$. The assignment $(s, F) \mapsto s \cdot F$ also clearly commutes with restrictions □

If $\mathscr{F}$ is an $\mathscr{O}_X$ module, then we denote by $\mathscr{F}^*$ the dual sheaf $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$. One may hope that taking stalks commutes with the $\underline{Hom}$, i.e. that some thing of the form:

$$\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \cong \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$

however this is rarely the case:

[81] See for example, Atiyah Macdonald Chapter 2, Proposition 2.12.

[82] In any abelian category, a functor of this form is called an internal hom functor, as it is an analogue of the true Hom functor, but has value in the abelian category, rather than the category of abelian groups.

[83] This can obviously defined similarly for general sheaves, or sheaves of abelian groups, etc.

[84] The morphisms gluing the $\mathscr{F}|_{U_i}$ together are just the identity morphisms, and similarly for the $\mathscr{G}|_{U_i}$.

Example 5.2.1. Let X be irreducible, and $\mathscr{O}_X$ be the constant sheaf with values in $\mathbb{Z}$ on X. Note since no finite intersection of open sets can be empty, this is the honest to god constant presheaf; clearly every sheaf of abelian groups on $\mathscr{O}_X$ is now canonically an $\mathscr{O}_X$ module. Let $\mathscr{F}$ be the sky scraper sheaf of $\mathbb{Z}$ at x, see Lemma 1.2.7. Supposing x is a closed point, then we claim that:

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{O}_U) = 0$$

for all $U \subset X$. If $x \notin U$ then $\mathscr{F}|_U = 0$ hence the claim; if $x \in U$, then $\mathscr{F}|_U$ is nonzero, however if $F \in \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{O}_U) = 0$, and $s \in \mathscr{F}|_U(V)$, we claim that:

$$F_V(s) = 0$$

for all $V \subset U$, and $s \in \mathscr{F}(V)$. Indeed, the restriction maps $\theta^V_W : \mathscr{O}_U(V) \to \mathscr{O}_U(W)$ are the identity, so let $W = V \smallsetminus \{x\}$, then $s|_W = 0$, hence we have that:

$$0 = F|_W(s|_W) = \theta^V_W \circ F_V(s)$$

so by injectivity, we have that $F_V(s) = 0$. It follows that F is identically zero on all $V \subset U$, hence F is the zero morphism. We have thus shown that:

$$\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)_x = 0 \ncong \mathbb{Z} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$

We will eventually show that the remedy for this is when $\mathscr{F}$ is finitely presented, which will imply that:

$$f^*\underline{Hom}_{\mathscr{O}_Y} \cong \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$$

Jus as in the category of A-modules, we have that $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, -)$ and $\underline{Hom}_{\mathscr{O}_X}(-, \mathscr{F})$ are functors. Indeed, we know where each should send objects, so let $F : \mathscr{G} \to \mathscr{H}$, then we have a morphism:

$$F^* : \underline{Hom}_{\mathscr{O}_X}(\mathscr{H}, \mathscr{F}) \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{G}, \mathscr{F})$$

given on opens by:

$$\begin{aligned} F^*_U : \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{H}|_U, \mathscr{F}|_U) &\longrightarrow \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{G}|_U, \mathscr{F}|_U) \\ G &\longmapsto G \circ F|_U \end{aligned}$$

which obviously commute with restriction maps. One easily checks that that $(F \circ G)^* = G^* \circ F^*$, and so $\underline{Hom}_{\mathscr{O}_X}(-, \mathscr{F})$ is a contravariant functor. Similarly, we have $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, -)$ is a covariant functor, sending F to:

$$F_* : \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{H})$$

given on opens by:

$$\begin{aligned} (F_*)_U : \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U) &\longrightarrow \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{H}|_U) \\ G &\longmapsto F|_U \circ G \end{aligned}$$

Our goal is to show that these functors are exact, just as in the case of A modules.

Proposition 5.2.1. *Let $\mathscr{F}$ an $\mathscr{O}_X$ module, then the functors $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, -)$ and $\underline{Hom}_{\mathscr{O}_X}(-, \mathscr{F})$ are left exact.*[85]

Proof. Let:

$$0 \longrightarrow \mathscr{G}_1 \xrightarrow{f_1} \mathscr{G}_2 \xrightarrow{f_2} \mathscr{G}_3$$

be an exact sequence of $\mathscr{O}_X$ modules. We first show that:

$$0 \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}_1) \xrightarrow{f_{1*}} \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}_2) \xrightarrow{f_{2*}} \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}_3)$$

[85] A contravariant functor is left (right) exact if it takes right (left) exact sequences to left (right) exact sequences.

is exact. It suffices to show that this exact on every open set,[86] i.e. that the following sequence of abelian groups is exact for every U:

$$0 \longrightarrow \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}_1|_U) \xrightarrow{(f_{1*})_U} \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}_2|_U) \xrightarrow{(f_{2*})_U} \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}_3|_U)$$

Suppose that $F \in \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}_1|_U)$ satisfies:

$$(f_{1*})_U(F) = f_1|_U \circ F = 0$$

In particular since $\ker f_1|_U = 0$, we must have that $\ker F = \mathscr{F}_U$, so F is the zero morphism, implying $(f_{1*})_U$ is injective. Now clearly if $F \in \operatorname{im}(f_{1*})_U$ then $(f_{2*})_U(F) = 0$; suppose that $F \in \ker(f_{2*})_U$, then we have that:

$$f_2|_U \circ F = 0$$

then we want to show that $F = f_1|_U \circ G$ for some $G \in \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}_2|_U)$. Note that since $\ker f_1|_U = 0$, we have that $\mathscr{G}_1|_U$ is canonically $\operatorname{im} f_1|_U = \ker f_2|_U$, i.e. $(\mathscr{G}_1|_U, f_1|_U)$ satisfies the universal property of the kernel. By the aforementioned, we have that there exists a unique map G such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}|_U \arrow[rr, "F"] \arrow[rrrr, bend left, "0"] \arrow[dr, "\exists ! G"'] & & \mathscr{G}_2|_U \arrow[rr, "f_2|_U"] & & G_3|_U \\
& \mathscr{G}_1|_U \arrow[ur, "f_1|_U"'] & & &
\end{tikzcd}$$

implying the claim.

Now let:

$$\mathscr{G}_1 \xrightarrow{f_1} \mathscr{G}_2 \xrightarrow{f_2} \mathscr{G}_3 \longrightarrow 0$$

be an exact sequence of $\mathscr{O}_X$ modules. By the same argument to show that:

$$0 \longrightarrow \underline{\mathcal{H}om}_{\mathscr{O}_X}(\mathscr{G}_3, \mathscr{F}) \xrightarrow{f_2^*} \underline{\mathcal{H}om}_{\mathscr{O}_X}(\mathscr{G}_2, \mathscr{F}) \xrightarrow{f_1^*} \underline{\mathcal{H}om}_{\mathscr{O}_X}(\mathscr{G}_1, \mathscr{F})$$

is exact, it suffices to show that we have an exact sequence of abelian groups:

$$0 \longrightarrow \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{G}_3|_U, \mathscr{F}|_U) \xrightarrow{(f_2^*)_U} \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{G}_2|_U, \mathscr{F}|_U) \xrightarrow{(f_1^*)_U} \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{G}_1, |_U, \mathscr{F}|_U)$$

Let $F \in \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{G}_3|_U, \mathscr{F}|_U)$ be such that $F \circ f_2|_U = 0$. It follows that $\ker F = \mathscr{G}_3|_U$ as $\operatorname{im} f_2|_U = \mathscr{G}_3$, so F is the zero map, hence $F = 0$, and $(f_2^*)_U$ is injective.

Now let $F \in \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{G}_2|_U, \mathscr{F}|_U)$, clearly if $F = G \circ f_2|_U$ then we have that $(f_1^*)_U(F) = 0$. Now suppose that F satisfies:

$$F \circ f_1|_U = 0$$

Note that since f_2 is surjective, we have that $(\mathscr{G}_3|_U, f_2|_U)$ is canonically $\operatorname{coker} f_1$, hence by the universal property of the cokernel, there exists a unique G such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{G}_1|_U \arrow[r, "f_1|_U"] & \mathscr{G}_2|_U \arrow[rr, "F"] \arrow[dr, "f_2|_U"'] & & \mathscr{F}|_U \\
& & \mathscr{G}_3|_U \arrow[ur, "G"'] &
\end{tikzcd}$$

Therefore $F = G \circ f_2|_U$ for a unique $G \in \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{G}_3, \mathscr{F}_U)$, hence we have proven exactness of the sequence. □

Using exactness, we will be able to show that the desired property holds on stalks in nice enough situations. First of all note that we have maps:

$$\operatorname{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}_U) \longmapsto \operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$
$$F \longmapsto F_x$$

[86] Note that exact sequences of sheaves don't *need* to be exact on open sets, but if they are exact on open sets then they are exact.

which obviously commute with restrictions, hence there is a unique morphism Ψ_x making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U) \arrow[rr] \arrow[dr] \arrow[ddr, bend right] & & \mathrm{Hom}_{\mathscr{O}_V}(\mathscr{F}|_V, \mathscr{G}|_V) \arrow[dl] \arrow[ddl, bend left] \\
& \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \arrow[d, "\Psi_x"] & \\
& \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x) &
\end{tikzcd}$$

We need the following lemma, which is an analogue of the result that $\mathrm{Hom}_A(A^I, M) \cong M^I$.

Lemma 5.2.2. *Let $\mathscr{F}$ be an $\mathscr{O}_X$ module, then:*

$$\underline{Hom}_{\mathscr{O}_X}(\mathscr{O}_X^n, \mathscr{F}) \cong \mathscr{F}^n$$

as $\mathscr{O}_X$ modules.

Proof. This is essentially obvious, and it suffices to prove that there is a natural[87] isomorphism:

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U^n, \mathscr{F}|_U) \cong \mathscr{F}(U)^n$$

for all U. First note by the universal property of the coproduct, we have that naturally:

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U^n, \mathscr{F}|_U) \cong \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_Y, \mathscr{F}|_U)^n$$

hence it suffices to show that:

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U, \mathscr{F}|_U) \cong \mathscr{F}(U)$$

Now define a morphism:

$$\begin{aligned} \Phi_U : \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U, \mathscr{F}|_U) &\longrightarrow \mathscr{F}(U) \\ F &\longmapsto F_U(1) \end{aligned}$$

where $1 \in \mathscr{O}_U(U)$ is the 'global' unit section. This is injective as if $F_U(1) = 0$, then for all $V \subset U$ and $s \in \mathscr{O}_U(V)$ we have, :

$$F_V(s) = s \cdot F_V(1) = s \cdot F_V(\theta_V^U(1)) = s \cdot F_U(1) = 0$$

implying that F is the zero morphism. This is surjective because if $a \in \mathscr{F}(U)$ then the map defined for all $V \subset U$:

$$\begin{aligned} F_V : \mathscr{O}_U(V) &\longrightarrow \mathscr{F}(V) \\ s &\longmapsto s \cdot a|_V \end{aligned}$$

defines a morphism of $\mathscr{O}_X$ modules. In particular $F_U(1) = a$, hence Φ_U is an isomorphism for all U, and clearly commutes with restrictions, implying the claim. ☐

We now have the following:

Proposition 5.2.2. *Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$ modules, and $x \in X$. If $\mathscr{F}$ is of finite type, then for all $\mathscr{O}_X$ modules $\mathscr{G}$, Ψ_x is injective. If $\mathscr{F}$ is in addition finitely presented, then for all $\mathscr{G}$ Ψ_x is an isomorphism.*

Proof. Let $[U, F] \in \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x$ and suppose that $\Psi_x([U, F]) = F_x = 0$. By shrinking U if we need to, there exist sections $s_1, \ldots, s_n$ of $\mathscr{F}(U)$ such that we have a surjection:

$$\mathscr{O}_U^n \longrightarrow \mathscr{F}|_U$$

[87] I.e. will commute with restriction maps.

In particular, the $s_{i,y}$ generate $\mathscr{F}_y$ as an $\mathscr{O}_{X,y}$ module for all $y \in U$. Since $F_x = 0$, we have that $F_x(s_{i,x}) = 0$ for all $i = 1, \ldots, n$. By taking n intersections, we can find an open neighborhood V of x such that $F_V(s_i|_V) = 0$ for all i. In particular, we have that for all $y \in V$ the $F_y = 0$ is the zero morphism, so $F|_V = 0$ as well. It follows that $[U, F] = [V, F|_V] = 0$, so Ψ_x is injective.

Now suppose that $\mathscr{F}$ is finitely presented. For every $x \in U$, we have an exact sequence:

$$\mathscr{O}_U^n \longrightarrow \mathscr{O}_U^m \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

Since taking stalks is exact, we have that:

$$\mathscr{O}_{U,x}^n \longrightarrow \mathscr{O}_{U,x}^m \longrightarrow \mathscr{F}_x \longrightarrow 0$$

is exact. Now taking $\mathrm{Hom}_{\mathscr{O}_{X,x}}(-, \mathscr{G}_x)$ we obtain an exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x) \longrightarrow \mathscr{G}_x^m \longrightarrow \mathscr{G}_x^n$$

Now applying $\underline{Hom}_{\mathscr{O}_X}(-, \mathscr{G})$ to the initial exact sequence yields:

$$0 \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{G}^m \longrightarrow \mathscr{G}^n$$

by Lemma 5.2.2. Taking stalks we get the following exact sequence:

$$0 \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \longrightarrow \mathscr{G}_x^m \longrightarrow \mathscr{G}_x^n$$

The map $\mathscr{G}_x^m \to \mathscr{G}_x^n$ is, up to isomorphism, the same in both instances; it follows that $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x$ and $\mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$ are both the kernel of the same map and thus isomorphic. □

Now that we have successfully found conditions in which the stalks of $\underline{Hom}_{\mathscr{O}_X}$ behave as desired we are ready to move on to the main and final goal of this chapter: proving a tensor hom adjunction for sheaves.

Theorem 5.2.2. *Let $\mathscr{O}_X$ and $\mathscr{O}'_X$ be sheaves of commutative rings on X. Let $\mathscr{F}, \mathscr{G} \in \mathrm{Mod}_{\mathscr{O}_X}$, and $\mathscr{G}, \mathscr{H} \in \mathrm{Mod}_{\mathscr{O}'_X}$. Then there is a naturals isomorphism of sheaves of abelian groups:*

$$\underline{Hom}_{\mathscr{O}_{X'}}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}, \mathscr{H}) \cong \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H}))$$

Before we begin with the proof of the above statement, we briefly describe how $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is an $\mathscr{O}'_X$ module, and how $\underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H})$ is an $\mathscr{O}_X$ module. On the level of presheaves, we have a canonical morphism:

$$\mathscr{O}'_X \times (\mathscr{F} \otimes^p_{\mathscr{O}_X} \mathscr{G}) \longrightarrow \mathscr{F} \otimes^p_{\mathscr{O}_X} \mathscr{G}$$

given on opens by:

$$\begin{aligned} \mathscr{O}'_X(U) \times (\mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)) &\longrightarrow \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U) \\ (s, f \otimes g) &\longmapsto f \otimes (s \cdot g) \end{aligned}$$

This obviously makes $\mathscr{F} \otimes^p_{\mathscr{O}_X} \mathscr{G}$ a presheaf of $\mathscr{O}'_X$ modules, so by sheafifying and taking the induced morphism, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ an $\mathscr{O}'_X$ module. To show that $\underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H})$ is an $\mathscr{O}_X$ module, for each $s \in \mathscr{O}_X(U)$, we first define the morphism of $\mathscr{O}_U$ modules $\phi_s : \mathscr{G}|_U \to \mathscr{G}|_U$ given on opens by:

$$\begin{aligned} \mathscr{G}|_U(V) &\longrightarrow \mathscr{G}|_U(V) \\ f &\longmapsto s|_V \cdot f \end{aligned}$$

We thus define a morphism:

$$\mathscr{O}_X \times \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H}) \longrightarrow \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H})$$

on open sets by:

$$\begin{aligned} \mathscr{O}_X(U) \times \mathrm{Hom}_{\mathscr{O}'_U}(\mathscr{G}|_U, \mathscr{H}|_U) &\longrightarrow \mathrm{Hom}_{\mathscr{O}'_U}(\mathscr{G}|_U, \mathscr{H}|_U) \\ (s, F) &\longmapsto F \circ \phi_s \end{aligned}$$

One easily checks that this assignment makes $\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{G}|_U, \mathscr{H}|_U)$ an $\mathscr{O}_X(U)$ module, and that these maps commute with restrictions, giving $\underline{Hom}_{\mathscr{O}'_X}(\mathscr{F}, \mathscr{G})$ the structure of an $\mathscr{O}'_X$ module. We now proceed with the proof, it will be very similar to Theorem 5.1.1:

Proof. We first wish to define a morphism:

$$\Psi : \underline{Hom}_{\mathscr{O}'_X}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}, \mathscr{H}) \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H}))$$

On open sets, this should be a morphism of abelian groups:

$$\Psi_U : \mathrm{Hom}_{\mathscr{O}'_U}(\mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U, \mathscr{H}|_U) \longrightarrow \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H})|_U)$$

Given a morphism of $\mathscr{O}'_X$ modules $f : \mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U \to \mathscr{H}|_U$, we obtain the following morphism of abelian groups:

$$\tilde{f} : f \circ \otimes : \mathscr{F}|_U \oplus \mathscr{G}|_U \to \mathscr{H}|_U$$

Using the above, we need to define a morphism $\mathscr{F}|_U \to \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H})|_U$. Let $s \in \mathscr{F}(V)$, then we define a morphism $s \lrcorner \tilde{f}|_V : \mathscr{G}|_V \to \mathscr{H}|_V$ on opens by:

$$\begin{aligned} \mathscr{G}(W) &\longrightarrow \mathscr{H}(W) \\ t &\longmapsto \tilde{f}(s|_W, t) \end{aligned}$$

which is automatically a morphism of $\mathscr{O}'_V$ modules. The assignment $\mathscr{F}(V) \longrightarrow \mathrm{Hom}_{\mathscr{O}'_V}(\mathscr{G}|_V, \mathscr{H}|_V)$ is also clearly a morphism of $\mathscr{O}_U(V)$ modules, and so defines a morphism of $\mathscr{O}_X$ modules $\mathscr{F}_U \to \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H})|_U$ which we denote by $(-)\lrcorner\tilde{f}$. Hence Ψ_U is given on opens by:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{O}'_U}(\mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U, \mathscr{H}|_U) &\longrightarrow \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H})|_U) \\ f &\longmapsto (-)\lrcorner\tilde{f} \end{aligned}$$

Suppose that $(-)\lrcorner\tilde{f} = 0$, then for all $V \subset U$ and $s \in \mathscr{F}(V)$ we have that $s\lrcorner\tilde{f}|_V : \mathscr{G}|_V \to \mathscr{H}|_V$ is the zero morphism. On global sections, this means that for all $s \in \mathscr{F}(V)$ and all $t \in \mathscr{G}(V)$, $\tilde{f}(s,t) = f \circ \otimes (s,t) = 0$. However this implies that the stalk map:

$$f_x : \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x \longrightarrow \mathscr{H}_x$$

is zero on simple tensors, hence f_x is zero. It follows that f is identically zero and Ψ_U is injective.

Now let $g \in \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H})|_U)$; we define a morphism:

$$\mathscr{F}|_U \oplus \mathscr{G}|_U \longrightarrow \mathscr{H}|_U$$

on open sets by:

$$\begin{aligned} \mathscr{F}(V) \oplus \mathscr{G}(V) &\longrightarrow \mathscr{H}(V) \\ (s,t) &\longmapsto (g_V(s))_V(t) \end{aligned}$$

Note that $g_V : \mathscr{F}(V) \longrightarrow \mathrm{Hom}_{\mathscr{O}'_V}(\mathscr{G}|_V, \mathscr{H}|_V)$, so $(g_V(s))_V : \mathscr{G}(V) \longrightarrow \mathscr{H}(V)$. As in Theorem 5.1.1, this morphism satisfies the minimal properties to factor through the tensor product over $\mathscr{O}_X$, namely being additive in both entries, and respecting the $\mathscr{O}_X$ module structure on $\mathscr{F}$ and $\mathscr{G}$. It follows that we get an induced morphism:

$$f : \mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U \longrightarrow \mathscr{H}|_U$$

After unraveling our definition of Ψ,, one easily checks that $\Psi_U(f)$ is equal to g, so Ψ_U is surjective, implying the claim. □

Notice now that by the above, Lemma 5.2.2, and Theorem 1.3.1 we easily have that:

$$\begin{aligned} \underline{Hom}_{\mathscr{O}_X}(f^*\mathscr{F}, \mathscr{G}) =& \underline{Hom}_{\mathscr{O}_X}(f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X, \mathscr{G}) \\ \cong& \underline{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{F}, \underline{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{G})) \\ \cong& \underline{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{F}, \mathscr{G}) \\ \cong& \underline{Hom}_{\mathscr{O}_Y}(\mathscr{F}, f_*\mathscr{G}) \end{aligned}$$

taking global sections we obtain:

Theorem 5.2.3. *Let $\mathscr{F}$ be an $\mathscr{O}_Y$ module, $\mathscr{G}$ an $\mathscr{O}_X$ module, and $f : X \to Y$ a morphism of ringed spaces. There is then a natural isomorphism:*

$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{F}, f_*\mathscr{G})$$

In other words f^* is the left adjoint of f_*.

5.3 Some Commutative Algebra: Localization of Modules

In Section 1.1 we laid the ground work in commutative algebra, namely the localization of a ring, to construct the structure sheaf of an affine scheme in Section 1.4. In this section, we do something remarkably similar for modules over a fixed ring A, so that in the next section we can easily construction modules over affine schemes. In particular, our goal is to develop a theory of localization for modules, and explore their properties. Most of this section comes from Atiyah-Macdonald.

Lemma 5.3.1. *Let $S \subset A$ be a multiplicatively closed subset. There exists an exact covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$ which we call the localization of a module.*

Proof. We impose an equivalence relation on the set $M \times S$ as follows: $(m_1, s_1) \sim (m_2, s_2)$ if and only if there exists a $t \in S$ such that:

$$t(s_2 m_1 - s_1 m_2) = 0$$

Essentially the same proof as in Proposition 1.1.2 shows that $M \times S/\sim$, which we denote by $S^{-1}M$ going forward, has the structure of an $S^{-1}A$ module. In particular, if $a/s \in S^{-1}A$, then we define:

$$[a, s] \cdot [m, t] = [am, st] \qquad \text{and} \qquad [m_1, t_1] + [m_2, t_2] = [t_2 m_1 + t_1 m_2, t_1 t_2]$$

which are easily checked to be well defined. We also denote the equivalence classes $[m, t]$ by m/t, and thus multiplication and addition are given by:

$$\frac{a}{s} \cdot \frac{m}{t} = \frac{am}{st} \qquad \text{and} \qquad \frac{m_1}{t_1} + \frac{m_2}{t_2} = \frac{t_2 m_1 + t_1 m_2}{t_1 t_2}$$

Now let $\phi : M \to N$ be an A module morphism; we want to define an $S^{-1}A$ morphism $\phi' : S^{-1}M \to S^{-1}N$. Since any such morphism must satisfy:

$$\begin{aligned} \phi'\left(\frac{m}{t}\right) =& \phi'\left(\frac{1}{t} \cdot \frac{m}{1}\right) \\ =& \frac{1}{t} \cdot \phi'\left(\frac{m}{1}\right) \end{aligned}$$

there is essentially one way to define this morphism, and that is as:

$$\phi'\left(\frac{m}{t}\right) = \frac{\phi(m)}{t}$$

We check that this well defined: suppose that $m_1/t_1 = m_2/t_2$, then there is an s satisfying:

$$s(t_1 m_2 - t_2 m_1) = 0$$

Since ϕ is a morphism of A modules, it follows easily that:

$$s(t_1 \phi(m_2) - t_2 \phi(m_1)) = 0$$

implying that:

$$\frac{\phi(m_1)}{t_1} = \frac{\phi(m_2)}{t_2}$$

hence ϕ' is well defined. Let $\psi : N \to P$ be another morphism of modules, and $m/t \in S^{-1}M$, then:

$$(\psi \circ \phi)'\left(\frac{m}{t}\right) = \frac{\psi(\phi(m))}{t} = \psi'\left(\frac{\phi(m)}{t}\right) = \psi'\left(\phi'\left(\frac{m}{t}\right)\right)$$

hence $(\psi \circ \phi)' = \psi' \circ \phi'$. Since we clearly have that Id' is the identity morphism $S^{-1}M \to S^{-1}M$, it follows that the assignment $M \mapsto S^{-1}M$ and $\phi \mapsto \phi'$ defines a covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$.

It remains to show that this functor is exact. Let:

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_2$$

be an exact sequence of A-modules, we claim that:

$$S^{-1}M_1 \xrightarrow{f_1'} S^{-1}M_2 \xrightarrow{f_2'} S^{-1}M_2$$

is exact. It is clear that $f_2' \circ f_1' = 0$, so we need only show that $\ker f_2' \subset \operatorname{im} f_1'$. Let $m_2/t_2 \in \ker f_2'$, then:

$$f_2'\left(\frac{m_2}{t_2}\right) = \frac{f_2(m_2)}{t_2} = 0$$

then there exists an $s \in S$ such that:

$$s \cdot f_2(m_2) = 0$$

In particular, we have that $f_2(s \cdot m_2) = 0$, so there is a unique element $m_1 \in M_1$ such that $f_1(m_1) = s \cdot m_2$. It follows that:

$$f_1'\left(\frac{m_1}{st_2}\right) = \frac{f_1(m_1)}{st_2} = \frac{s \cdot m_2}{s \cdot t_2} = \frac{m_2}{t_2}$$

hence the sequence is exact. □

We call this functor localization, and as in the ring case if we S is the multiplicatively closed subset generated by $f \in A$, and if $S = \mathbb{A} \setminus \mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} A$ we denote $S^{-1}M$ by M_g and $M_\mathfrak{p}$ respectively. Moreover, note that we have a well defined localization map $\pi : M \to S^{-1}M$, sending m to $m/1$.

Lemma 5.3.2. *The kernel of the map $\pi : M \to S^{-1}M$ is precisely:*

$$\{m \in M : \exists s \in S, s \cdot m = 0\}$$

In particular, if A is an integral domain, and M is torsion free, then $\ker \pi = 0$.

Proof. If $m/1 = 0$ then by definition there exists an $s \in S$ such that $s \cdot m = 0$. If A is an integral domain, and M has zero torsion, then for all $a \in A$ and all $m \in M$ we have that $a \cdot m = 0$ implies either m or a is equal to zero implying the claim. □

We now show that localization behaves well with submodules, and quotients:

Lemma 5.3.3. *Let M be an A modules, N_1 and N_2 submodules of M, and $S \subset A$ a multiplicatively closed set. Then the following hold:*

i) If $\pi : M \to S^{-1}M$ is the localization map then $S^{-1}N_1 = \langle \pi(N) \rangle \subset S^{-1}M$.

ii) $S^{-1}(N_1 \cap N_2) = S^{-1}(N_1) \cap S^{-1}(N_2)$

iii) $S^{-1}(N_1 + N_2) = S^{-1}N_2 + S^{-1}N_2$

iv) There is a natural isomorphism $S^{-1}(M/N_1) \cong S^{-1}M/S^{-1}N_1$.

Proof. For i), note that $S^{-1}N_1$ is easily identified as a submodule of $S^{-1}M$ as the inclusion morphism $\iota : N_1 \to M$ gets sent to a morphism $\iota' : S^{-1}N_1 \to S^{-1}M$ satisfying:

$$\iota'\left(\frac{n}{s}\right) = \frac{\iota(n)}{s} = \frac{n}{s} \in S^{-1}M$$

so it is also an inclusion. now if $n/s \in S^{-1}N$ we have that

$$n/s = (1/s) \cdot (n/s) \in \langle \pi(N) \rangle$$

If $m/s \in \langle \pi(N) \rangle$, then for some $a/s \in S^{-1}A$ we have $m/s = (a/s) \cdot (n/1)$, however $a \cdot n \in N_1$ as N_1 is an A submodule/ It follows that $an/s \in S^{-1}N_1$ implying i).

For ii) if $n/s \in S^{-1}(N_1 \cap N_2)$, then by i) we can take $n/1$ to be such that $n \in N_1 \cap N_2$. In particular, $n/s \in S^{-1}N_1$ and $n/s \in S^{-1}N_2$ hence $n/s \in S^{-1}N_1 \cap S^{-1}N_2$. Conversely, if $n/s \in S^{-1}N_1 \cap S^{-1}N_2$, then $n/s \in S^{-1}N_i$ for each i. It follows that we can take n to be such that $n \in N_1 \cap N_2$ so $n/s \in S^{-1}(N_1 \cap N_2)$ by i), implying ii).

For *iii*), let $n/s \in S^{-1}(N_1 + N_2)$, then $n \in N_1 + N_2$ hence $n = n_1 + n_2$ for $n_i \in N_i$. It follows that $n/s = n_1/s + n_2/s \in S^{-1}N_1 + S^{-1}N_2$, giving us the first inclusion. If $n/s \in S^{-1}N_1 + S^{-1}N_2$ then we can write n/s as $n_1/s_1 + n_2/s_2$ where $n_i \in N_i$. Now:

$$\frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{s_2 n_1 + s_1 n_2}{s_1 s_2} \in S^{-1}(N_1 + N_2)$$

because $s_2 n_1 + s_1 n_1 \iota N_1 \cap N_2$.

Finally, for *iv*), we have an exact sequence:

$$0 \longrightarrow N_1 \longrightarrow M \longrightarrow M/N_1 \longrightarrow 0$$

so the functor S^{-1} gives us an exact sequence:

$$0 \longrightarrow S^{-1}N_1 \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N_1) \longrightarrow 0$$

implying that $S^{-1}(M/N_1) \cong S^{-1}M/S^{-1}N_1$ as desired. □

Alternatively to the construction in Lemma 5.3.1, we can view $S^{-1}M$ as a tensor product. Indeed, localization makes $S^{-1}A$ an A modules, so we could define $S^{-1}M$ as $M \otimes_A S^{-1}A$, one just has to check that this is an equivalent definition.

Proposition 5.3.1. *There is a natural isomorphism of $S^{-1}A$ modules:*

$$M \otimes_A S^{-1}A \cong S^{-1}M$$

Proof. Note that that we have an A bilinear morphism:

$$\begin{aligned} M \times S^{-1}A &\longrightarrow S^{-1}M \\ (m, a/s) &\longmapsto (m \cdot a)/s \end{aligned}$$

which then descends to an A linear morphism:

$$\phi : M \otimes_A S^{-1}A \longrightarrow S^{-1}M$$

This easily seen to be $S^{-1}A$ linear, where $M \otimes_A S^{-1}A$ has the obvious structure of an $S^{-1}A$ modules. Moreover, it is clearly surjective, as if $m/s \in S^{-1}M$, then we have that $\phi(m \otimes 1/s) = m/s$. Let:

$$\alpha = \sum_{i=1}^{n} m_i \otimes (a_i/s_i) \in \ker \phi$$

Then note that:

$$\alpha = \sum_{i=1}^{n} a \cdot m_i \otimes (\frac{1}{s_i})$$

If $t_i = s_1 \cdots \hat{s}_i \cdots s_n$, then $1/s_i = t_i/s$, so:

$$\alpha = \sum_{i=1}^{n} a \cdot m_i \otimes (t_i/s) = \left(\sum_{i=1}^{n} a \cdot t_i \cdot m_i \right) \otimes (1/s)$$

Let:

$$n = \sum_{i=1}^{n} a \cdot t_i \cdot m_i$$

then we have that:

$$n/s = 0$$

implying there is some $u \in S$ such that $u \cdot n = 0$. However, we can then write:

$$\alpha = n \otimes \frac{u}{su} = (un) \otimes \frac{1}{s} = 0$$

so ϕ is injective as well. □

Note that the above along with Lemma 5.1.1 implies that $S^{-1}A$ is a flat[88] A module. We also have the following obvious corollary:

Corollary 5.3.1. *Let M_1 and M_2 be A modules, then:*

$$S^{-1}(M_1 \otimes_A M_2) \cong S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2$$

Proof. By Proposition 5.3.1, we have that:

$$\begin{aligned} S^{-1}(M_1 \otimes_A M_2) \cong& (M_1 \otimes_A M_2) \otimes_A S^{-1}A \\ \cong& M_1 \otimes_A (M_2 \otimes_A S^{-1}A) \\ \cong& M_1 \otimes_A (S^{-1}A \otimes_{S^{-1}A} S^{-1}M_2) \\ \cong& (M_1 \otimes_A S^{-1}A) \otimes_{S^{-1}A} S^{-1}M_2 \\ \cong& S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2 \end{aligned}$$

□

We end our short foray into commutative algebra by proving some 'local to global' properties of A modules:

Proposition 5.3.2. *Let M be an A module, then the following are equivalent:*

i) M is the zero module.

ii) $M_\mathfrak{p}$ is the zero module for all $\mathfrak{p} \in \operatorname{Spec} A$.

iii) $M_\mathfrak{m}$ is the zero module for all $\mathfrak{m} \in |\operatorname{Spec} A|$

Proof. Clearly $i) \Rightarrow ii)$, and $ii) \Rightarrow iii)$, so it suffices to show $iii) \Rightarrow i)$. Suppose that $M_\mathfrak{m}$ is the zero module for all $\mathfrak{m}$, and let $m \in M$. Let $I \subset A$ be the ideal defined by:

$$I = \{a \in A : a \cdot m = 0\}$$

If $I = A$ then $m = 0$, otherwise $I \subset \mathfrak{m}$ for some $\mathfrak{m} \in |\operatorname{Spec} A|$. In this case, we have that $m/1 = 0 \in A_\mathfrak{m}$, hence there is some $s \notin \mathfrak{m}$ satisfying $s \cdot m = 0$. This implies that $s \in I$, but $I \subset \mathfrak{m}$, so $I \not\subset \mathfrak{m}$ and thus $I = A$, $m = 0$. Since this holds for arbitrary m we have that M is the zero module. □

This then implies the following:

Corollary 5.3.2. *Let:*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$$

be a sequence of A modules, then the following are equivalent:

i) The sequence of A modules is exact.

ii) For all $\mathfrak{p} \in \operatorname{Spec} A$ the localized sequence is exact.

iii) For all $\mathfrak{m} \in |\operatorname{Spec} A|$ the localized sequence is exact.

In particular, a morphism of A modules is injective or surjective if and only if the localized morphism is injective or surjective for all $\mathfrak{m} \in |\operatorname{Spec} A|$.

Proof. We clearly have $i) \Rightarrow ii) \Rightarrow iii)$ since localization is an exact functor. Now suppose that that:

$$M_{1\mathfrak{m}} \xrightarrow{f_{1\mathfrak{m}}} M_{2\mathfrak{m}} \xrightarrow{f_{2\mathfrak{m}}} M_{3\mathfrak{m}}$$

is exact for all $\mathfrak{m}$. Since $f_{2\mathfrak{m}} \circ f_{1\mathfrak{m}} = 0$ for all $\mathfrak{m}$, and $f_{2\mathfrak{m}} \circ f_{1\mathfrak{m}} = (f_2 \circ f_1)_\mathfrak{m}$, we have that $\operatorname{im}(f_2 \circ f_1)_\mathfrak{m} = 0$ for all $\mathfrak{m}$, hence $\operatorname{im} f_2 \circ f_1 = 0$ by Proposition 5.3.2. It follows that $\operatorname{im} f_1 \subset \ker f_2$. In particular, we have that there is a well defined quotient $\ker f_2 / \operatorname{im} f_1$, and by Lemma 5.3.3 we have that

$$(\ker f_2 / \operatorname{im} f_1)_\mathfrak{m} \cong (\ker f_2)_\mathfrak{m} / (\operatorname{im} f_1)_\mathfrak{m} \cong \ker f_{2m} / \operatorname{im} f_{1\mathfrak{m}} = 0$$

so by Proposition 5.3.2 we that $\ker f_2 / \operatorname{im} f_1 = 0$ implying the claim. □

[88] I.e. the functor $\otimes_A S^{-1}A$ is exact.

A further consequence of the above is that flatness is a local property:

Proposition 5.3.3. *Let M be an A module then the following are equivalent:*

i) M is a flat A module.

ii) For all $\mathfrak{p} \in \operatorname{Spec} A$ $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$ module.

iii) For all $\mathfrak{m} \in |\operatorname{Spec} A|$ $M_\mathfrak{m}$ is a flat $A_\mathfrak{m}$ module.

Proof. Suppose that M is a flat module, and $f : N \to P$ an injective morphism of $A_\mathfrak{p}$ modules. We want to show that the induced map

$$f' : N \otimes_{A_\mathfrak{p}} M_\mathfrak{p} \longrightarrow P \otimes_{A_\mathfrak{p}} M_\mathfrak{p}$$

is also injective. Now $\pi : A \to S^{-1}A$ gives every $A_\mathfrak{p}$ module an A module structure, such that f is also an A module morphism. Now observe that we have the following isomorphisms:

$$N \otimes_{A_\mathfrak{p}} M_\mathfrak{p} \cong N \otimes_{A_\mathfrak{p}} (A_\mathfrak{p} \otimes_A M) \cong N \otimes_A M$$

so up to isomorphism the morphism $N \otimes_{A_\mathfrak{p}} M_\mathfrak{p} \to P \otimes_{A_\mathfrak{p}} M_\mathfrak{p}$ is the map $N \otimes_A M \to P \otimes_A M$ induced by tensoring with M. It follows that since M is flat that f' is flat hence $M_\mathfrak{p}$ is flat.

Clearly $ii) \Rightarrow iii)$. Assuming $iii)$ let $f : N \to P$ an injective morphism, and $f' : N \otimes_A M \to P \otimes_A M$ the induced map. By Corollary 5.3.1, for all $\mathfrak{m} \in |\operatorname{Spec} A|$, up to isomorphism $f'_\mathfrak{m} : (N \otimes_A M)_\mathfrak{p} \to (P \otimes_A M)_\mathfrak{m}$ is the morphism:

$$N_\mathfrak{p} \otimes_{A_\mathfrak{p}} M_\mathfrak{p} \longrightarrow P_\mathfrak{p} \otimes_{A_\mathfrak{p}} M_\mathfrak{p}$$

induced by $f_\mathfrak{m} : N_\mathfrak{m} \to P_\mathfrak{m}$ and tensoring with $M_\mathfrak{m}$. It follows that $\ker f'_\mathfrak{m} = (\ker f')_\mathfrak{m} = 0$. Since this holds for all $\mathfrak{m}$, we have that $\ker f' = 0$ so M is flat. □

5.4 Quasicoherent Sheaves Over a Scheme

In this section we develop the theory of quasicoherent sheaves over a scheme. Since the scheme structure on X is generally fixed, we use $\operatorname{QCoh}(X)$ to refer to the category of quasicoherent $\mathscr{O}_X$ modules, breaking from our notation in the previous section. Our main goal in this section is to associate to each A module a quasicoherent sheaf over $\operatorname{Spec} A$, and show that every quasicoherent sheaf over X is locally of this form. Using this, we will show that $\operatorname{QCoh}(X)$ is an abelian category, and explore a connection between quasicoherent sheaves of ideals of $\mathscr{O}_X$, and closed subschemes of X.

Lemma 5.4.1. *Let M be an A module, then there is a quasicoherent sheaf $\widetilde{M}$ on $\operatorname{Spec} A$ satisfying $\widetilde{M}(U_g) \cong M_g$. In particular $\widetilde{M}_\mathfrak{p} \cong M_\mathfrak{p}$, and the assignment $M \mapsto \widetilde{M}$ defines a covariant functor* $\operatorname{Mod}_A \to \operatorname{Mod}_{\operatorname{Spec} A}$.

Proof. We define a sheaf on a basis by $\mathcal{F}(U_g) = M_g$. The restriction maps are those induced by identifying $M_g \cong M \otimes_A A_g$ and taking $\operatorname{Id} \otimes \theta^{U_g}_{U_h}$, where $\theta^g_h : A_g \to A_h$[89] are the usual restriction maps. It is clear that this defines a presheaf on a basis. Specifically, since $U_h \subset U_g$, we have that there exists an $k \in \mathbb{Z}^+$ and $a \in A$ such that $a \cdot h = g^k$, so these restriction maps are given by:

$$\begin{aligned} \theta^g_h : M_g &\longrightarrow M_h \\ \frac{m}{g^n} &\longmapsto \frac{m \cdot a^n}{h^{nk}} \end{aligned}$$

The same exact argument as in Proposition 1.4.3, but with replacing elements in A_g with elements in M_g demonstrates that this indeed defines a sheaf on a base. Due to the similarity of the argument, we elect to not reproduce it here.

To see that $\widetilde{M}_\mathfrak{p}$ is uniquely isomorphic to $M_\mathfrak{p}$, it suffices to show that $\mathcal{F}_\mathfrak{p}$ is isomorphic to $M_\mathfrak{p}$, however this argument is virtually identical to the one in Proposition 1.4.4, replacing $A_\mathfrak{p}$, and A_g with $M_\mathfrak{p}$ and M_g.

Finally, let $f : M \to N$ be a morphism of A modules, then on each distinguished open we get an induced morphism $M_g \to N_g$, given by $f \otimes \operatorname{Id}_{A_g}$. This map clearly commutes with restriction maps,

[89] We employ the same notation as in Proposition 1.4.3.

hence by Theorem 1.4.1 we have a unique morphism of $\mathscr{O}_{\operatorname{Spec} A}$ modules $\tilde{f} : \widetilde{M} \to \widetilde{N}$. In particular, the stalk map $f_{\mathfrak{p}}$ is given by the induced map $M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ up to a unique isomorphism. Moreover, since localization is a functor, we have that $\widetilde{(f \circ g)} = \tilde{f} \circ \tilde{g}$ hence the assignment $M \mapsto \widetilde{M}$ defines a covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_{\operatorname{Spec} A}$ as desired. □

We have the following borderline immediate corollary:

Corollary 5.4.1. *For all A modules M, the sheaf of $\mathscr{O}_{\operatorname{Spec} A}$ modules is quasicoherent. In particular, the assignment $M \mapsto \widetilde{M}$ is a functor $\mathrm{Mod}_A \to \mathrm{QCoh}(\operatorname{Spec} A)$.*

Proof. Let I be a possibly infinite indexing set such that:

$$f : \bigoplus_{i \in I} A \longrightarrow M$$

is a surjection. In particular, we could easily take I to have cardinality of M, take some bijection $h : I \to S$, and take f to be a direct sum of maps of the form:

$$\begin{aligned} f_i : A &\longrightarrow M \\ 1 &\longmapsto h(i) \end{aligned}$$

For the same reason, we easily obtain an indexing set J such that we have a surjection:

$$\bigoplus_{j \in J} A \longrightarrow \ker f$$

We thus have an exact sequence:

$$\bigoplus_{j \in J} A \longrightarrow \bigoplus_{i \in I} A \longrightarrow M \longrightarrow 0$$

Which induces a sequence of $\mathscr{O}_{\operatorname{Spec} A}$ modules by Lemma 5.3.1:

$$\mathscr{O}_{\operatorname{Spec} A}^J \longrightarrow \mathscr{O}_{\operatorname{Spec} A}^I \longrightarrow \widetilde{M} \longrightarrow 0$$

On stalks this given by:

$$\bigoplus_{j \in J} A_{\mathfrak{p}} \longrightarrow \bigoplus_{i \in I} A_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow 0$$

which is exact since localization is an exact functor by Lemma 5.2.1. It follows that the original sequence of $\mathscr{O}_{\operatorname{Spec} A}$ modules is exact, and the $\widetilde{M}$ is quasicoherent. □

Our first major goal is to prove that this functor is an equivalence of categories. We begin with the following lemma:

Lemma 5.4.2. *Let M be an A module, and $\mathscr{F}$ a sheaf of $\mathscr{O}_{\operatorname{Spec} A}$ modules. Any morphism $M \to \mathscr{F}(\operatorname{Spec} A)$ of A modules induces a morphism of $\mathscr{O}_{\operatorname{Spec} A}$ modules $\widetilde{M} \to \mathscr{F}$. Moreover, every such morphism of sheaves is induced by it's action on global sections, $M \to \mathscr{F}(\operatorname{Spec} A)$.*

Proof. Let $\phi : M \to \mathscr{F}(\operatorname{Spec} A)$ be an A module morphism. We define a morphism on distinguished opens by:

$$\begin{aligned} \psi_{U_g} : M_g &\longrightarrow \mathscr{F}(U_g) \\ \frac{m}{g^k} &\longrightarrow \frac{1}{g^k} \cdot (\phi(m)|_{U_g}) \end{aligned}$$

where we are using the fact that $\mathscr{F}(U_g)$ is an A_g module. If this morphism is well defined for each g, then it clearly commutes with restrictions, so suppose that $m/g^k = n/g^l$, then there exists an $L \in \mathbb{Z}^+$ such that:

$$g^L(g^l m - g^k n) = 0$$

Now note that:

$$\frac{1}{g^k} \cdot (\phi(m)|_{U_g}) - \frac{1}{g^k}(\phi(n)|_{U_g}) = \frac{1}{g^{k+l}} \cdot (\phi(g^l m - g_n^k))$$

Multiplying by $1 = g^L/g^L$ yields:

$$\frac{1}{g^{L+k+l}} \cdot (\phi(g^L(g^l m - g_n^k))) = 0$$

implying that ψ_{U_g} is well defined as desired. It follows from Theorem 1.4.1 that there is an induced morphism of $\mathscr{O}_{\operatorname{Spec} A}$ modules $\psi : \widetilde{M} \to \mathscr{F}$.

Now let $\psi : \widetilde{M} \to \mathscr{F}$ be a morphism of $\mathscr{O}_{\operatorname{Spec} A}$ modules, and set $\phi = \psi_{\operatorname{Spec} A}$. We need to show that:

$$\psi_{U_g}\left(\frac{m}{g^k}\right) = \frac{1}{g_k} \cdot \phi(m)|_{U_g}$$

Since ψ_{U_g} is a morphism of A_g modules, we have that:

$$\begin{aligned} \psi_{U_g}\left(\frac{m}{g^k}\right) &= \frac{1}{g^k} \cdot \psi_{U_g}(m/1) \\ &= \frac{1}{g^k} \cdot \psi_{U_g}(m|_{U_g}) \\ &= \frac{1}{g^k} \cdot \phi(m)|_{U_g} \end{aligned}$$

implying the claim. □

With this we can show the following:

Lemma 5.4.3. *Suppose that $\mathscr{F}$ is an $\mathscr{O}_{\operatorname{Spec} A}$ module such that there exists an exact sequence:*

$$\mathscr{O}_{\operatorname{Spec} A}^I \longrightarrow \mathscr{O}_{\operatorname{Spec} A}^J \longrightarrow \tilde{F} \longrightarrow 0$$

Then there is an isomorphism $\widetilde{M} \cong \mathscr{F}$, where $M \cong \mathscr{F}(\operatorname{Spec} A)$.

Proof. We first note that $\mathscr{O}_{\operatorname{Spec} A}^I$ is the $\mathscr{O}_{\operatorname{Spec} A}$ module induced by A^I. Taking global sections, we get a morphism:

$$\phi_A : A^I \to A^J$$

and set $M = \operatorname{coker} \phi_A$. Since on global sections, we have that the composition:

$$A^J \longrightarrow A^I \longrightarrow \tilde{F}(\operatorname{Spec} A)$$

is equal to zero, there exists a unique morphism $\psi : M \to \mathscr{F}(\operatorname{Spec} A)$ which by Lemma 5.4.2 induces a unique morphism $\tilde{\psi} : \widetilde{M} \to F$. Since the morphism ϕ_A is the one which induces the morphism $\mathscr{O}_{\operatorname{Spec} A}^I \to \mathscr{O}_{\operatorname{Spec} A}^J$, we have that the following diagram commutes:

$$\begin{tikzcd} \mathscr{O}_{\operatorname{Spec} A}^I \arrow[r] & \mathscr{O}_{\operatorname{Spec} A}^J \arrow[rr] \arrow[dr] & & \mathscr{F} \\ & & \widetilde{M} \arrow[ur] & \end{tikzcd}$$

Talking stalks we obtain the following commutative diagram:

$$\begin{tikzcd} A_{\mathfrak{p}}^I \arrow[r] & A_{\mathfrak{p}}^J \arrow[rr] \arrow[dr] & & \mathscr{F}_{\mathfrak{p}} \\ & & M_{\mathfrak{p}} \arrow[ur] & \end{tikzcd}$$

Now, $M_{\mathfrak{p}}$ and $\mathscr{F}_{\mathfrak{p}}$ are both the cokernel of the morphism $A_{\mathfrak{p}}^I \to A_{\mathfrak{p}}^J$, so the morphism $M_{\mathfrak{p}} \to \mathscr{F}_{\mathfrak{p}}$ is the unique isomorphism which makes the above diagram commute. It follows that $\tilde{\psi}$ is an isomorphism implying the claim. □

The preceding lemma demonstrates that for a very specific class of quasicoherent $\mathscr{O}_{\operatorname{Spec} A}$ modules, we have the desired claim. We now show that this holds in generality:

Proposition 5.4.1. *The functor* $\operatorname{Mod}_A \to \operatorname{QCoh}(\operatorname{Spec} A)$ *given by sending* M *to* $\widetilde{M}$ *is essentially surjective*[90]

Proof. Let $\mathscr{F} \in \operatorname{QCoh}(\operatorname{Spec} A)$, then every point has a neighborhood U such that there exists an exact sequence:

$$\mathscr{O}_U^I \longrightarrow \mathscr{O}_U^J \longrightarrow \tilde{F}|_U \longrightarrow 0$$

Without loss of generality, we can take $U = U_{f_i}$ for $f_i \in A$, and since $\operatorname{Spec} A$ is quasicompact, we can take finitely many to cover $\operatorname{Spec} A$. By Lemma 5.4.3, it follows that we have a cover $\{U_{f_i}\}_{i=1}^n$ of $\operatorname{Spec} A$ such that:

$$\mathscr{F}|_{U_{f_i}} \cong \widetilde{M_i}$$

These isomorphisms induce isomorphisms $\phi_{ij} : \widetilde{M_i}|_{U_{f_i f_j}} \to \widetilde{M_j}|_{U_{f_i f_j}}$ which trivially satisfy the cocycle condition. Denote by ψ_{ij} the isomorphisms $\widetilde{M_i}(U_{f_i f_j}) \to \tilde{M}_j(U_{f_i f_j})$ induced by taking global sections of ϕ_{ij}. Up to isomorphism we can view the global sections on $\mathscr{F}$ as:

$$\mathscr{F}(\operatorname{Spec} A) = \left\{ (m_i) \in \prod_{i=1}^n M_i : \psi_{ij}(m_i|_{U_{f_i f_j}}) = m_j|_{U_{f_i f_j}} \right\}$$

Note that for each i we naturally have:

$$\begin{aligned} \widetilde{M_i}(U_{f_i f_j}) =& M_i \otimes_{A_{f_i}} A_{f_i f_j} \\ \cong& M_i \otimes_{A_{f_i}} A_{f_i} \otimes_A A_{f_j} \\ \cong& M_i \otimes_A A_{f_j} \\ \cong& (M_i)_{f_j} \end{aligned}$$

so the restriction maps are localization maps, $m_i|_{U_{f_i f_j}} = m_i/1 \in (M_i)_{f_j}$, and the ψ_{ij} are isomorphisms $(M_i)_{f_j} \to (M_j)_{f_i}$. With this we have that up to isomorphis:

$$\mathscr{F}(\operatorname{Spec} A) = \left\{ (m_i) \in \prod_{i=1}^n M_i : \psi_{ij}(m_i/1) = m_j/1 \right\}$$

Moreover, we have that $\mathscr{F}(\operatorname{Spec} A)$ is the kernel of the morphism:

$$\begin{aligned} \bigoplus_{i=1}^n M_i &\longrightarrow \bigoplus_{i,k=1}^n (M_i)_{f_k} \\ (m_i) &\longrightarrow (\psi_{ij}(m_i/1) - m_k/1) \end{aligned}$$

Since localization is exact, and commutes with finite products, we have that if $M = \mathscr{F}(\operatorname{Spec} A)$, then:

$$M_{f_j} = \ker \left(\bigoplus_{i=1}^n (M_i)_{f_j} \longrightarrow \bigoplus_{i,k=1}^n (M_i)_{f_j f_k} \right)$$

where if $\psi'_{ik} : (M_i)_{f_j f_k} \to (M_k)_{f_i f_j}$ is the induced morphism, then the above map is given by:

$$(m_i/f_j^{l_i}) \longmapsto \left(\psi'_{ik} \left(\frac{m_i f_k^{l_i}}{(f_k f_j)^{l_i}} \right) - \frac{m_k f_i^{l_i}}{(f_j f_i)^{l_i}} \right)$$

It follows that:

$$M_{f_j} = \left\{ (m_i/f_j^{k_i}) \in \prod_i (M_i)_{f_j} : \psi'_{ik} \left(\frac{m_i f_k^{l_i}}{(f_k f_j)^{l_i}} \right) = \frac{m_k f_i^{l_i}}{(f_j f_i)^{l_i}} \right\}$$

[90] Recall this implies that any object $\mathscr{F} \in \operatorname{QCoh}(\operatorname{Spec} A)$ is isomorphic to $\widetilde{M}$ for some M.

Let ξ be the morphism $\widetilde{M} \to \mathscr{F}$ be the morphism induced by the identity map $M \to \mathscr{F}(\operatorname{Spec} A)$. The map $\xi_{U_{f_j}}$ is then given by:

$$\begin{aligned} \xi_{U_{f_j}} : M_{f_j} &\longrightarrow M_j \\ (m_i/f_j^{k_i}) &\longrightarrow (1/f_j^{k_j}) \cdot m_j \end{aligned}$$

Suppose that $(m_i/f_j^{k_i}) \mapsto 0$, and let $K = \max\{k_i\}$. Note that $(f_j^{K-k_i} m_i/1) \mapsto 0$ if and only if the original element does, so it suffices to consider an element in $\ker \xi_{U_{f_j}}$ of the form $(m_i/1)$.

Now let $m_j \in M_j$, we need to define elements in $(M_i)_{f_j}$, and do so by noting that there exist $m_i \in M_i$ and $k_i \in \mathbb{Z}^+$ such that:

$$\psi_{ji}(m_j/1) = \frac{m_i}{f_j^{k_i}} \in (M_i)_{f_j}$$

We claim that the sequence $(m_i/f_j^{k_i}) \in \prod (M_i)_{f_i}$ actually lies in M_{f_i}. Since the ϕ_{ij} satisfy the cocycle condition, we have that for all i, j, l:

$$\psi'_{jl} = \psi'_{il} \circ \psi'_{ji}$$

Now consider:

$$\begin{aligned} \psi'_{il}\left(\frac{m_i f_l^{k_i}}{(f_l f_j)^{k_i}}\right) =& \psi'_{il}\left(\frac{m_i}{f_j^{k_i}}\Big|_{U_{f_i f_j f_l}}\right) \\ =& \psi'_{il}\left(\psi_{ji}(m_j/1)|_{U_{f_i f_j f_l}}\right) \\ =& \psi'_{il}\left(\psi'_{ji}\left(\frac{m_j}{1}\Big|_{U_{f_i f_j f_l}}\right)\right) \\ =& \psi'_{jl}\left(\frac{m_j}{1}\Big|_{U_{f_i f_j f_l}}\right) \\ =& \psi_{jl}\left(\frac{m_j}{1}\right)\Big|_{U_{f_i f_j f_l}} \\ =& \frac{m_l f_i^{kl}}{(f_i f_j)^{k_l}} \end{aligned}$$

hence $(m_i/f_j^{k_i}) \in M_{f_i}$. It is clear that $(m_i/f_j^{k_i})$ maps to m_j, hence $\xi_{U_{f_j}}$ is surjective, and thus an isomorphism as desired. In particular, the sheaf morphism $\xi|_{U_{f_j}}$ is determined by $\xi_{U_{f_j}}$, and is thus an isomorphism. Since ξ is locally an isomorphism on an open cover, it follows that ξ is an isomorphism, and thus $\mathscr{F} \cong \tilde{M}$ as desired. □

We can now prove our first main result of the section:

Theorem 5.4.1. *The functor* $\operatorname{Mod}_A \to \operatorname{QCoh}(\operatorname{Spec} A)$ *is an equivalence of categories. In particular* $\operatorname{QCoh}(\operatorname{Spec} A)$ *is an abelian category.*

Proof. It suffices to show that the functor is fully faithful, as it essentially surjective by the preceding proposition. Let M and N be A modules, then the morphism:

$$\begin{aligned} \operatorname{Hom}_A(M, N) &\longrightarrow \operatorname{Hom}_{\operatorname{Spec} A}(\widetilde{M}, \widetilde{N}) \\ f &\longmapsto \tilde{f} \end{aligned}$$

is surjective by Lemma 5.4.2. Suppose that $f \mapsto 0$, then by Lemma 5.4.1, we have that up to a unique isomorphism $\tilde{f}_{\mathfrak{p}} = f_{\mathfrak{p}}$, hence $f_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} A$. It follows that since $(\operatorname{im} f)_{\mathfrak{p}} = \operatorname{im} f_{\mathfrak{p}} = 0$, we have that f is the zero morphism by Proposition 5.3.2, implying the equivalence.

It is now obvious that $\operatorname{QCoh}(\operatorname{Spec} A)$ is an abelian category, as for all $\tilde{f} : \widetilde{M} \to \widetilde{N}$ we have that $\ker \tilde{f} \cong \widetilde{\ker f}$ and $\operatorname{coker} \tilde{f} \cong \widetilde{\operatorname{coker} f}$. □

An immediate corollary is that $\operatorname{QCoh}(X)$ is an abelian category when X is a scheme.

Corollary 5.4.2. *Let X be a scheme and $\mathscr{F}$ be a quasicoherent $\mathscr{O}_X$ module, then then for every open $U = \operatorname{Spec} A \subset X$ there exists an A module such that $\mathscr{F}|_U \cong \widetilde{M}$. Moreover* $\operatorname{QCoh}(X)$ *is an abelian category.*

Dimension and Divisors

6.1 Some Commutative Algebra: Krull Dimension

Let M be a smooth manifold, and recall that the dimension of M (if M is of pure dimension that is) is defined to be the dimension of the Euclidean space it is locally homeomorphic to. That is, if U is an open neighborhood of $p \in M$ and $\phi : U \to V \subset \mathbb{R}^n$, is a coordinate chart, then the dimension of M is n. In particular, we also have that the dimension as a vector space over $\mathbb{R}$ of the tangent space at a point is equal to the dimension of M for al $p \in M$.

We wish to develop a theory of dimension for schemes which mimics the above behavior in the category of smooth manifolds; that is for 'nice enough' schemes [91] we want our notion of dimension to be determined by the dimension of an open affine, as well as by the stalk at a closed point $x \in X$. In particular, we will also want single point schemes to have dimension zero, and our classical examples, $\mathbb{P}^n_k$ and $\mathbb{A}^n_k$, to have dimension n.

Since the category of affine schemes is anti-equivalent to the category of commutative rings, we will first develop the dimension theory for commutative rings.

Definition 6.1.1. Let A be a commutative ring; a strictly increasing finite chain of prime ideals:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

has **length n**[92]. Let $L(A) \subset \mathbb{N}$ be the ordered set consisting of the lengths of all finite increasing chains of prime ideals; we define the **Krull dimension** of a commutative ring A, denoted $\dim A$, to be $\sup L(A)$ if it exists, and to be infinite if there is no least upper bound[93].

One might quickly jump to the conclusion that all rings of finite dimension are Noetherian, or, equivalently, that any non-Noetherian ring will have infinite dimension. While the study of Krull dimension of Noetherian rings will prove a fruitful endeavor, as the next example shows, the former is not the case.

Example 6.1.1. Let $A = k[x_0, x_1, \ldots,]/\left\langle x_0^2, x_1^2, \ldots\right\rangle$, then we claim that A contains only one prime ideal. Note that A is clearly not Noetherian. The prime ideals of A are in bijection with prime ideals containing $I = \left\langle x_0^2, x_1^2, \ldots\right\rangle$. That is every prime ideal can be identified with a prime ideal of A lying in the closed set $V(I) \subset \operatorname{Spec} A$. We have that $V(I) = V(\sqrt{I})$, and that each $x_i \in \sqrt{I}$ as $x_i^2 \in I$. It follows that $\langle x_0, x_1, \ldots\rangle \subset \sqrt{I}$, so $\sqrt{I} = \langle x_0, x_1, \ldots,\rangle$ and is thus maximal. Therefore, $V(I)$ consists of a single point, and thus $A = k[x_0, x_1, \ldots,]/\left\langle x_0^2, x_1^2, \ldots\right\rangle$ has one prime ideal, so $\dim A = 0$.

We would also like to show the existence of a Noetherian ring of infinite dimension. However, the construction of such a ring was historically an elusive endeavor, and requires more machinery than we have on hand. Therefore, such an example appears later on in the section[94], but we stress that there Noetherian does not imply finite dimensional.

Example 6.1.2. We want to determine the dimension of $\mathbb{Z}$. Every non zero prime ideal in $\mathbb{Z}$ is is maximal, hence the only prime that can possible be contained in a non zero prime is the zero ideal. It follows that every chain of increasing prime ideals is of one of two forms:

$$\langle 0\rangle \qquad \text{or} \qquad \langle 0\rangle \subset p\mathbb{Z}$$

where p is prime. It follows that $L(\mathbb{Z}) = \{0, 1\}$ which has least upper bound 1 hence $\dim \mathbb{Z} = 1$.

[91] To be defined later.

[92] We are essentially counting number of inclusions, not the number of prime ideals.

[93] Note that since $L(A) \subset \mathbb{N}$, if $\sup L(A)$ exists, then $\sup A \in L(A)$.

[94] See Example 6.1.3.

In particular, if A is a PID, then every non-zero prime ideal is maximal, so $L(A) = \{0, 1\}$ hence $\dim A = 1$. Note that for any field $k[x]$ is a PID, hence $\dim k[x] = 1$.

In order to determine the dimension of more complicated rings it will be convenient to determine equivalent definitions of dimension.

Definition 6.1.2. Let $\mathfrak{p} \in \operatorname{Spec} A$, and let $L(\mathfrak{p})$ be the set consisting of the lengths of all strictly increasing chains of prime ideals ending with $\mathfrak{p}$. We define the **height of** $\mathfrak{p}$, denoted $\operatorname{ht}(\mathfrak{p})$, to be $\sup L(\mathfrak{p})$ if it exists, and infinite otherwise.

We have the following characterization of height zero prime ideals:

Lemma 6.1.1. *Let $\mathfrak{p} \in \operatorname{Spec} A$, then $\operatorname{ht}(\mathfrak{p}) = 0$ if and only if $\mathfrak{p}$ is minimal*[95] *over $\langle 0 \rangle$.*

Proof. Let $\mathfrak{p}$ be a minimal prime ideal over 0, then by definition, if $\mathfrak{q} \subset \mathfrak{p}$, we have that $\mathfrak{q} = \mathfrak{p}$, hence the only chain of prime ideals ending with $\mathfrak{p}$ is the trivial chain $\mathfrak{p}$. It follows that $\operatorname{ht}(\mathfrak{p}) = 0$.

Let $\operatorname{ht}(\mathfrak{p}) = 0$, and suppose $\mathfrak{q} \subset \mathfrak{p}$. If this inclusion is strict, then we have that $\operatorname{ht}(\mathfrak{p}) \geq 1$, hence we must have that $\mathfrak{q} = \mathfrak{p}$ implying that $\mathfrak{p}$ is minimal over $\langle 0 \rangle$. $\square$

While we have used localization throughout this text, we have not yet had need to determine what $\operatorname{Spec} S^{-1}A$ is in terms of of prime ideals of A. We do so now:

Lemma 6.1.2. *Let S be a multiplicatively closed set, then their exists a bijection between $\operatorname{Spec} S^{-1}A$ and prime ideals of A such that $S \cap \mathfrak{p} = \emptyset$.*

Proof. This is entirely analogous to Proposition 1.1.3. We define the maps, and leave the rest of the proof as an exercise to the reader.

Let $\mathfrak{P}$ denote the set of prime ideals of A such that $S \cap \mathfrak{p} = \emptyset$; we define a set map:

$$\begin{aligned} f : \mathfrak{P} &\longrightarrow \operatorname{Spec} S^{-1}A \\ \mathfrak{p} &\longmapsto \langle \pi(\mathfrak{p}) \rangle \end{aligned}$$

where $\pi : A \to S^{-1}A$ is the localization map. For this map to be well defined, we need to check that this is a prime ideal. Let $a/s, b/t \in S^{-1}A$ such that $ab/(ts) \in f(\mathfrak{p})$, it follows that:

$$\frac{ab}{ts} = \frac{p}{u}$$

for some $u \in S$, and some $p \in \mathfrak{p}$. There then exists an element $v \in S$ such that:

$$v(uab - pts) = 0$$

It follows that $abuv = ptsv \in \mathfrak{p}$, so $ab \in \mathfrak{p}$, hence either a or $b \in \mathfrak{p}$ implying either a/s or $b/t \in f(\mathfrak{p})$, so $f(\mathfrak{p})$ is prime.

We define an inverse map by:

$$\begin{aligned} g : \operatorname{Spec} S^{-1}A &\longrightarrow \mathfrak{P} \\ \mathfrak{q} &\longmapsto \pi^{-1}(\mathfrak{q}) \end{aligned}$$

This is clearly prime, so we need to check that $g(\mathfrak{q}) \cap S = \emptyset$. Suppose other wise, then there is some $s \in S$ such that $s \in \pi^{-1}(\mathfrak{q})$. It follows that $s/1 \in \pi(\pi^{-1}(\mathfrak{q})) \subset \mathfrak{q}$, implying that $\mathfrak{q} = S^{-1}A$ a contradiction.

$\square$

Note that these maps are inclusion preserving, so if $\mathfrak{p} \subset \mathfrak{q} \in \mathfrak{P}$, then $f(\mathfrak{p}) \subset f(\mathfrak{q})$, and similarly for g. With the above characterization we can show the following:

Proposition 6.1.1. *Let $\mathfrak{p} \in \operatorname{Spec} A$, then $\operatorname{ht}(\mathfrak{p}) = \dim A_{\mathfrak{p}}$.*

Proof. We first show that $\operatorname{ht}(\mathfrak{p})$ is infinite if and only if $\dim A_{\mathfrak{p}}$ is infinite. Suppose that $\operatorname{ht}(\mathfrak{p})$ is infinite, then for any strictly increasing finite chain of prime ideals ending with $\mathfrak{p}$ of length n:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$$

[95] As in Theorem 3.4.3

we can find a sequence:

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_m = \mathfrak{p}$$

such that $m > n$. Each of these ideals is contained in $\mathfrak{p}$, hence $(A \smallsetminus \mathfrak{p}) \cap \mathfrak{q}_i = \emptyset$, and similarly for each $\mathfrak{p}_i$. It follows that

$$f(\mathfrak{p}_0) \subset f(\mathfrak{p}_1) \subsetneq \cdots \subsetneq f(\mathfrak{p}_n) = \mathfrak{m}_\mathfrak{p}$$

is a chain of prime ideals of length n in $A_\mathfrak{p}$. Here f is the map from the preceding lemma, and $\mathfrak{m}_\mathfrak{p}$ is the unique maximal ideal in $A_\mathfrak{p}$.

Suppose that $\dim A_\mathfrak{p}$ is finite, then $\sup L(A_\mathfrak{p})$ exists, then there exists an $n \in L(A)$, such that that for all $m \in L$, we have that $n \geq m$. In particular, n corresponds to a chain of prime ideals:

$$\mathfrak{q}_0' \subsetneq \cdots \subsetneq \mathfrak{q}_n' = \mathfrak{m}_\mathfrak{p}$$

where we must end with $\mathfrak{m}_\mathfrak{p}$ as otherwise there exists a chain of length $n+1$ due to the fact that $\mathfrak{m}_\mathfrak{p}$ contains every ideal of $A_\mathfrak{p}$. It follows that:

$$g(\mathfrak{q}_0') \subset \cdots \subsetneq g(\mathfrak{q}_n') = \mathfrak{p}$$

is a chain of length n. However, since $\operatorname{ht}(\mathfrak{p})$ is infinite, we can take $m > n$ and find a chain:

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_m = \mathfrak{p}$$

Then:

$$f(\mathfrak{q}_0) \subset f(\mathfrak{q}_1) \subsetneq \cdots \subsetneq f(\mathfrak{q}_m) = \mathfrak{m}_\mathfrak{p}$$

is a chain of prime ideals in $A_\mathfrak{p}$ of length $m > n$, hence there exists $m \in L(A)$ such that $m > n$ a contradiction. It follows that if $\operatorname{ht}(\mathfrak{p})$ is infinite, then $\dim A_\mathfrak{p}$ is infinite as well.

Now suppose that $\dim A_\mathfrak{p}$ is infinite, then as before, given $m \in L(A_\mathfrak{p})$, we can always find an $n \in L(A_\mathfrak{p})$ such that $n > m$. Suppose that $\operatorname{ht}(\mathfrak{p})$ is finite, and let $n = \sup L(\mathfrak{p})$. This corresponds to a chain of prime ideals ending with $\mathfrak{p}$ of length n:

$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$$

It follows that:

$$f(\mathfrak{p}_0) \subsetneq \cdots \subsetneq f(\mathfrak{p}_n) = \mathfrak{m}_\mathfrak{p}$$

is a chain of prime ideals of length n in $A_\mathfrak{p}$. Since $\mathfrak{m}_\mathfrak{p}$ is the unique maximal ideal of $A_\mathfrak{p}$, and $\dim A_\mathfrak{p}$ is infinite, we have that there exists a chain of prime ideals of length $m > n$ ending with $m_\mathfrak{p}$:

$$\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_m = \mathfrak{m}_\mathfrak{q}$$

so:

$$g(\mathfrak{q}_0) \subsetneq \cdots \subsetneq g(\mathfrak{q}_m) = \mathfrak{p}$$

is a chain of prime ideals in A terminating with $\mathfrak{p}$ of length m. It follows that $m \in L(\mathfrak{p})$, and is greater than n hence we must have that no such n is complete.

Now suppose that $\dim A_\mathfrak{p}$ is finite, then by the above we equivalently have that $\operatorname{ht}\mathfrak{p}$ is finite as well. Let $\dim A_\mathfrak{p} = n$ and $\operatorname{ht}\mathfrak{p} = m$. Suppose that:

$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_m = \mathfrak{p}$$

is a strictly in increasing chain of prime ideals of length m terminating with $\mathfrak{p}$, then we have that:

$$f(\mathfrak{p}_0) \subsetneq \cdots \subsetneq f(\mathfrak{p}_m) = \mathfrak{m}_\mathfrak{p}$$

is a strictly increasing chain of prime ideals of length m in $A_\mathfrak{p}$. It follows that $m \leq n$. Now let:

$$\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n$$

be a strictly increasing chain of prime ideals of length n in $A_{\mathfrak{p}}$. We know that $\mathfrak{q}_n = \mathfrak{m}_{\mathfrak{p}}$, as otherwise there exists a chain of length $n+1$. It follows that:

$$g(\mathfrak{q}_0) \subsetneq \cdots \subsetneq g(\mathfrak{q}_n) = \mathfrak{p}$$

is a strictly increasing chain of prime ideals in A terminating with $\mathfrak{p}$ of length n. It follows that $n \leq m$, hence we must have that $m = n$ implying the claim. ☐

Now let $H(A)$ be the set defined by:

$$H(A) = \{\mathrm{ht}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} A\}$$

where if $\mathrm{ht}(\mathfrak{p})$ is infinite, we replace it with the symbol ∞. Note that $\mathbb{N} \cup \{\infty\}$ carries a total order by declaring that $\infty > m$ for all $m \in \mathbb{N}$. It follows that $H(A) \subset \mathbb{N} \cup \{\infty\}$ carries a natural ordering, and that $\sup H(A) = \infty$ if and only if there exists a prime ideal of infinite height, or $\sup H(A)$ contains only prime ideals of finite height, but $H(A)$ is unbounded as a subset of $\mathbb{N}$. Our next result will characterize the Krull dimension of a ring in terms of the heights of prime ideals.

Proposition 6.1.2. *The Krull dimension of A is finite if and only if $\sup H(A) \neq \infty$. In particular if $\sup H(A) \neq \infty$, or $\dim A$ is finite, then $\dim A = \sup H(A)$.*

Proof. For the first claim, we will instead show the contrapositive; i.e. that $\dim A$ is infinite if and only if $\sup H(A) = \infty$.

Suppose that $\sup H(A) = \infty$, then there either exists a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ such that $\mathrm{ht}(\mathfrak{p})$ is infinite, or every prime ideal of A has finite height, but $H(A)$ is infinite. In the first case, it follows that that for all $n \in L(A)$ we can find an increasing chain of prime ideals of length $m > n$ which terminates with $\mathfrak{p}$, so $\dim A$ cannot be finite. In the latter case, it follows that for any $n \in L(A)$ we can find a $\mathfrak{q} \in \operatorname{Spec} A$ such that $\mathrm{ht}(\mathfrak{q}) > n$, but then $\mathrm{ht}(\mathfrak{q}) \in L(A)$ so $\sup L(A)$ does not exist, and $\dim A$ cannot be finite.

Now suppose $\dim A$ is infinite. For all $n \in H(A)$, we can find an increasing chain of prime ideals:

$$\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_m = \mathfrak{q}$$

where $m > n$. It follows that $\mathrm{ht}(\mathfrak{q})$ is either infinite, in which case $\sup H(A) = \infty$ and we are done, or $\mathrm{ht}(\mathfrak{q})$ is finite but greater than n. In the latter case, since n was arbitrary, we have that $H(A)$. so by definition $\sup H(A) = \infty$.

To prove the second claim, suppose that either $\sup H(A) \neq \infty$, or $\dim A$ is finite. In both cases, by the first statement we have that $\sup H(A) = m$ and $\dim A = n$ for $m, n \in \mathbb{N}$. Now if $\sup H(A) = m$, we have that that $\mathrm{ht}(\mathfrak{p}) \leq m$ for all $\mathfrak{p} \in \operatorname{Spec} A$. Since $\dim A = n$, we have that there exists a chian of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$, which clearly has height n. It follow that $\mathrm{ht}(\mathfrak{p}) = n \leq m$. Now, similarly, we know that there exists a prime $\mathfrak{p}$ of height m, but this must also be less than or equal to n, hence $n \leq m$ and $m \leq n$ implying the claim. ☐

Now note that if we take $H_{\mathfrak{m}}(A) \subset H(A)$ to be the subset of heights of maximal ideals then same result holds. We now prove the following lemma:

Lemma 6.1.3. *Let $\dim A = n$, and $\mathfrak{p} \in \operatorname{Spec} A$, then:*

i) The quotient ring $A/\mathfrak{p}$ is finite dimensional and satisfies:

$$\dim A/\mathfrak{p} \leq \dim A - \mathrm{ht}(\mathfrak{p})$$

with equality if every maximal chain of prime ideals has the same length.

ii) If every maximal chain of prime ideals in A has the same length, then $A/\mathfrak{p}$ is a ring where every maximal chain of prime ideals has the same length.

iii) If every maximal chain of prime ideals has the same length, and $\mathfrak{p}$ is a maximal ideal then $\dim A_{\mathfrak{p}} = \dim A$.

Proof. Note that there is an inclusion preserving bijection $\operatorname{Spec} A/\mathfrak{p} = \mathbb{V}(\mathfrak{p})$. Moreover, $\mathbb{V}(\mathfrak{p})$ consists of all prime ideals which contain $\mathfrak{p}$, hence every chain of prime ideals $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_k$ in $A/\mathfrak{p}$ can be viewed as a chain of prime ideals $\mathfrak{p} \subset \mathfrak{p}_0 \subset \cdots \mathfrak{p}_k$ in A, which must have length less than or equal to n. It follows that at minimum that $A/\mathfrak{p}$ has finite dimension less than or equal to n.

Now let $\mathrm{ht}(\mathfrak{p}) = k$, then we have that there is a chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}$ of length k. Furthermore, let $\dim A/\mathfrak{p} = l$, then by the above discussion there is a chain of prime ideal $\mathfrak{p} = \mathfrak{q}_0 \subset \cdots \mathfrak{q}_l$ in A of length l. We can glue these chains together to get a chain in A of length $l + k$ which must be less than or equal to n. It follows that:

$$\dim A/\mathfrak{p} + \mathrm{ht}(\mathfrak{p}) \leq \dim A$$

implying the inequality.

Suppose that every maximal chain of prime ideals is the same length m; in particular we then have that $H_{\mathfrak{m}}(A) = \{m\}$, $\dim A = m$, and $\mathrm{ht}(\mathfrak{m}) = m$ for all maximal ideals of A. Let $\dim A/\mathfrak{p} = l$, then there exists a chain of prime ideals containing $\mathfrak{p}$, $\mathfrak{q}_0 \subset \cdots \mathfrak{q}_l$, which must have $\mathfrak{q}_0 = \mathfrak{p}$, and $\mathfrak{q}_l = \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, as otherwise we would have $\dim A/\mathfrak{p} > l$. We can extend this to a maximal chain of prime ideals in A:

$$\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_k = \mathfrak{p} = \mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_l = \mathfrak{m}$$

where $\mathrm{ht}(\mathfrak{p}) \geq k$, and by assumption $k + l = m$. It follows that:

$$\dim A \geq \mathrm{ht}(\mathfrak{p}) + \dim A/\mathfrak{p} \geq k + l = \dim A$$

hence $\mathrm{ht}(\mathfrak{p}) + \dim A/\mathfrak{p} = \dim A$, implying $i)$.

For $ii)$, suppose that there was a maximal chain of prime ideals of length $k < \dim A/\mathfrak{p} = l$. Then this corresponds to a chain of prime ideals containing $\mathfrak{p}$, $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_k$, such that $\mathfrak{q}_0 = \mathfrak{p}$ and $\mathfrak{q}_k$ is maximal. We can extend this to a maximal chain of prime ideals for $\mathfrak{p}_0 \subset \cdots \mathfrak{p}_n = \mathfrak{p} = \mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_k$ which must satisfy $k + n = \dim A$ since every maximal chain in A has the same length. It follows that since $\dim A = \dim A/\mathfrak{p} + \mathrm{ht}(\mathfrak{p})$, that $n > \mathrm{ht}(\mathfrak{p})$ a contradiction. Clearly, there can't be a maximal chain of prime ideals of length greater than the dimension, implying $ii)$.

For $iii)$, we have that if every maximal chain of prime ideals has length m, then $H_{\mathfrak{m}}(A) = m$, hence $\mathrm{ht}(\mathfrak{m}) = \dim A_{\mathfrak{m}} = m$ for all $\mathfrak{m}$. It follows that $\dim A_{\mathfrak{m}} = \dim A$ for all $\mathfrak{m}$ as well, as desired. $\square$

Before moving onwards, where we will consider dimension theory in more restrictive cases, we begin our construction of an infinite dimensional Noetherian ring. We first need the following lemma from Atiyah and Bott:

Lemma 6.1.4. *Let A be a ring such that $A_{\mathfrak{m}}$ is Noetherian for all all maximal ideals $\mathfrak{m}$, and for all $a \neq 0 \in A$, we have that a lies in finitely many $\mathfrak{m}$. Then A is Noetherian.*

Proof. Let $I \subset A$ be an ideal, then their exist finitely many maximal ideals $\{\mathfrak{m}_i\}_{i=1}^n$ which contain I, as otherwise there would be some a which lies in infinitely many $\mathfrak{m}$. Let $a \in A$, then for all $1 \leq i \leq n$ we have that $a \in \mathfrak{m}_i$, and that there exist finitely many more $\mathfrak{m}_i$ such that $a \in \mathfrak{m}_i$ for all $1 \leq i \leq n + k$ for some k. We thus obtain the set $\{\mathfrak{m}_i\}_{i=1}^{n+k}$. Choose elements $b_j \in I$ such that $b_j \notin \mathfrak{m}_{n+j}$ for $1 \leq j \leq k$; moreover we have that if $\pi_{\mathfrak{m}_i} : A \to A_{\mathfrak{m}_i}$ is the localization map, then $\langle \pi_{\mathfrak{m}_i}(I) \rangle$ is finitely generated from all i. For each i, there thus exist elements $\{c_{j_i}\}_{j_i=1}^{n_i}$ in a whose image generate $\langle \pi_{\mathfrak{m}_i}(I) \rangle$. Let:

$$J = \langle a, b_l, c_{j_i} : 1 \leq l \leq k, 1 \leq i \leq n, 1 \leq j_i \leq n_i \rangle$$

We wish to show that $\langle \pi_{\mathfrak{m}}(I) \rangle = \langle \pi_{\mathfrak{m}}(J) \rangle$ for all maximal ideals $\mathfrak{m}$. Set $I_{\mathfrak{m}} = \langle \pi_{\mathfrak{m}}(I) \rangle$, and $J_{\mathfrak{m}} = \langle \pi_{\mathfrak{m}}(J) \rangle$. For $1 \leq i \leq n$, this is true as J contains elements which map to the generators of $I_{\mathfrak{m}}$ by construction. For each $\mathfrak{m}_{n+j}$, $1 \leq j \leq k$ this is true as both I and J contain elements (namely b_j) which map to invertible elements in $A_{\mathfrak{m}_{n+j}}$, so the ideals $I_{\mathfrak{m}}$ and $J_{\mathfrak{m}}$ are the whole ring. For any other maximal ideal, $\mathfrak{m}$, we have that $a \notin \mathfrak{m}$ so again the ideals $I_{\mathfrak{m}}$ and $J_{\mathfrak{m}}$ are the whole ring, and it follows that for all $\mathfrak{m} \in |\operatorname{Spec} A|$, we have that $I_{\mathfrak{m}} = J_{\mathfrak{m}}$.

Consider the identity map $\mathrm{Id} : A \to A$. This clearly descends to an A-module homomorphism $\iota : J \to I$. Moreover, for each maximal ideal, we have that Id induces the identity map $A_{\mathfrak{m}} \to A_{\mathfrak{m}}$, which again induces a unique, well defined $A_{\mathfrak{m}}$ module homomorphism $\iota : J_{\mathfrak{m}} \to I_{\mathfrak{m}}$. This map is clearly injective as the it comes from the restriction of an injective map, and moreover, it is surjective as $J_{\mathfrak{m}} = I_{\mathfrak{m}}$, and ι is an A-module homomorphism, so this map is an isomorphism for all $\mathfrak{m} \in |\operatorname{Spec} A|$. Note that ι is also injective, as $J \subset I$ by construction, so we need only show that ι is surjective. Consider the following exact sequence:

$$J \to I \to \operatorname{coker} \iota \to 0$$

This then gives rise to an exact sequence on stalks:

$$J_{\mathfrak{m}} \to I_{\mathfrak{m}} \to (\operatorname{coker} \iota)_{\mathfrak{m}} \to 0$$

but here $\iota_{\mathfrak{m}}$ is surjective, so $(\operatorname{coker} \iota)_{\mathfrak{m}} = \operatorname{coker} \iota_{\mathfrak{m}} = 0$ for all $\mathfrak{m}$.

Now suppose for the sake of contradiction that $\operatorname{coker} \iota \neq 0$. Let $x \in \operatorname{coker} \iota$, and define the ideal:

$$I' = \{a \in A : a \cdot x = 0\}$$

We have that I' is contained in some maximal ideal $\mathfrak{m}$, so consider $I'_{\mathfrak{m}}$. Then, $x/1 \in (\operatorname{coker} \iota)_{\mathfrak{m}}$, but this must be equal to zero as $(\operatorname{coker} \iota)_{\mathfrak{m}} = 0$. This implies that there exists a $y \in A \setminus \mathfrak{m}$ such that $x \cdot y = 0$. However, this means that $y \in I'$ by definition, a contradiction as $I' \subset \mathfrak{m}$. It follows that $\operatorname{coker} \iota = 0$, so ι is surjective, and thus the restriction of the identity map to J takes J to I, implying $I = J$. Therefore, I is finitely generated, and since I was arbitrary A is Noetherian. □

We will also need the following result, known as the prime avoidance lemma.

Lemma 6.1.5. *Let $I \subset A$ be an ideal, and $I \subset \bigcup_{i \in L} \mathfrak{p}_i$ for some finite indexing set L. Then for some i we have that $I \subset \mathfrak{p}_i$.*

Proof. We first assume that $L = \{1, \ldots, n\}$, is arbitrary and proceed by induction. The case where $N = 1$ is obvious. Now suppose that $n = 2$, and that $I \not\subset \mathfrak{p}_1$ and $I \not\subset \mathfrak{p}_2$. Then there exists $a, b \in I$ such that $a \notin \mathfrak{p}_1$ and $b \notin \mathfrak{p}_2$, consequently, we have that $a \in \mathfrak{p}_2$ and $b \in \mathfrak{p}_1$ as otherwise $I \not\subset \mathfrak{p}_1 \cup \mathfrak{p}_2$ and we are done. We claim that $a + b \notin \mathfrak{p}_1$ and $a + b \notin \mathfrak{p}_2$. Indeed, if $a + b \in \mathfrak{p}_1$, then $a + b - b \in \mathfrak{p}_1$ so $a \in \mathfrak{p}_1$, and similarly for $\mathfrak{p}_2$. It follows that $I \not\subset \mathfrak{p}_1 \cup \mathfrak{p}_2$, so by the contrapositive we have that $I \subset \mathfrak{p}_1$ or $I \subset \mathfrak{p}_2$.

Now let $L = \{1, \ldots, n\}$, and assume the result holds for $L' = \{1, \ldots, n-1\}$. If the product:

$$I \cdot \mathfrak{p}_1 \cdots \mathfrak{p}_{n-1} = \langle a \cdot p_1 \cdots p_{n-1} : a \in I, p_1 \in \mathfrak{p}_1, \ldots, p_{n-1}\mathfrak{p}_n \rangle$$

is contained in $\mathfrak{p}_n$, then we have that $I \subset \mathfrak{p}_n$ or $P = (\mathfrak{p}_1 \cdots \mathfrak{p}_{n-1}) \subset \mathfrak{p}_n$. Indeed, if $a \in I$ and $p \in P$ such that $a, p \notin \mathfrak{p}_n$, then their product is in $I \cdot P \subset \mathfrak{p}_n$, contradicting the fact that $\mathfrak{p}_n$ is prime. If $I \subset \mathfrak{p}_n$ then we are done. If $P \subset \mathfrak{p}_n$, then we have that by induction $\mathfrak{p}_i \subset \mathfrak{p}_n$ for some i. If this is the case, then $I \subset \bigcup_{j \neq i \in L} \mathfrak{p}_j$, so by the inductive hypothesis we are done. We may thus assume that $I \cdot P \not\subset \mathfrak{p}_n$.

Furthermore, if for all $a \in I$ we have that $a \in \mathfrak{p}_i$ for some $i \in L'$, then $I \subset \bigcup_{i \in L'} \mathfrak{p}_i$ hence by the inductive hypothesis we are done. So we may further suppose that there exists an element $a \in I$ such that $a \notin \mathfrak{p}_i$ for all $i \in L'$. Now, if $a \notin \mathfrak{p}_n$, then we have that $I \not\subset \bigcup_{i \in L} \mathfrak{p}_i$ so by the contrapositive we are done. Hence we may also assume that $a \in \mathfrak{p}_n$.

Suppose that $a \in \mathfrak{p}_n$, and $I \cdot P \not\subset \mathfrak{p}_n$, but $I \not\subset \mathfrak{p}_i$ for all i. Take an element $b \in I \cdot P$ such that $b \notin \mathfrak{p}_n$; then we claim that $a + b \notin \mathfrak{p}_i$ for all $i \in L$. Indeed, if $a + b \in \mathfrak{p}_i$ for some $i \in L'$, then since $b \in \mathfrak{p}_i$ for all $i \in L'$, we have that $a + b - b \in \mathfrak{p}_i$ a contradiction. Similarly, if $a + b \in \mathfrak{p}_n$ then $a + b - a \in \mathfrak{p}_n$, another contradiction. It, follows that $a + b \notin \mathfrak{p}_i$ for all $i \in L$, hence $I \not\subset \bigcup_{i \in L} \mathfrak{p}_i$, so by the contrapositive we have that $I \subset \mathfrak{p}_i$ for some $i \in L$, implying the claim.

□

We now finally construct our example:

Example 6.1.3. Let $A = k[x_0, x_1, \ldots]$, and define the prime ideals:

$$\mathfrak{p}_i = \langle x_{2^i+1}, x_{2^i+2}, \ldots, x_{2^{i+1}} \rangle$$

for all $i > 0$. We set:

$$S = \bigcap_{i \geq 1} (A \setminus \mathfrak{p}_i)$$

Note that S is multiplicatively closed, indeed if $a, b \in S$, then $a, b, \in A \setminus \mathfrak{p}_i$ for all i. Since $A \setminus \mathfrak{p}_i$ is multiplicatively closed, we have that $a \cdot b \in A \setminus \mathfrak{p}_i$.

We first claim that $S^{-1}A$ is infinite dimensional. Note that for any i we have the following chain:

$$\langle 0 \rangle \subset \langle x_{2^i+1} \rangle \subset \langle x_{2^i+1}, x_{2^i+2} \rangle \subset \cdots \subset \langle x_{2^i+1}, x_{2^i+2}, \ldots, x_{2^{i+1}} \rangle = \mathfrak{p}_i$$

We claim that this is of length $2^{i+1} - 2^i$. Indeed, there are $2^{i+1} - 2^i$ elements which generate $\mathfrak{p}_i$, thus there are $2^{i+1} - 2^i - 1$ inclusions in the above chain ignoring the zero ideal, and when we include the zero ideal inclusion, we get $2^{i+1} - 2^i$ as the length. It follows that $\mathrm{ht}(\mathfrak{p}_i) > 2^{i-1} - 2^i$, which is a strictly increasing sequence of integers. We obtain that for $n \in \mathbb{N}$, we can find and i such that $\mathrm{ht}(\mathfrak{p}_i) > n$, so, via the inclusion preserving bijection from Lemma 6.1.2, we obtain that $\dim S^{-1}A = \infty$.

We will now make use of Lemma 6.1.4 and Lemma 6.1.5 to show that $S^{-1}A$ is Noetherian. Set $S^{-1}\mathfrak{p}_i = \langle \pi(\mathfrak{p}_i) \rangle$, where π is the localization map. We first show that any $f \in k[x_0, x_1, \dots]$ is contained in finitely many $\mathfrak{p}_i$. Indeed, since f is a finite sum of polynomials, there is a maximum j such that x_j appears in the polynomial f. We claim that $f \notin \mathfrak{p}_k$ for any $k \geq j$. Indeed, if $f \in \mathfrak{p}_k$, then there must be a $2^k + m$ for some m such that x_{2^k+m} appears in the polynomial f. However $2^k + m > j$, hence f cannot lie in $\mathfrak{p}_k$. Since there are finitely many $\mathfrak{p}_i$ such that $i < j$, it follows that f must lie in finitely many, perhaps 0, prime ideals of the form $\mathfrak{p}_i$.

Now let $f/g \in S^{-1}A$, and suppose that f/g lies in infinitely many prime ideals of the form $S^{-1}\mathfrak{p}_i$. It follows that $f/1$ lies in infinitely many $S^{-1}\mathfrak{p}_i$, so f lies in infinitely many $\mathfrak{p}_i$, a clear contradiction. It follows that all f/g must lie in finitely many $S^{-1}\mathfrak{p}_i$.

We first claim that:

$$(S^{-1}A)_{S^{-1}\mathfrak{p}_j} \cong A_{\mathfrak{p}_j}$$

for all j. Indeed, note that:

$$S = A \setminus \bigcup_{i \geq 1} \mathfrak{p}_i$$

So consider the localization $\pi_j : A \to A_{\mathfrak{p}_i}$. If $s \in S$, then $s \notin \bigcup_{i \geq 1} \mathfrak{p}_i$, so $s \notin\in \mathfrak{p}_i$ for all i; in particular, $s \notin \mathfrak{p}_j$ so the image of is invertible. It follows that there is a unique map such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\pi_j"] \arrow[d, "\pi"] & A_{\mathfrak{p}_j} \\ S^{-1}A \arrow[ur, "\phi"] & \end{tikzcd}$$

We claim that $A_{\mathfrak{p}_j}$ satisfies the universal property of localization with the localization map given by ϕ. Indeed, let $\psi : S^{-1}A \to B$ be a homomorphism such that for all $b \in \psi(S^{-1}A \setminus S^{-1}\mathfrak{p}_j)$, we have that b is invertible. By the universal property, there is then a unique map $A \to B$ that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\pi"] & B \\ S^{-1}A \arrow[ur, "\psi"] & \end{tikzcd}$$

Let $a \in A \setminus \mathfrak{p}_j$, then $a/1 \in S^{-1}A \setminus S^{-1}\mathfrak{p}_j$, hence $\beta(a) = \psi(a/1)$ is invertible. It follows there exists a unique map α such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\pi_j"] & B \\ A_{\mathfrak{p}_j} \arrow[ur, "\alpha"] & \end{tikzcd}$$

Now, we need only check that $\alpha \circ \phi = \psi$. Recall that the localizations π and π_j are epimorphisms. In particular, we have that:

$$\alpha \circ \phi \circ \pi = \alpha \circ \pi_j = \beta$$

whilst:

$$\psi \circ \pi = \beta$$

hence $\alpha \circ \phi = \psi$ as desired. It follows that $A_{\mathfrak{p}_j}$ is canonically isomorphic to $(S^{-1}A)_{S^{-1}\mathfrak{p}_j}$.

Now let $k(\mathfrak{p}_j^c)$ be the field of fractions of $k[x_i : x_i \notin \mathfrak{p}_j]$. Moreover, set

$$k(\mathfrak{p}_j^c)[\mathfrak{p}_j] = k(\mathfrak{p}_j^c)[x_{2^j+1}, \ldots, x_{2^{j+1}}]$$

We claim that:

$$A_{\mathfrak{p}_i} \cong (k(\mathfrak{p}_j^c)[\mathfrak{p}_j])_{\mathfrak{p}_j'}$$

where

$$\mathfrak{p}_j' = \langle x_{2^j+1}, \ldots, x_{2^{j+1}} \rangle \subset k(\mathfrak{p}_j^c)[\mathfrak{p}_j]$$

There is an obvious inclusion $\iota_A : A \to k(\mathfrak{p}_j^c)[\mathfrak{p}_j]$, so compose this with the localization map $\pi_c : k(\mathfrak{p}_j^c)[\mathfrak{p}_j] \to (k(\mathfrak{p}_j^c)[\mathfrak{p}_j])_{\mathfrak{p}_j'}$. Note that $\iota^{-1}(\mathfrak{p}_j') \subset \mathfrak{p}_j$, hence if $a \notin \mathfrak{p}_j$, we have that $\iota_A(a) \notin \mathfrak{p}_j'$. It follows that $\pi_c \circ \iota_A(a)$ is invertible, hence there exists a unique map α such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\pi_c \circ \iota_A"] \arrow[d, "\pi_j"'] & k(\mathfrak{p}_j^c)[\mathfrak{p}_j]_{\mathfrak{p}_j'} \\ A_{\mathfrak{p}_j} \arrow[ur, dashed, "\alpha"'] & \end{tikzcd}$$

Note that $k(\mathfrak{p}_j^c) = k[\mathfrak{p}_j^c]_0$. There is a canonical map $k[\mathfrak{p}_j^c] \to A_{\mathfrak{p}_j}$ given by inclusion, composed with localization. Every nonzero element in $k[\mathfrak{p}_j^c]$ then maps to invertible element of $A_{\mathfrak{p}_j}$ hence we obtain the following unique diagram:

$$\begin{tikzcd} k[p_j^c] \arrow[r, "\pi_j \circ \iota_k"] \arrow[d, "\pi_0"'] & A_{\mathfrak{p}_j} \\ k(\mathfrak{p}_j^c) \arrow[ur, dashed, "\beta"'] & \end{tikzcd}$$

Note that β is injective as $k(\mathfrak{p}_j^c)$ is a field. Now, we adjoin the variables $\{x_{2^j+1}, \ldots, x_{2^{j+1}}\}$, and obtain a unique map β', such the that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\pi_j"] \arrow[d, "\iota_A"'] & A_{\mathfrak{p}_j} \\ k(\mathfrak{p}_j^c)[\mathfrak{p}_j] \arrow[ur, dashed, "\beta'"'] & \end{tikzcd}$$

Note that β' restricted to the subfield $k(\mathfrak{p}_j^c)$ is just β, and that $\beta'(x_{2^j+m}) = \pi_j(x_{2^j+m})$. Moreover, we have that the unique maximal ideal $\mathfrak{m}_{\mathfrak{p}_j} \subset A_{\mathfrak{p}_j}$ is generated by $\{x_{2^j+1}/1, \ldots, x_{2^{j+1}}/1\}$. It follows that $(\beta')^{-1}(\mathfrak{m}_{\mathfrak{p}_j}) \subset \mathfrak{p}_j'$, hence if $c \notin \mathfrak{p}_j'$, then $\beta'(c) \notin \mathfrak{m}_{\mathfrak{p}_j}$. Therefore, there exists a unique map ξ such the the following square commutes:

$$\begin{tikzcd} A \arrow[r, "\pi_j"] \arrow[d, "\iota"'] & A_{\mathfrak{p}_j} \\ k(\mathfrak{p}_j^c)[\mathfrak{p}_j] \arrow[r, "\pi_c"] \arrow[ur, dashed, "\beta'"] & k(\mathfrak{p}_j^c)[\mathfrak{p}_j]_{\mathfrak{p}_j'} \arrow[u, dashed, "\xi"'] \end{tikzcd}$$

We check that $\alpha \circ \xi = \mathrm{Id}$. Using the fact that localization maps are epimorphisms, we have that:

$$\alpha \circ \xi \circ \pi_c = \alpha \circ \beta'$$

By identifying ι_A as the tensor product of two epimorphisms, $\pi_0 \otimes \mathrm{Id} : k[\mathfrak{p}_j^c] \otimes_k k[\mathfrak{p}_j] \to k(\mathfrak{p}_j^c) \otimes_k k[\mathfrak{p}_j] \cong k(\mathfrak{p}_j^c)[\mathfrak{p}_j]$, we see that ι_A is also an epimorphism. It follows that:

$$\alpha \circ \beta' \circ \iota_A = \alpha \circ \pi_j = \pi_c \circ \iota_A$$

whilst:

$$\mathrm{Id} \circ \pi_c \circ \iota_A = \pi \circ \iota_A$$

hence $\alpha \circ \xi = \mathrm{Id}$ as desired. To see that $\xi \circ \alpha = \mathrm{Id}$, we examine:

$$\xi \circ \alpha \circ \pi_j = \xi \circ \pi_c \circ \iota_A = \beta' \circ \iota_A = \pi_j$$

whilst:

$$\mathrm{Id} \circ \pi_j = \pi_j$$

hence $\xi \circ \alpha = \mathrm{Id}$ as desired. It follows that:

$$(S^{-1}A)_{S^{-1}\mathfrak{p}_j} = k(\mathfrak{p}_j^c)[\mathfrak{p}_j]_{\mathfrak{p}_j'}$$

which is the localization of the Noetherian ring $k(\mathfrak{p}_j^c)[\mathfrak{p}_j]$, so $(S^{-1}A)_{S^{-1}\mathfrak{p}_j}$ is indeed Noetherian.

We now check that each $S^{-1}\mathfrak{p}_i$ is maximal. We claim that $S^{-1}A/S^{-1}\mathfrak{p}_i$ is a field for all i. We need only check that every nonzero element $[a/s] \in S^{-1}A/S^{-1}\mathfrak{p}_i$ has an inverse. Since $[a/s]$ is non zero, we may assume that $a/s \notin S^{-1}\mathfrak{p}_i$, in particular $a/1 \notin S^{-1}\mathfrak{p}_i$. It follows that $a \notin \mathfrak{p}_i$. If a contains a monomial cx_{2^i+1}, with $c \in k$ non zero, then $a \notin \mathfrak{p}_j$ for any $j \neq i$ as well, hence $a \in S$. If a contains no such polynomial, then we consider $a + x_{2^i+1}$ which cannot lie in $\mathfrak{p}_i$ as this would imply a does, and clearly cannot lie in $\mathfrak{p}_j$ for any j, so $a + x_{2^i+1} \in S$. If $a \in S$, then $a/1$ is invertible, so $[a/s]$ is invertible as well. If $a \notin S$, then $a + x_{2^i+1} \in S$, hence we see that:

$$[a/s] \cdot [s/(a + x_{2^i+1})] = [(a + x_{2^i+1})/s] \cdot [s/(a + x_{2^i+1})] = [1]$$

so every nonzero $[a/s]$ is invertible. It follows that $S^{-1}\mathfrak{p}_i$ is maximal for all i.

Finally we show that $S^{-1}\mathfrak{p}_i$ are the only maximal ideals of $S^{-1}A$. Suppose that $\mathfrak{m}$ is a maximal ideal of $S^{-1}A$, then, in particular, we have that $\mathfrak{m}$ corresponds to a prime ideal such that $\mathfrak{q}$ such that $S \cap \mathfrak{q} = \emptyset$. In other words, we have that $\mathfrak{q} \subset \bigcup_i \mathfrak{p}_i$. We need to now prove a generalized form of Lemma 6.1.5, i.e. that this implies that $\mathfrak{q} \subset \mathfrak{p}_i$ for some i. Our approach will mimic this lemma; that is, we will assume that $\mathfrak{q} \not\subset \bigcup_{i \in L} \mathfrak{p}_i$ for any finite L, and show that this implies $\mathfrak{q} \not\subset \bigcup_i \mathfrak{p}_i$. For all $f \in A$, consider the following set:

$$L_f = \{n \in \mathbb{N} : f \in \mathfrak{p}_i\}$$

By our earlier work, we know this is a finite set; let $f \in \mathfrak{q}$, if $L_f \cap L_g \neq \emptyset$ for all $g \in \mathfrak{q}$, then we claim that $\mathfrak{q} \subset \bigcup_{L_f} \mathfrak{p}_i$. Indeed, if $g \in \mathfrak{q}$, and $L_g \cap L_f \neq \emptyset$, then there exist such $i \in L_f$ such that $g \in \mathfrak{p}_i$, hence $\mathfrak{q} \subset \bigcup_{L_f} \mathfrak{p}_i$. It follows, by the contrapositive, that if $\mathfrak{q} \not\subset \bigcup_{i \in L} \mathfrak{p}_i$ for all there must exist some h such that $L_f \cap L_h = \emptyset$. Let $n \in L_h$, and m be the highest degree monomial of f. Then we claim that $f + x_{2^n+1}^{m+1}h \notin \mathfrak{p}_i$ for all i. Note that $L_{x_{2^n+1}^{m+1}h} = L_h$ as $x_{2^n+1}^{m+1} \in \mathfrak{p}_n$, and $h \in \mathfrak{p}_n$, so $x_{2^n+1}^{m+1}h \in \mathfrak{p}_i$ for all i such that $h \in \mathfrak{p}_i$. Note that for all $i \in L_f \cup L_h$, we have that $f + x_{2^n+1}^{m+1}h \notin \mathfrak{p}_i$, as if $f + x_{2^n+1}^{m+1}h \in \mathfrak{p}_i$ for some $i \in L_f \cup L_h$, then i in either L_f or L_h, hence either $f \in \mathfrak{p}_i$ or $x_{2^n+1}^{m+1}h \in \mathfrak{p}_i$, and in either case we obtain that both f and $x_{2^n+1}^{m+1}h$ are in $\mathfrak{p}_i$ contradicting the fact that $L_f \cap L_h = \emptyset$. Now suppose that $i \notin L_f \cup L_h$; since $x_{2^n+1}^{m+1}$ is one degree higher than the highest degree monomial in f, there can be no combination of monomials in the sum $f + x_{2^n+1}^{m+1}h$. Since $i \notin L_f \cup L_h$, it follows that there must a monomial in f which does not lie in $\mathfrak{p}_i$, and since there is no combination of monomials, we must have that the same monomial appears in $f + x_{2^n+1}^{m+1}h$. Therefore, $f + x_{2^n+1}^{m+1}h \notin \mathfrak{p}_i$, as $g \in \mathfrak{p}_i$ implies that each monomial of $g \in \mathfrak{p}_i$ because $\mathfrak{p}_i$ is generated by monomials. It follows that if $\mathfrak{q} \not\subset \bigcup_{i \in L} \mathfrak{p}_i$ for any finite set, then $\mathfrak{q} \not\subset \bigcup_i \mathfrak{p}_i$. By the contrapositive, if $q \subset \bigcup_i \mathfrak{p}_i$, then there exists some finite set L such that $\mathfrak{q} \subset \bigcup_{i \in L} \mathfrak{p}_i$. Lemma 6.1.5 then implies that $\mathfrak{q} \subset \mathfrak{p}_i$ for some i, hence $\mathfrak{m} = S^{-1}\mathfrak{q} \subset S^{-1}\mathfrak{p}_i$, but $S^{-1}\mathfrak{p}_i$ is not the whole ring, so $\mathfrak{m} = S^{-1}\mathfrak{p}_i$.

In conclusion, we have shown that the maximal ideals of $S^{-1}A$ are precisely $S^{-1}\mathfrak{p}_i$, that every element $f/g \in S^{-1}A$ is contained in finitely many $S^{-1}\mathfrak{p}_i$, and that $S^{-1}A_{S^{-1}\mathfrak{p}_i}$ is Noetherian for all i. By Lemma 6.1.4, we have that $S^{-1}A$ is Noetherian, hence $S^{-1}A$ is a Noetherian infinite dimensional ring.

To actually begin calculating the dimensions of our favorite rings, i.e. polynomial rings over a field and their quotients, we will need to study Noether normalization. In particular, this will eventually allow us to calculate the dimension of finitely generated k-algebras. We first review some field theory.

Let K/k be a field extension, i.e. $k \subset K$. Recall that a field extension is algebraic if every element in K is the root of some polynomial in $k[x]$. Moreover, a an extension is finite if K is a finite dimensional k vector space. Every finite extension is algebraic, however there exists non algebraic extensions, which are known as transcendental extensions. Indeed, consider the field extension $\mathbb{Q}(\pi)$, that is the smallest field containing $\mathbb{Q}$ and π. This is not an algebraic field extension as π is not the root of any polynomial in $\mathbb{Q}[x]$[96]. We also see that $\mathbb{Q}(\pi)/\mathbb{Q}$ is an infinite field extension. Indeed, we claim that $\{\pi, \dots, \pi^n\}$ is a $\mathbb{Q}$-linear independent set for all n. Suppose there exist not identically zero $a_i/b_i \in \mathbb{Q}$ such that:

$$\frac{a_1}{b_1}\pi + \cdots + \frac{a_n}{b_n}\pi^n = 0$$

but this now implies that π is the root of the polynomial:

$$p(x) = \frac{a_1}{b_1}x + \cdots + \frac{a_n}{b_n}x^n$$

which is obviously false, so it follows that $a_i/b_i = 0$ for all i. Therefore, $\{\pi, \dots, \pi^n\}$ is a linearly independent set for all values of n, and $\mathbb{Q}(\pi)$ can clearly not be finite dimensional.

Corollary 6.1.1. *Let K/k be a field extension which is not algebraic. Then K is an infinite dimensional vector space.*

Proof. Since K/k is not an algebraic extension, there exists some $\alpha \in K$ such that α is not the root of any polynomial in $k[x]$. The argument for $\mathbb{Q}(\pi)$ applies to K proves the claim. □

Despite transcendental extensions being infinite dimensional vector spaces over the base field, we can still obtain finite numbers from them. Let $K/L/k$ and $K/F/k$ be intermediate field extensions, and denote by $L \cdot F$ the smallest field extension of k which contains both L and F. We write that $L \sim F$ if $L \cdot F$ is an algebraic extension of L and F.

Definition 6.1.3. Let K/k be a field extension; a **transcendence basis** for K is a set of algebraically independent[97] elements S, such that the smallest field extension of k containing S, denoted $k(S)$, satisfies $k(S) \sim K$. The **transcendence degree** K, denoted $\operatorname{tdeg}_k K$, is the cardinality of S.

Assuming that all of this well defined for the moment, we move to the following example:

Example 6.1.4. We immediately see that if K/k is algebraic, then $K \sim k$, so clearly $\operatorname{tdeg}_k K = 0$. Similarly, if $K = \mathbb{Q}(\pi)$ and $k = \mathbb{Q}$, then $\operatorname{tdeg}_{\mathbb{Q}} \mathbb{Q}(\pi) = 1$. Further, if $K'/K/k$, and K' is algebraic over K, then we have $\operatorname{tdeg}_k K = \operatorname{tdeg}_k K'$.

Example 6.1.5. Let $A = k[x_1, \dots, x_n]$, and $K = \operatorname{Frac}(A)$, it's field of fractions. We claim that $\operatorname{tdeg}_k K = n$; let $S = \{x_1, \dots, x_n\}$, then these are algebraically independent over k essentially by the definition of the polynomial ring. We need only check that that $K/k(S)$ is an algebraic extension, but this is easily seen to be true as $k(S) = K$. Indeed, the smallest field which contains S must also contain every polynomial in the x_i, hence every element of A must be invertible in $k(S)$, so $k(S) = A_\eta = K$.

In a sense, the transcendence degree of a field extension is measuring how much the field extension fails to be algebraic. We also note that the transcendence degree of A_η, is what we would expect the Krull dimension of A to be; this connection between transcendence degree and Krull dimension will be made clear with the results to come, but we first we check that all of this makes sense.

Lemma 6.1.6. *Let K/k be a field extension, then a transcendence basis S exists.*

Proof. First note that if there is no nonempty algebraically independent subset of K then k is algebraic. Indeed, this would imply that $\{x\}$ is not an algebraically independent subset, hence the homomorphism:

$$\begin{aligned} k[y] &\longrightarrow K \\ y &\longmapsto x \end{aligned}$$

is not injective, so x is algebraic. It follows that every element of K is algebraic so K/k is an algebraic field extension. In this case, $S = \emptyset$ is a transcendence basis.

Supposing that K/k is not algebraic, let S be an algebraically independent set, and T a subset of K containing S which generates K/k (as a k algebra). Let:

$$\mathscr{B} = \{B \subset K : B \text{ is algebraically independent and } S \subset B \subset T\}$$

[96]This a hard fact to prove

[97]$S \subset K$ is algebraically independent if the map $k[y_s : s \in S] \to K$ given by $y_s \mapsto s$ is injective.

We have that $\mathscr{B}$ is partially order by inclusion, and $\mathscr{B}$ is non empty as it contains S. Take any totally ordered subset $\mathscr{B}' \subset \mathscr{B}$, and set:

$$T' = \bigcup_{B \in \mathscr{B}'} B$$

We see that T' contains S, and is contained in T. We claim T' is algebraically independent, as if it isn't then we the map:

$$k[x_t : t \in T'] \longrightarrow K$$

is not injective, so some $p \in k[x_t : t \in T']$ maps to zero. Since polynomial rings consist of finite sums of finite monomials, it follows that there exists $\{t_1, \dots, t_n\} \subset T'$ such that $p(t_1, \dots, t_n) = 0$. However, since this set is finite, and $\mathscr{B}'$ is totally ordered, we must have that there is some $B \in \mathscr{B}'$ which contains each t_i. This would imply B is not algebraically independent though, hence T' must be algebraically independent as well. It follows that every chain in $\mathscr{B}$ has an upper bound, so by Zorn's lemma there exists a maximal element $S' \in \mathscr{B}$.

We claim that $K/k(S')$ is algebraic; if it was not, then there is some $\alpha \in K$ which is not the root of a polynomial in $k(S')[x]$. It follows that $S' \cup \{\alpha\}$ is then algebraically independent, contradicting the maximality of S', so no such element can exist implying the claim. ☐

Lemma 6.1.7. *Let K/k be a field extension, then the relation $\sim$ is an equivalence relation, and* $\operatorname{tdeg}_k K$ *is well defined.*

Proof. It is clear that for $K/L/k$, and $K/F/k$, we have that $L \sim F \Leftrightarrow F \sim L$, and $L \sim L$, hence the relation is both symmetric and reflexive. We check that this relation is transitive, and thus an equivalence relation.

First note that $L \sim F$ if and only if every element of L is algebraic over F, and every element of F is algebraic over L. Indeed, suppose $L \sim F$, then $x \in L \subset L \cdot F$ is algebraic over F and vice versa. Now, conversely, suppose that every element of L is algebraic over F, and every element of F is algebraic over L, and let $x \in L \cdot F$. In particular, since $L \cdot F$ is the smallest field extension of k containing L and F, hence x can be written as a sum $\sum_i l_i f_i$, where $l_i \in L$ and $f_i \in F$. Each l_i is algebraic over F by assumption, hence each $l_i f_i$ is algebraic over F, so x is algebraic over F. Similarly x is algebraic over L, hence $L \cdot F$ is an algebraic extension of L and F as desired.

Let $L \sim F \sim E$. Let $x \in L$, then x is algebraic over F so must be algebraic over E, hence every element in L is algebraic over E, and vice versa. It follows that $\sim$ is an equivalence relation as desired.

Suppose that $k(S) \sim K$, and S is not finite. If S' is any other transcendence basis, then we must have that $k(S') \sim k(S)$ by the transitivity property of $\sim$. For each $s' \in S'$, there must be a finite set $S_{s'} \subset S$ such that s' is algebraic over $k(S_{s'})$. Set:

$$T = \bigcup_{s' \in S} S_i$$

We have that $T \subset S$; suppose that $S \not\subset T$, then there is some $s \in S \setminus T$ which is algebraic over $k(S')$. However, by construction, $k(S')$ is algebraic over $k(T)$, hence s is algebraic over T. There is then some polynomial $p \in k(T)[x]$ such that $p(s) = 0$, but since $T \subset S$, this implies that S is not algebraically independent. If S' is finite then T is a finite collection of finite sets, and thus finite, so S is finite as well, hence S' is not finite. We then have that:

$$|S| = \left| \bigcup_{s' \in S'} S_{s'} \right| = |S'|$$

as S' is infinite and each $S_{s'}$ is finite.

Now note that if S is finite, then by the above argument we must have that S' is also finite. Let $S = \{s_1, \dots, s_n\}$, and $S' = \{t_1, \dots, t_m\}$, and without loss of generality suppose that $m \leq n$. We proceed via induction on m. If $m = 0$, then S' is empty, and K/k is algebraic, hence $n = 0$ as well. If $m > 0$, then $k(S) \sim k(S')$ so every element of S is algebraic over S'. It follows that we must have that there is an irreducible[98] polynomial $p \in k[y_1, \dots, y_{n+1}]$ such that $p(s_1, \dots, s_n, t_m) = 0$. Since t_m is not algebraic

[98]If it was not irreducible, S or S' would not be algebraically independent.

over k, p cannot be a polynomial entirely in y_{n+1}, so assume that p uses y_n without loss of generality. Let $T = (s_1, \ldots, s_{n-1}, t_m)$, then we claim that $K/k(T)$ is algebraic. To do so, note that $k(T, s_n)/k(T)$ is algebraic as s_n is the root of $p(s_1, \ldots, s_{n-1}, \cdot, t_m) \in k(T)[y_n]$, and that $K/k(T, s_n)$ is algebraic as $S \subset T \cup \{s_n\}$. We thus have the following chain of algebraic extensions:

$$K/k(T, s_n)/k(T)$$

implying that $K/k(T)$ is algebraic. We want to show that T is a transcendence basis; if T is not algebraically independent, then there would be an irreducible polynomial $q \in k[y_1, \ldots, y_n]$ such that $q(s_1, \ldots, s_{n-1}, t_m) = 0$ which must involve y_n as $\{s_1, \ldots, s_{n-1}\}$ is algebraically independent. This implies that t_m is algebraic over $k(s_1, \ldots, s_{n-1})$, so:

$$k(T, s_n)/k(T)/k(s_1, \ldots, s_{n-1})$$

is a chain of algebraic extensions. This implies that s_n is algebraic over $k(s_1, \ldots, s_{n-1})$ which is obviously impossible as S is algebraically independent.

Since T is algebraically independent we have that T is a transcendence basis for K. Now consider $K/k(t_m)$, then we have that $k(S') = (k(t_m))(t_1, \ldots, t_{m-1})$ and $K/K(S')$ is algebraic so $\{t_1, \ldots, t_{m-1}\}$ is a transcendence basis for $K/k(t_m)$. Furthermore, we have that $k(T) = (k(t_m))(s_1, \ldots, s_{n-1})$, and $K/k(T)$ is algebraic, so $\{s_1, \ldots, s_{n-1}\}$ is a a transcendence basis for $K/k(t_m)$. By the inductive hypothesis $n - 1 = m - 1$, hence $n = m$ and we must have $|S| = |S'|$ in the finite case as well.

It follows that $\operatorname{tdeg}_k K$ is independent of our choice of transcendence basis as desired. □

The following result mimics the fact that for finite field extensions $K/L/k$ we have:

$$\dim_k K = \dim_L K + \dim_k L$$

Lemma 6.1.8. *Let $K/L/k$ be field extensions with finite transcendence degrees, then:*

$$\operatorname{tdeg}_k K = \operatorname{tdeg}_L K + \operatorname{tdeg}_k L$$

Proof. Let $S \subset L$ be a transcendence basis for L/k, and $T \subset K$ be a transcendence basis for K/L. We claim that $S \cup T$ is a transcendence basis for K/k. We first show that $K/k(S \cup T)$ is algebraic; examine the following tower of field extensions:

$$K/L(T)/k(S \cup T)$$

Note that $K/L(T)$ is algebraic, so we need only show that $L(T)/k(S \cup T)$ is algebraic. Any element in $L(T)$ can be written as:

$$\sum_i l_i t_i$$

where $l_i \in L$, and $t_i \in T$. Since $L/k(S)$ is algebraic, we have that each l_i is the root of some polynomial in $k(S)[x]$. However, this polynomial also exists in $k(S \cup T)[x]$, hence each l_i, viewed as an element of $L(T)$, is algebraic over $k(S \cup T)$. Each t_i is also algebraic, as the polynomial $x - t_i \in k(S \cup T)[x]$ is a polynomial which has t_i as a root. It follows that any element of $L(T)$ is the sum of products of algebraic elements, and is thus algebraic, so $L(T)/k(S \cup T)$ is an algebraic extension. Since towers of algebraic extensions are algebraic, $K/k(S \cup T)$ is algebraic.

Now let $S = \{s_1, \ldots, s_n\}$, and $T = \{t_1, \ldots, t_m\}$, and consider the homomorphism:

$$\begin{aligned} k[x_1, \ldots, x_n, y_1, \ldots, y_m] &\longrightarrow K \\ x_i &\longmapsto s_i \\ y_i &\longmapsto t_i \end{aligned}$$

If $S \cup T$ is not algebraically independent, then there is some polynomial p in $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ which this map sends to zero. Consider the polynomial:

$$q' = p(s_1, \ldots, s_n, \cdot, \ldots, \cdot)$$

i.e. the polynomial $q' \in K[y_1, \ldots, y_m]$ given by evaluating p on $\{s_1, \ldots, s_m\}$. The coefficients of q' are multiples of elements of k with elements of L, hence the coefficients of q' lie in L, meaning we have that $q' \in L[y_1, \ldots, y_m] \subset K[y_1, \ldots, y_m]$. If q' is identically zero, then the polynomial $q'' = p(\cdot, \ldots, \cdot, 1, \ldots, 1) \in k[x_1, \ldots, x_n]$ has a root at $(s_1, \ldots, s_n)$, implying that S is not algebraically independent, a contradiction. It follows that q' is not identically zero, however, q' then has $(t_1, \ldots, t_m)$ as a root so T is not algebraically independent, another contradiction. We thus see that no such q can exist, hence $S \cup T$ is algebraically independent as desired.

By the above, we have that $S \cup T$ is a transcendence basis, hence:

$$\operatorname{tdeg}_k K = |S \cup T| = |S| + |T| = \operatorname{tdeg}_L K + \operatorname{tdeg}_k L$$

as desired. $\square$

The following lemma will prove useful, and generalizes Example 6.1.5:

Lemma 6.1.9. *Let A be an integral[99] finitely generated k algebra. Then if $K = \operatorname{Frac}(A)$, $\operatorname{tdeg}_k K$ is finite.*

Proof. Since A is finitely generated, and an integral domain, there exists a $\mathfrak{p} \in \mathbb{A}_k^m$ such that:

$$A = k[t_1, \ldots, t_m]/\mathfrak{p}$$

Denote by a_i the image of t_i in A, then clearly $K = k(a_1, \ldots, a_m)$. Let[100]:

$$n = \max\{|B| : B \subset \{a_1, \ldots, a_m\},\ B \text{ is an algebraically independent set over } k\}$$

We claim that $\operatorname{tdeg}_k K = n$; indeed without loss of generality we can take $\{a_1, \ldots, a_n\}$ to be an algebraically independent set, and since for any $i \neq 1, \ldots, n$, the set $\{a_1, \ldots, a_n, a_i\}$ is algebraically dependent, we have that $k(a_1, \ldots, a_n, a_{n+1}, \ldots, a_m)/k(a_1, \ldots, a_n)$ is algebraic. It follows that $\{a_1, \ldots, a_n\}$ is transcendence basis for K, and thus $\operatorname{tdeg}_k K = n$. $\square$

With this notion of transcendence degree we prove the following theorem, known as the Noether Normalization:

Theorem 6.1.1. *Let A be an integral finitely generated k algebra, and K it's field of fractions, as in Lemma 6.1.9. If $\operatorname{tdeg}_k K = n$, then there exists an algebraically independent subset $\{\alpha_1, \ldots, \alpha_n\} \subset A$ over k, such that A is a finite extension of $k[y_1, \ldots, y_n]$.*

Proof. Since A is a finitely generated k algebra, and an integral domain, we can write:

$$A = k[t_1, \ldots, t_m]/\mathfrak{p}$$

for some $\mathfrak{p} \in \mathbb{A}_k^m$, and $m \geq 0$. Denote by a_i the image of t_i under the above projection. By Lemma 6.1.9 we have that $n \leq m$; we proceed by induction on m. The base case, $m = n$, immediately implies that $\{a_1, \ldots, a_m\}$ is a transcendence basis for K, thus $\mathfrak{p} = \langle 0 \rangle$, and so A is trivially a finite extension of $k[y_1, \ldots, y_{n=m}]$.

Now suppose that $m > n$, and we have proven that if $B = k[u_1, \ldots, u_{m-1}]/\mathfrak{q}$, and $\operatorname{tdeg}_k \operatorname{Frac}(B) = n$ then B is a finite extension of $k[y_1, \ldots, y_n]$. Since $n < m$, we have that the map:

$$\begin{aligned} k[t_1, \ldots, t_m] &\longrightarrow A \\ p &\longmapsto p(a_1, \ldots, a_m) \end{aligned}$$

is not injective. Let p lie in the kernel of the above homomorphism. Moreover, for $i \neq m$ set $b_i = a_i - a_m^{r_i}$ for some r_i. Note then that:

$$p(b_1 + a_m^{r_1}, \ldots, b_{m-1} + a_m^{r_{m-1}}, a_m) = 0$$

Let B be the subalgebra generated by b_i, then we want to show that the polynomial $q \in B[x]$ defined by:

$$q(x) = p(b_1 + x^{r_1}, \ldots, b_{m-1} + x^{r_{m-1}}, x)$$

[99] As in A is an integral domain, not that A is integral over k.

[100] Note that such an $n \leq m$ as there are only finitely many subsets of $a_1, \ldots, a_m$, the cardinality of each being bounded above by m.

is monic for some r_i. Now note that:

$$p = \sum_{i_1 \cdots i_m} k_{i_1 \cdots i_m} y_1^{i_1} \cdots y_m^{i_m}$$

so:

$$q(x) = \sum_{i_1 \cdots i_m} k_{i_1 \cdots i_m} (b_1 + x^{r_1})^{i_1} \cdots (b_{m-1} + x^{r_{m-1}})^{i_{m-1}} x^{i_m}$$

We can thus choose $\{r_1, \ldots, r_{m-1}\}$ so that the highest degree term of $q(x)$ is contained in the single monomial:

$$k_{i_1 \cdots i_m} (b_1 + x^{r_1})^{j_1} \cdots (b_{m-1} + x^{r_{m-1}})^{j_{m-1}} x^{j_m}$$

for some $j_1 \cdots j_m$. The polynomial:

$$q(x) = \sum_{i_1 \cdots i_m} \frac{k_I}{j_1 \cdots j_m} (b_1 + x^{r_1})^{i_1} \cdots (b_{m-1} + x^{r_{m-1}})^{i_{m-1}} x^{i_m}$$

is then monic, and satisfies $q(a_m) = 0$. It follows by Corollary 3.9.1 that A is an integral B algebra.

We now note that A is a finitely generated B algebra; indeed, we have that each $a_i \neq a_m$ is the image of $b_i + a_m^{r_i}$, and that $a_m = b_m$. By Proposition 3.9.1 we now have that A is a finite B algebra.

Since B is a subalgebra of an integral domain, B is an integral domain. Let:

$$\begin{aligned} k[u_1, \ldots, u_{m-1}] &\longrightarrow B \subset A \\ u_i &\longmapsto b_i \end{aligned}$$

then $B \cong k[u_1, \ldots, u_{m-1}]/\mathfrak{q}$ for some prime ideal $\mathfrak{q}$. In particular, $\mathrm{Frac}(B) = k(b_1, \ldots, b_{m-1})$. In fact, we have that:

$$K = k(a_1, \ldots, a_m)/k(b_1, \ldots, b_{m-1})$$

is algebraic, as $q \in k(b_1, \ldots, b_{m-1})[x]$ as well. It follows by Lemma 6.1.8 that:

$$\mathrm{tdeg}_k K = \mathrm{tdeg}_{\mathrm{Frac}(B)} K + \mathrm{tdeg}_k \mathrm{Frac}(B) = \mathrm{tdeg}_k \mathrm{Frac}(B)$$

hence by the inductive hypothesis there exists a finite extension:

$$k[y_1, \ldots, y_n] \to B$$

By Lemma 3.9.1 we have that the composition:

$$k[y_1, \ldots, y_n] \to B \to A$$

makes A a finite $k[y_1, \ldots, y_n]$ algebra. Since each map is injective, A is a finite extension of $k[y_1, \ldots, y_n]$, and letting α_i denote the image of y_i provides us with an algebraically independent subset of A, implying the claim. □

We will need the following lemma to make the connection between transcendence degree, and the Krull dimension of finitely generated integral domains.

Lemma 6.1.10. *Let $\phi : B \to A$ be an integral extension, then* $\dim A = \dim B$.

Proof. The fact that ϕ is an integral extension, means that Going Up (i.e. Lemma 3.10.4) holds for the induced map $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$. In particular, if we have a chain of prime ideals of B:

$$\langle 0 \rangle \subset \mathfrak{p}_1 \subset \cdots \mathfrak{p}_n$$

then by inductively applying Going Up, we obtain a chain of prime ideals in A:

$$\langle 0 \rangle \subset \mathfrak{q}_1 \subset \cdots \mathfrak{q}_n$$

where $\phi^{-1}(\mathfrak{q}_i) = \mathfrak{p}_i$. Note that $\mathfrak{q}_i \neq \mathfrak{q}_{i+1}$, hence $\dim B \leq \dim A$.

Now note that given a chain of prime ideals in A:

$$\langle 0\rangle \subset \mathfrak{q}_1 \subset \cdots \mathfrak{q}_n$$

we obtain a chain of prime ideals in B given by:

$$\langle 0\rangle \subset \mathfrak{p}_1 \subset \cdots \mathfrak{p}_n$$

where $\mathfrak{p}_i = \phi^{-1}(\mathfrak{q}_i)$. If we can show that $\phi^{-1}(\mathfrak{q}_i) \neq \phi^{-1}(\mathfrak{q}_{i+1})$ for all i, then we will have $\dim A \leq \dim B$ and be done.

We have that:

$$f^{-1}(\mathfrak{p}) = \operatorname{Spec} k_{\mathfrak{p}} \otimes_B A$$

we claim that $k_{\mathfrak{p}} \otimes_B A$ is a zero dimensional ring. Indeed, by Proposition 3.9.2, integral morphisms are preserved by base change, hence $k_{\mathfrak{p}} \to k_{\mathfrak{p}} \otimes_B A$ is integral. Moreover, it is injective as $k_{\mathfrak{p}}$ is a field. It thus suffices to show that any integral extension $k \to A$ implies $\dim A = 0$. Let $\mathfrak{p} \subset A$ be a prime, then we claim that $A/\mathfrak{p}$ is a field, and thus every prime is maximal. Note that the composition $k \to A/\mathfrak{p}$ is now an integral extension of k into an integral domain. Let $[a] \in A/\mathfrak{p}$ be nonzero, then we have that there exists a monic polynomial of smallest possible degree with coefficients in k satisfying:

$$[a]^n + c_{n-1}[a]^{n-1} + \cdots + c_0 = 0$$

Since $A/\mathfrak{p}$ is an integral domain, we then have that $c_0 = 0$, as otherwise $[a] = 0$, or the polynomial is not of smallest degree. In particular we have that:

$$1 = -c_0^{-1}([a]^n + c_{n-1}[a]^{n-1} + \cdots + c_1[a])$$

so:

$$[a]^{-1} = -c_0^{-1}([a]^{n-1} + c_{n-2}[a]^{n-1} + \cdots + c_1)$$

implying that $A/\mathfrak{p}$ is a field. It follows that every prime ideal is maximal and thus $\dim A = 0$. In particular, we have that $k_{\mathfrak{p}} \otimes_B A$ is zero dimensional ring, so if $\phi^{-1}(\mathfrak{q}_i) = \phi^{-1}(\mathfrak{q}_{i+1})$ then we cannot have $\mathfrak{q}_i \subset \mathfrak{q}_{i+1}$ as this would imply that $\dim k_{\mathfrak{p}} \otimes_B A$ has dimension greater than zero. □

The following result is our *entire motivation* of going over the notion of transcendence degree:

Theorem 6.1.2. *Let A be an integral finitely generated k algebra, and K it's field of fractions as in Lemma 6.1.9; then* $\dim A = \operatorname{tdeg}_k K$.

Proof. We prove this on induction of $\operatorname{tdeg}_k K$. If $\operatorname{tdeg}_k K = 0$, then we have by Noether Normalization a finite extension $k \to A$. Proposition 3.9.1 and Lemma 6.1.10 then imply the base case.

Supposing this holds for transcendence degrees less than n, suppose that $\operatorname{tdeg}_k K = n$. Again by Noether Normalization, we have a finite extension $k[x_1, \ldots, x_n] \to A$. It thus suffices to show that $\dim k[x_1, \ldots, x_n] = n$ by Lemma 6.1.10. Note that $\dim k[x_1, \ldots, x_n] \geq n$ as we always have the following chain of ideals:

$$\langle 0\rangle \subset \langle x_1\rangle \subset \cdots \subset \langle x_1, \ldots, x_n\rangle$$

Now suppose there exists a chain of prime ideals:

$$\langle 0\rangle \subset \mathfrak{p}_1 \cdots \mathfrak{p}_m$$

where $m > n$. Then take an irreducible element $f \in \mathfrak{p}_1$, and construct the chain of prime ideals:

$$\langle 0\rangle \subset \langle f\rangle \cdots \mathfrak{p}_m$$

We see that $\dim k[x_1, \ldots, x_n]/\langle f\rangle$ has dimension at least $m - 1 \geq n$. We claim this is a contradiction, as $\operatorname{tdeg}_k \operatorname{Frac}(k[x_1, \ldots, x_n]/\langle f\rangle) = n - 1$. Indeed, without loss of generality assume that x_n occurs in f, then with $B = k[x_1, \ldots, x_n]/\langle f\rangle$, we claim that $\{b_i = [x_i]\}_{i=1}^{n-1}$ is a transcendence basis for $\operatorname{Frac}(B)/k$. We see that this algebraically independent as the map:

$$\begin{aligned} k[y_1, \ldots, y_{n-1}] &\longrightarrow B \subset \operatorname{Frac}(B) \\ y_i &\longmapsto b_i \end{aligned}$$

Suppose that $p \mapsto 0 \in B$, then in particular, if:

$$p = \sum_{i_1 \cdots i_{n-1}} k_{i_1 \cdots i_{n-1}} y_1^{i_1} \cdots y_n^{i_{n-1}}$$

we have that:

$$\sum_{i_1 \cdots i_{n-1}} k_{i_1 \cdots i_{n-1}} [x_1]^{i_1} \cdots [x_{n-1}]^{i_{n-1}} = 0 \Rightarrow \sum_{i_1 \cdots i_{n-1}} k_{i_1 \cdots i_{n-1}} x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} \in \langle f \rangle$$

which is impossible by construction. We claim that $\mathrm{Frac}(B)/k(b_1, \ldots, b_{n-1})$ is algebraic. Indeed, if:

$$f = \sum_{i_1 \cdots i_n} c_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

let $g \in k(b_1, \ldots, b_{n-1})[x]$ be given by:

$$g = \sum_{i_1 \cdots i_n} c_{i_1 \cdots i_n} b_1^{i_1} \cdots b_{n-1}^{i_{n-1}} x^{i_n}$$

then clearly $g(b_n) = 0$, so $\mathrm{Frac}(B)/k(b_1, \ldots, b_{n-1})$ is indeed algebraic. It follows that $\{b_1, \ldots, b_{n-1}\}$ is a transcendence basis, thus $\dim B = n - 1$, contradicting the existence of a chain of prime ideals in $k[x_1, \ldots, x_n]$ of length $m > n$. Therefore, $\dim k[x_1, \ldots, x_n] \leq n$, implying equality, and so $\dim A = n$ as well. □

We end this section by noting we now have a slick proof of Zariski's lemma:

Theorem 6.1.3. *Let A be a finitely generated k algebra, and $\mathfrak{m} \in |\operatorname{Spec} A|$. Then $k_{\mathfrak{m}}/k$ is a finite extension of k.*

Proof. Note that the residue field $k_{\mathfrak{m}}$ is given precisely by $A/\mathfrak{m}$. In particular, we know that $k_{\mathfrak{m}}$ has dimension 0 as it is a field, and that $k_{\mathfrak{m}}$ is a finitely generated k algebra, via the composition:

$$k \hookrightarrow A \to k_{\mathfrak{m}}$$

The field of fractions of $k_{\mathfrak{m}}$ is then obviously $k_{\mathfrak{m}}$, hence we have that $\mathrm{tdeg}_k\, k_{\mathfrak{m}} = 0$. In particular, $k_{\mathfrak{m}}/k$ is an algebraic, i.e. integral extension, and is finitely generated, hence by Proposition 3.9.2 we have that $k_{\mathfrak{m}}/k$ is finite. □

6.2 Dimension of Schemes